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A REMARK ON LOCAL BEHAVIOR OF CHARACTERISTIC FUNCTIONS

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It is shown that, if, for a distribution function F, $1 - F(x) + F(-x)$ varies regularly at ∞ with exponent α, $0 > \alpha > -1$, then $|\mathrm{Im}\,\phi(t)| = O(1 - \mathrm{Re}\,\phi(t))$ $(t \to 0)$, where ϕ is the characteristic function of F. Versions for $\alpha \leqq -1$ are also given.

1. Let F be a distribution function and $\phi = \chi + i\psi$ its characteristic function. Kesten (1972) page 715, has asked when

$$(*) \qquad |\psi(t)| = O(1 - \chi(t)) \quad \text{as} \quad t \to 0+ .$$

Some sufficient but not necessary conditions are given in this note; in particular, if $H(x) = 1 - F(x) + F(-x)$ varies regularly with exponent $\alpha < 0$, then (*), or an extended version, holds. The extended version is designed to account for cases when (*) fails, superficially, because moments of F are nonzero; for example, if F has a nonzero mean and finite variance, (*) fails. This behavior is easily explained by the Theorem given below. An example for which (*) fails essentially has not been found.

2. Let $H(x) = 1 - F(x) + F(-x)$ and $G(x) = 1 - F(x) - F(-x)$ be the tail sum and tail difference of F; then

$$\chi(t) = 1 + \int_{0+}^{\infty} (1 - \cos tx)\, dH(x) \qquad \text{and} \qquad \psi(t) = -\int_0^\infty \sin tx\, dG(x) .$$

Introduce the symmetric mean $\mu_1{}^* = -\int_0^\infty x\, dG(x) = \lim_{T\to+\infty} \int_{-T}^{T} x\, dF(x)$ and the second moment $\mu_2 = -\int_0^\infty x^2\, dH(x) = \int_{-\infty}^{\infty} x^2\, dF(x)$. Exclude the trivial case $G \equiv 0$ (F symmetric) and exclude the case where F has bounded support, so that $H(x) \neq 0$ for any x.

THEOREM. *Let H vary regularly with index α. Then*

(i) *For* $0 > \alpha > -1$, $|\psi(t)| = O(1 - \chi(t))$, $(t \to 0+)$.
(ii) *For* $\alpha = -1$, $|\psi(t) + t\int_0^{1/t} x\, dG(x)| = O(1 - \chi(t))$, $(t \to 0+)$.
(iii) *For* $-1 > \alpha \geqq -2$, $|\psi(t) - t\mu_1{}^*| = O(1 - \chi(t))$, $(t \to 0+)$.
(iv) *For* $-2 > \alpha > -3$, $|\psi(t) - t\mu_1{}^*| = O[-(1 - \chi(t) - (t^2/2)\mu_2)]$, $(t \to 0+)$.

PROOF. Now,

$$1 - \chi(t) = -\int_0^\infty (1 - \cos tx)\, dH(x) \geqq -\int_{\varepsilon/t}^{1/t} (1 - \cos tx)\, dH(x)$$
$$\geqq (1 - \cos \varepsilon) H(\varepsilon t^{-1})[1 - H(t^{-1})/H(\varepsilon t^{-1})] ,$$

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where $0 < \varepsilon < 1$. That H is of index α means that $H(t^{-1})/H(\varepsilon t^{-1}) \to \varepsilon^{-\alpha} < 1$ $(t \to 0+)$ provided $\alpha < 0$. Choose δ small enough for $\varepsilon^{-\alpha} + \delta < 1$; then t can be chosen so small that $H(t^{-1})/H(\varepsilon t^{-1}) \leqq \varepsilon^{-\alpha} + \delta < 1$. Hence

$$1 - \chi(t) \geqq H(\varepsilon t^{-1})(1 - \cos \varepsilon)(1 - \varepsilon^{-\alpha} - \delta) > 0 .$$

Also,

$$\begin{aligned}|\psi(t)| &= |-\textstyle\int_0^\infty \sin tx\, dG(x)| = t|\int_0^\infty \cos txG(x)\,dx| \\ &\leqq t|\textstyle\int_0^{\varepsilon/t} \cos txG(x)\,dx| + t|\int_{\varepsilon/t}^\infty \cos txG(x)\,dx| ,\end{aligned}$$

after integrating by parts. Further,

$$t|\textstyle\int_0^{\varepsilon/t} \cos txG(x)\,dx| \leqq t \int_0^{\varepsilon/t} H(x)\,dx = \varepsilon H(\varepsilon t^{-1}) \int_0^1 H(x\varepsilon t^{-1})/H(\varepsilon t^{-1})\,dx ,$$

after noting that $|G| \leqq H$. Finally

$$\begin{aligned}t|\textstyle\int_{\varepsilon/t}^\infty \cos txG(x)\,dx| &= t|[1 - F(\varepsilon t^{-1})] \textstyle\int_{\varepsilon/t}^{\xi_1} \cos tx\,dx - F(-\varepsilon t^{-1}) \int_{\varepsilon/t}^{\xi_2} \cos tx\,dx| \\ &\leqq 2[1 - F(\varepsilon t^{-1}) + F(-\varepsilon t^{-1})] = 2H(\varepsilon t^{-1})\end{aligned}$$

by the second mean value theorem, where $\xi_1, \xi_2 \geqq \varepsilon/t$. Thus

$$\frac{|\psi(t)|}{1 - \chi(t)} \leqq [\varepsilon \textstyle\int_0^1 [H(x\varepsilon t^{-1})/H(\varepsilon t^{-1})]\,dx + 2][(1 - \cos \varepsilon)(1 - \varepsilon^{-\alpha} - \delta)]^{-1}$$

so that, taking $t \to 0+$ and $\delta \to 0+$, (cf. Feller (1966) page 281),

$$\limsup_{t\to 0+} \frac{|\psi(t)|}{1 - \chi(t)} \leqq [\varepsilon/(\alpha + 1) + 2][(1 - \cos \varepsilon)(1 - \varepsilon^{-\alpha})]^{-1}$$

provided $0 > \alpha > -1$. This completes the proof of (i). Now consider

$$\begin{aligned}&|\psi(t) + t \textstyle\int_0^{1/t} x\,dG(x)| \\ &\quad = |-t \textstyle\int_0^{\varepsilon/t} (1 - \cos tx)G(x)\,dx - t \int_{\varepsilon/t}^{1/t} G(x)\,dx \\ &\qquad + G(t^{-1}) + t \textstyle\int_{\varepsilon/t}^\infty \cos txG(x)\,dx| \\ &\quad \leqq \varepsilon \textstyle\int_0^1 (1 - \cos \varepsilon x)H(\varepsilon t^{-1}x)\,dx + (1 - \varepsilon)H(\varepsilon t^{-1}) + H(\varepsilon t^{-1}) + 2H(\varepsilon t^{-1})\end{aligned}$$

using the same arguments as before. Using the same estimate for $1 - \chi(t)$, it will be seen that

$$\limsup_{t\to 0+} \frac{|\psi(t) + t \int_0^{1/t} x\,dG(x)|}{1 - \chi(t)} \leqq [\varepsilon^3/(\alpha + 3) + 4][(1 - \cos \varepsilon)(1 - \varepsilon^{-\alpha})]^{-1} ,$$

valid for $-1 \geqq \alpha > -3$. This proves the result for $\alpha = -1$. If $-1 > \alpha > -3$, in addition

$$\psi(t) - t\mu_1{}^* = \psi(t) + t \textstyle\int_0^{1/t} x\,dG(x) + t \int_{1/t}^\infty x\,dG(x) ,$$

while

$$\begin{aligned}|t \textstyle\int_{1/t}^\infty x\,dG(x)| &= |G(t^{-1}) + t \textstyle\int_{1/t}^\infty G(x)\,dx| \\ &\leqq H(\varepsilon t^{-1})(1 + \textstyle\int_1^\infty H(xt^{-1})/H(\varepsilon t^{-1})\,dx)\end{aligned}$$

and $\int_1^\infty H(xt^{-1})/H(\varepsilon t^{-1})\,dx \leqq \int_1^\infty H(\varepsilon xt^{-1})/H(\varepsilon t^{-1})\,dx \to -(\alpha + 1)^{-1}$ as $t \to 0+$.

This proves (iii). For $-2 > \alpha > -3$, the estimate for χ is sharpened in exactly the same way:

$$-\left(1 - \chi(t) - \frac{t^2}{2}\mu_2\right) = -\int_0^\infty \left(\frac{t^2x^2}{2} - 1 + \cos tx\right) dH(x)$$
$$\geqq (\tfrac{1}{2}\varepsilon^2 - 1 + \cos\varepsilon)(1 - \varepsilon^{-\alpha} - \delta)H(\varepsilon t^{-1})$$

as before. It is clear that the process can be repeated for values of $\alpha \leqq -3$.

REMARKS. (i) Referring to Feller (1966) page 313, shows that (*), or an extended version, holds for any distribution in the domain of attraction of a stable law.

(ii) An examination of the proof of the Theorem shows that all that is actually required is that $H(t^{-1})/H(\varepsilon t^{-1})$ be bounded away from 1 as $t \to 0+$ for $0 < \varepsilon < 1$, and that $|G(x\varepsilon t^{-1})|/H(\varepsilon t^{-1}) \leqq g(x)$ where $g \in L\ (0, 1)$. Both of these are satisfied if there exist constants c, $c' > 0$ such that $c'\varepsilon^\alpha \leqq H(\varepsilon x)/H(x) \leqq c\varepsilon^\alpha$ for x large and $0 < \varepsilon < 1$. A condition of this type holds for semistable F (Pillai (1971)). The condition is related to the concept of "dominated variation" (Feller (1966)).

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