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LOWER CLASS SEQUENCES FOR THE SKOROHOD-STRASSEN APPROXIMATION SCHEME

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Let $S_n = X_1 + \cdots + X_n$ where $\{X_k\}_{k\geq 1}$ is a sequence of independent, identically distributed random variables with mean zero and variance one. By the Skorohod representation S_n has the same distribution as $\xi(U_n)$ where ξ is standard Brownian motion. We find increasing sequences of real numbers $\{c_n\}$ and $\{d_n\}$ such that

$$\limsup_{n\to\infty} \frac{\xi(U_n) - \xi(n)}{(c_n \lg n)^{\frac{1}{2}}} = \infty \quad \text{a.s.}$$

and

$$\limsup_{n\to\infty} \frac{\xi(U_n) - \xi(n)}{(d_n \lg n)^{\frac{1}{2}}} = 0 \quad \text{a.s.}$$

We conclude with an example which explicitly gives the sequences $\{c_n\}$ and $\{d_n\}$ in terms of the original random variables $\{X_k\}$.

1. Introduction. Let $\{X_n\}_{n\geq 1}$ be independent random variables with the same distribution; make the normalizations $E(X_n) = 0$, $E(X_n^2) = 1$; and let $S_n = X_1 + \cdots + X_n$. Different embeddings of S_n into Brownian motion have been constructed (by Skorohod, Breiman, Root, Dubins and Monroe) with the following common properties (see Breiman (1968) pages 276–278). There exists a probability space $(\Omega, \mathscr{B}, P)$ with a Brownian motion $\xi(t)$ (normalized so that $E[\xi(t)] = 0$ and $E[\xi^2(t)] = t$) and a sequence of nonnegative, independent, identically distributed random variables $\{T_i\}_{i\geq 1}$ defined on it such that the following conditions hold.

$$\{\xi(\textstyle\sum_{i=1}^n T_i)\}_{n\geq 1} \quad \text{has the same distribution as} \quad \{S_n\}_{n\geq 1}\,. \tag{1.1}$$

$$E(T_n) = E(X_n^2) = 1\,. \tag{1.2}$$

Recent proofs of the law of the iterated logarithm rely on such a Skorohod-type embedding. Let $U_n = \sum_{i=1}^n T_i$, $V_n = U_n - n$, and $W_n = \xi(U_n) - \xi(n)$. Using the strong law of large numbers on V_n it is shown (see Breiman (1968) pages 291–292) that

$$\limsup_{n\to\infty} W_n/(n \lg\lg n)^{\frac{1}{2}} = 0 \quad \text{a.s.} \tag{1.3}$$

where $\lg n = \log_e n$.

Kiefer (1969) considered a more specialized case. Assuming $E(T_i - 1)^2 = \beta < \infty$ he shows that

$$\limsup_{n\to\infty} W_n/((2\beta n \lg\lg n)^{\frac{1}{2}} \lg n)^{\frac{1}{2}} = 1 \quad \text{a.s.} \tag{1.4}$$

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Notice that he finds a finite, positive number for the lim sup. His proof relies on the law of the iterated logarithm applied to the sequence $\{V_n\}$. Kostka (1972) generalized this result. Essentially he shows that if there exists a sequence of numbers $\{c_n\}_{n\geq 1}$ such that

(1.5) $$\limsup_{n\to\infty} |V_n|/c_n = 1 \quad \text{a.s.}$$

then

(1.6) $$\limsup_{n\to\infty} W_n/(c_n \lg n)^{\frac{1}{2}} = 1 \quad \text{a.s.}$$

Kesten (1972) showed, in essence, that the normalizing sequence c_n in (1.5) exists only if the T_i belong to the domain of partial attraction of the normal law. The purpose of this paper is to prove equations analogous to (1.6) without assuming the existence of an exact normalizing sequence c_n as given in (1.5).

Assuming

(1.7) $$\limsup_{n\to\infty} |V_n|/d_n = 0 \quad \text{a.s.}$$

and

(1.8) $$\limsup_{n\to\infty} V_n/c_n = \infty \quad \text{a.s.}$$

then basically we show that

(1.9) $$\limsup_{n\to\infty} W_n/(c_n \lg n)^{\frac{1}{2}} = \infty \quad \text{a.s.}$$

and

(1.10) $$\limsup_{n\to\infty} W_n/(d_n \lg n)^{\frac{1}{2}} = 0 \quad \text{a.s.}$$

Statement (1.10) is an easy upper class result for W_n. Statement (1.9) gives a lower class result for W_n and indicates how exact $(d_n \lg n)^{\frac{1}{2}}$ is as an upper bound for the fluctuations of W_n.

We conclude with an example which explicitly gives the sequences $\{c_n\}$ and $\{d_n\}$ in terms of the original random variables $\{X_n\}$.

2. The results. The following is an easy upper class result (see Strassen (1967) or Kostka (1972)) about the fluctuations in a Skorohod-type embedding versus the Brownian motion.

(2.1) PROPOSITION. *Suppose* $\limsup_{n\to\infty} |V_n|/d_n = 0$ *with probability one where* $\{d_n\}$ *is a sequence of positive numbers, then* $\limsup_{n\to\infty} W_n/(d_n \lg n)^{\frac{1}{2}} = 0$ a.s. *where* $\lg n = \log_e n$.

Under additional assumptions on the random times T_i, we can prove lower class equations of the form (1.9). In the following theorem a regularly varying sequence means a regularly varying function whose domain is restricted to the positive integers. The theorem can be proved for more general sequences, but these are sufficient for most examples.

(2.2) THEOREM. *Suppose there exist sequences of real numbers* $\{c_n\}$ *and* $\{d_n\}$ *which are regularly varying with exponent* m, $0 < m < 1$, *and satisfy* $c_n/n^m \uparrow$,

$d_n/n^m \uparrow$, and $(c_n \lg n)/d_n \uparrow \infty$ as $n \to \infty$. Furthermore suppose that

$$\limsup_{n\to\infty} |V_n|/d_n = 0 \quad \text{a.s.} \tag{2.3}$$

and

$$\limsup_{n\to\infty} V_{n_k}/c_{n_k} = \infty \quad \text{a.s.} \tag{2.4}$$

where $\{n_k\}$ is a subsequence of the integers such that $n_k \geqq \gamma^k$ for some $\gamma \geqq 2$ and

$$\textstyle\sum_{i=1}^r (n_i)^p = O((n_r)^p)$$

for some p, $m < p < 1$. Then

$$\limsup_{n\to\infty} W_n/(c_n \lg n)^{\frac{1}{2}} = \infty \quad \text{a.s.} \tag{2.5}$$

and

$$\limsup_{n\to\infty} W_n/(d_n \lg n)^{\frac{1}{2}} = 0 \quad \text{a.s.} \tag{2.6}$$

PROOF. Statement (2.6) follows immediately from Proposition (2.1). To prove statement (2.5) for $0 < m < p < 1$ let

$$F_{r,p} = \left\{\max_{1\leqq i\leqq (n_r)^p} |U_{n_r+i} - U_{n_r} - i| < \frac{\varepsilon}{2} c_{n_r}\right\}. \tag{2.7}$$

We will now show that $\overline{F_{r,p}}$, the complement of $F_{r,p}$, occurs only finitely often a.s. as $r \to \infty$. By the assumption on the sequence $\{T_i\}$

$$|\textstyle\sum_{i=1}^m (T_i - 1)| \leqq 2d_m \tag{2.8}$$

for all but a finite number of m a.s. Thus if (2.8) holds for two values of $m = M$ and $m = k > M$, then

$$\begin{aligned} |\textstyle\sum_{i=M+1}^k (T_i - 1)| &\leqq |\textstyle\sum_{i=1}^k (T_i - 1)| + |\textstyle\sum_{i=1}^M (T_i - 1)| \\ &\leqq 2d_k + 2d_M . \end{aligned}$$

In particular

$$\max_{n_r<k\leqq n_r+(n_r)^p} |\textstyle\sum_{i=n_r}^k (T_i - 1)| \leqq 4d_{k_r} \tag{2.9}$$

for all but a finite number of n_r a.s. where $k_r = \sum_{i=1}^r (n_i)^p$. But by the assumption on n_r

$$k_r = \textstyle\sum_{i=1}^r (n_i)^p \leqq C(n_r)^p \tag{2.10}$$

for some $C > 0$. Now, $d_n = n^m L(n)$ where $L(n)$ is slowly varying and thus

$$d_{k_r} = (k_r)^m L(k_r) = O[(n_r)^{pm} L(n_r)] = O[(n_r)^{pm+\alpha}]$$

for $\alpha > 0$ since the slow variation of $L(n)$ implies $L(n) < n^\alpha$ for $n \geqq N(\alpha)$. Thus

$$4d_{k_r} \leqq \frac{\varepsilon}{2}(n_r)^m < \frac{\varepsilon}{2} c_{n_r}$$

for r sufficiently large. Thus only finitely many $\overline{F_{r,p}}$ occur a.s. Let $K > 0$ be an arbitrarily large fixed constant. Now the event

$$K^2 c_{n_r} < U_n - n < (1+\varepsilon) d_{n_r} \tag{2.11}$$

for all n such that $n_r \leqq n \leqq n_r + (n_r)^p$ occurs for infinitely many r a.s.

Let J_r = integral part of $(n_r)^p/4d_{n_r}$ and for $0 \leqq i < J_r$ define

$$\begin{aligned} n'_{r,i} &= n_r + \operatorname{int}(2id_{n_r}) \\ n''_{r,i} &= n_r + \operatorname{int}(2id_{n_r} + K^2 c_{n_r}) \\ A_{r,i} &= \{\xi(n''_{r,i}) - \xi(n'_{r,i}) > K(p-m)^{\frac{1}{2}}(c_{n_r} \lg n_r)^{\frac{1}{2}}\} \\ B_{r,i} &= \{\sup_{0 \leqq x \leqq (1+\varepsilon)d_{n_r} - K^2 c_{n_r}} |\xi(n''_{r,i} + x) - \xi(n''_{r,i})| < \mu(c_{n_r} \lg n_r)^{\frac{1}{2}}\} \end{aligned}$$

where $0 < \mu < 1$.

$$\begin{aligned} C_{r,i} &= A_{r,i} \cap B_{r,i} \\ Q_r &= \textstyle\bigcup_{i=1}^{J_r} C_{r,i} . \end{aligned}$$

Suppose (2.11) holds for $n = n'_{r,i}$, $0 \leqq i < J_r$, then

$$\begin{aligned} n''_{r,i} &\leqq n'_{r,i} + K^2 c_{n_r} \\ &\leqq U_{n'_{r,i}} \\ &\leqq n'_{r,i} + (1+\varepsilon) d_{n_r} \\ &\leqq n''_{r,i} + (1+\varepsilon) d_{n_r} - K^2 c_{n_r} . \end{aligned}$$

This together with $C_{r,i}$ entails

$$\xi(U_{n'_{r,i}}) - \xi(n'_{r,i}) > (K(p-m)^{\frac{1}{2}} - \mu)(c_{n_r} \lg n_r)^{\frac{1}{2}}$$

which gives the desired result of the theorem. Thus it is sufficient to show $P(\bar{Q}_r$ occurs i.o.$) = 0$ which we proceed to do.

$$\begin{aligned} P(\bar{Q}_r) &= P(\textstyle\bigcap_{i=1}^{J_r} \overline{C_{r,i}}) = \prod_{i=1}^{J_r} P(\overline{C_{r,i}}) \\ P(\overline{C_{r,i}}) &= P(\overline{A_{r,i}} \cup \overline{B_{r,i}}) \\ &= P(\overline{A_{r,i}}) + P(\overline{B_{r,i}}) - P(\overline{A_{r,i}} \cap \overline{B_{r,i}}) \\ &= P(\overline{A_{r,i}}) + P(\overline{B_{r,i}}) - P(\overline{A_{r,i}})P(\overline{B_{r,i}}) \\ &= 1 - P(A_{r,i}) + P(\overline{B_{r,i}}) - (1 - P(A_{r,i}))P(\overline{B_{r,i}}) \\ &= 1 - P(A_{r,i})[1 - P(\overline{B_{r,i}})] . \end{aligned}$$

By Gaussian tail estimates

$$P(\overline{A_{r,i}}) \sim \left\{1 - \exp\left(\frac{-K^2(p-m)}{2K^2}(\lg n_r)(1+o(1))\right)\right\}$$

and since $P(\sup_{0 \leqq t \leqq T} \xi(t) \geqq b) = 2P(\xi(T) \geqq b)$, $b > 0$,

$$P(\overline{B_{r,i}}) \sim 2 \exp\left\{\left(\frac{-\mu^2 c_{n_r} \lg n_r}{2g_{n_r}}\right)(1+o(1))\right\}$$

where $g_{n_r} = (1+\varepsilon)d_{n_r} - K^2 c_{n_r}$.

Thus

$$P(\overline{C_{r,i}}) \sim 1 - \left\{\exp\left(\frac{-(p-m)}{2}\right)(\lg n_r)(1+o(1))\right\}$$
$$\times \left\{1 - 2\exp\left[\left(\frac{-\mu^2 c_{n_r} \lg n_r}{2g_{n_r}}\right)(1+o(1))\right]\right\}.$$

Since $g_{n_r} = (1+\varepsilon)d_{n_r} - K^2 c_{n_r} \leqq 3d_{n_r}$ and $(c_n \lg n)/d_n \uparrow \infty$

$$P(\overline{C_{r,i}}) \leqq 1 - \frac{1}{2}\left(\frac{1}{n_r}\right)^{((p-m)/2)(1+o(1))}$$

for r sufficiently large. Thus for large r

$$\lg P(\bar{Q}_r) \leqq \frac{(n_r)^p}{4d_{n_r}}\left(-\frac{1}{2}\left(\frac{1}{n_r}\right)^{((p-m)/2)(1+o(1))}\right)$$
$$\leqq -\frac{1}{8}\,\frac{(n_r)^{p-m}}{L(n_r)(n_r)^{((p-m)/2)(1+o(1))}}.$$

Since $0 < m < p < 1$ and $L(n)$ is slowly varying,

$$\lg P(\bar{Q}_r) \leqq -\tfrac{1}{8}(n_r)^q \qquad \text{where } q > 0.$$

Thus

$$P(\bar{Q}_r) \leqq 1/e^{\frac{1}{8}(n_r)^q}$$

and by Borel–Cantelli only finitely many $\bar{Q}_r$ occur a.s. This completes the proof.

It is of interest to give the sequences $\{c_n\}$ and $\{d_n\}$ in terms of the original random variables $\{X_i\}$. Sawyer (preprint) relates the asymptotic behavior of the tail distribution of X_1 to the asymptotic behavior of the tail distribution of the stopping time T, which comes from the Skorohod–Breiman representation of X_1. As a special case he shows that if $X_1 \geqq -M$ and $P(X_1 \geqq t) \sim c/t^q$ where $q > 1$, then $P(T_1 \geqq s) \sim c'/s^{q/2}$ where c', c, and M are positive constants. We can use this result and the following theorem due to Feller (1946) to find the sequence $\{d_n\}$ in terms of the original random variables $\{X_i\}$.

(2.12) THEOREM (Feller). *Let $\{Y_n\}_{n \geqq 1}$ be a sequence of independent, identically distributed random variables such that $E(|Y_1|) < \infty$, $E(Y_1) = 0$, and for some $0 < \delta < 1$, $E(|Y_1|^{1+\delta}) = \infty$. Furthermore, let $\{d_n\}$ be a sequence of numbers for which there exists an ε with $0 < \varepsilon < 1$ such that $c_n/n^{1/(1+\varepsilon)} \uparrow$, $c_n/n \downarrow$, and let $S_n = Y_1 + \cdots + Y_n$. Then $|S_n| > c_n$ infinitely often* a.s. *if and only if $|Y_n| > c_n$ infinitely often* a.s.

(2.13) EXAMPLE OF PROPOSITION (2.1). Assume $E(X_i) = 0$, $E(X_i^2) = 1$, $X_i \geqq -M$, and $P(X_i \geqq t) \sim c/t^{\frac{5}{2}}$. Then Sawyer's result says $P(T_i \geqq s) \sim c'/s^{\frac{5}{4}}$. By Feller's theorem

$$\limsup_{n\to\infty} |V_n|/(n \lg n (\lg\lg n)^2)^{\frac{4}{5}} = 0 \quad \text{a.s.}$$

Thus Proposition (2.1) gives

$$\limsup_{n\to\infty} W_n/(n(\lg n)^{\frac{9}{4}}(\lg\lg n)^2)^{\frac{2}{5}} = 0 \quad \text{a.s.}$$

Notice that we cannot use Feller's theorem to find the sequence $\{c_n\}$ in Theorem (2.2) since condition (2.4) requires information about the fluctuations of V_n along a geometric-like subsequence. To establish a result of this type, the following two lemmas are used. The first, which is a generalized Borel–Cantelli lemma, is proved by Lamperti (1963).

(2.14) LEMMA. *Let $D_1, D_2, \cdots$ be events in a sample space, and suppose that*

$$\sum_{n=1}^{\infty} P(D_n) = \infty .$$

Suppose also that for some constants N and $C < \infty$

$$P(D_n D_m) \leqq CP(D_n)P(D_m)$$

for all $n, m > N$. Then

$$P(D_n \text{ occur i.o.}) > 0 .$$

The following lemma, which appears here in a slightly more general form, is proved in Kostka (1973).

(2.15) LEMMA. *Let $\{Y_n\}_{n\geqq 1}$ be a sequence of independent, indentically distributed random variables and $S_n = Y_1 + \cdots + Y_n$. Assume $\{b_n\}$ is an increasing sequence of real numbers such that $nP(Y_1 > b_n) \to 0$ as $n \to \infty$. Assume also that $P(S_n > 0) \geqq \varepsilon > 0$. Then*

$$P(S_n > b_n) \geqq CnP(Y_1 > b_n)$$

for some $C > 0$.

(2.16) PROPOSITION. *Let $\{Y_n\}_{n\geqq 1}$ be independent, identically distributed random variables with mean zero such that $P(Y_1 \geqq s) \sim c'|s^{\frac{5}{4}}$ as $s \to \infty$. Let $S_n = Y_1 + \cdots + Y_n$. Then*

$$S_{2^k} \geqq K(2^k \lg 2^k \lg\lg 2^k)^{\frac{4}{5}} \quad \text{i.o.} \quad \text{a.s.}$$

for any $K > 0$.

(Note: A direct application of Feller's theorem above merely gives $S_n \geqq K(n(\lg n)(\lg\lg n))^{\frac{4}{5}}$ i.o. a.s.)

PROOF. Let $\gamma(k) = K(2^k \lg 2^k \lg\lg 2^k)^{\frac{4}{5}}$ and $D_k = \{\omega : S_{2^{k-1}} > 0, S_{2^k} - S_{2^{k-1}} \geqq \gamma(k)\}$. Then for $k > l$

$$\begin{aligned} D_k D_l &= \{\omega : S_{2^{k-1}} > 0, S_{2^k} - S_{2^{k-1}} \geqq \gamma(k), S_{2^l} > 0, S_{2^l} - S_{2^{l-1}} \geqq \gamma(l)\} \\ &\subset \{\omega : S_{2^{k-1}} > 0, S_{2^k} - S_{2^{k-1}} \geqq \gamma(k), S_{2^l} - S_{2^{l-1}} \geqq \gamma(l)\} . \end{aligned}$$

Thus, $P(D_k D_l) \leqq P(D_k)P(S_{2^l} - S_{2^{l-1}} \geqq \gamma(l)) \leqq cP(D_k)P(D_l)$ for some $c > 0$. Since by Lemma (2.15)

$$\sum_k P(D_k) \geqq C \sum_k 2^k P(X_1 > \gamma(k)) = \infty ,$$

Lemma (2.14) gives $P(S_{2^k} \geqq \gamma(k) \text{ i.o.}) > 0$. By the Hewitt–Savage zero-one law, this probability must be one which is the desired result of the proposition.

(2.17) EXAMPLE OF THEOREM (2.2). Assume as in Example (2.13) that $E(X_i) = 0$, $E(X_i^2) = 1$, $X_i \geqq -M$, and $P(X_i \geqq t) \sim c/t^{\frac{5}{2}}$. Then Sawyer's result again gives $P(T_i \geqq s) \sim c'/s^{\frac{5}{4}}$. Example (2.13) says

$$\limsup_{n\to\infty} W_n/(n(\lg n)^{\frac{9}{4}}(\lg\lg n)^2)^{\frac{2}{5}} = 0 \quad \text{a.s.}$$

Proposition (2.16) gives

$$\limsup_{n\to\infty} V_{2^n}/(2^n \lg 2^n \lg\lg 2^n)^{\frac{4}{5}} = \infty \quad \text{a.s.}$$

Thus Theorem (2.2) is applicable and yields

$$\limsup_{n\to\infty} W_n/(n(\lg n)^{\frac{9}{4}}(\lg\lg n))^{\frac{2}{5}} = \infty \quad \text{a.s.}$$

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