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LLN FOR WEAKLY STATIONARY PROCESSES ON LOCALLY COMPACT ABELIAN GROUPS

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A generalization of the weak law of large numbers is proved in the case of a process indexed by a second countable locally compact Abelian group. An application is given to random measures.

1. Introduction and statement of results. Let G be a locally compact, second countable, Abelian group, let $\mathscr{B}$ denote the Borel σ-field of subsets of G, and let μ be any fixed Haar measure on $(G, \mathscr{B})$. Following Debes *et al.* (1970) we define a sequence $\{K_n\}$ of subsets to be *regular* if the following conditions hold: (i) K_n is compact for each n, (ii) $\mu(K_n) > 0$ for sufficiently large n and (iii) if U is any symmetric, relatively compact, open neighborhood of 0 then

$$\lim_{n\to\infty} \frac{\mu((K_n)_U)}{\mu(K_n + U)} = 1 ,$$

where $(K_n)_U = \{x \in G : x + U \subset K_n\}$.

Debes *et al.* (1970) have shown that G always possesses at least one regular sequence of sets.

A process $W(x)$, $x \in G$ is called weakly stationary if, for $x, y \in G$, $EW(x)^2 < \infty$ and $EW(x)W(x + y)$ is independent of x and continuous in y.

THEOREM 1. *Suppose $W(x)$, $x \in G$ is weakly stationary. Then there is a random variable Λ such that for any regular sequence $\{K_n\}$ of subsets of G,*

$$\lim_{n\to\infty} \frac{1}{\mu(K_n)} \int_{K_n} W(x + z)\, d\mu(x) = \Lambda$$

uniformly for $z \in G$.

Let A be a random measure on $(G, \mathscr{B})$. It is called stationary if for $B_1, \cdots, B_m \in \mathscr{B}$ the distribution of $(A(B_1 + x), \cdots, A(B_m + x))$ is independent of $x \in G$. Let $\mathscr{C}$ denote the class of relatively compact Borel subsets of G. Then A is called weakly stationary if for $B, C \in \mathscr{C}$, $EA(C)^2 < \infty$ and $EA(B + x)A(B + x + y)$ is independent of x and continuous in y.

In Debes *et al.* (1970), it was proved that if A is stationary with $EA(C) < \infty$, $C \in \mathscr{C}$ then there is a random variable Λ such that for $\{K_n\}$ regular

$$\lim_{n\to\infty} \sup_z E\left|\frac{A(K_n + z)}{\mu(K_n)} - \Lambda\right| = 0 .$$

We prove the analogous result for weakly stationary random measures.

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THEOREM 2. *Suppose A is a weakly stationary random measure on G. Then there is a random variable Λ such that for $\{K_n\}$ regular*

$$\lim_{n\to\infty} \sup_z E\left|\frac{A(K_n + z)}{\mu(K_n)} - \Lambda\right|^2 = 0 .$$

2. Proofs.

LEMMA 1. *Let $\hat{G}$ denote the dual group of G. Then for any regular sequence $\{K_n\}$ of subsets of G,*

$$\lim_{n\to\infty} \frac{1}{\mu(K_n)} \int_{K_n} \langle\gamma, x\rangle\, d\mu(x) = 1_{\{\hat{0}\}}(\gamma) \qquad (\gamma \in \hat{G}) , \tag{1}$$

where 1_B denotes the indicator function of the set B.

PROOF. Note that if $\{K_n\}$ is regular then for any $x \in G$,

$$\lim_{n\to\infty} \frac{\mu((K_n + x)\Delta K_n)}{\mu(K_n)} = 0 \tag{2}$$

where Δ denotes the symmetric difference operation.

If $\gamma = \hat{0}$ then it is clear that (1) holds. So assume $\gamma \neq \hat{0}$ and choose $x_0 \in G$ such that $\langle\gamma, x_0\rangle \neq 1$. Then,

$$\begin{aligned}\langle\gamma, x_0\rangle \int_{K_n} \langle\gamma, x\rangle\, d\mu(x) &= \int_{K_n+x_0} \langle\gamma, x\rangle\, d\mu(x) \\ &= \int_{K_n} \langle\gamma, x\rangle\, d\mu(x) + \int_{(K_n+x_0)\setminus K_n} \langle\gamma, x\rangle\, d\mu(x) \\ &\qquad - \int_{(K_n\setminus(K_n+x_0))} \langle\gamma, x\rangle\, d\mu(x) .\end{aligned}$$

From this and (2) it follows that as $n \to \infty$

$$\left|\frac{1}{\mu(K_n)} \int_{K_n} \langle\gamma, x\rangle\, d\mu(x)\right| \cdot |\langle\gamma, x_0\rangle - 1| \leqq \frac{\mu((K_n + x_0)\Delta K_n)}{\mu(K_n)} \to 0 .$$

PROOF OF THEOREM 1. Let $\mathscr{W}$ be the closed linear manifold of $L^2(\Omega, \mathscr{F}, P)$ spanned by the process $W(x)$, $x \in G$. Since the process is weakly stationary it induces on $\mathscr{W}$ a group U_y, $y \in G$, of unitary operators via $U_y W(x) = W(x + y)$. By the generalized theorem of Stone (Loomis (1973) page 147),

$$U_y = \int_{\hat{G}} \langle\gamma, y\rangle\, dE_\gamma ; \qquad y \in G .$$

Set $\phi(d\gamma) = dE_\gamma W(0)$. Then

$$W(x) = \int_{\hat{G}} \langle\gamma, x\rangle \phi(d\gamma) ; \qquad x \in G .$$

By Fubini's Theorem for random measures

$$\frac{1}{\mu(K_n)} \int_{K_n} W(x + z)\, d\mu(x) = \int_{\hat{G}} \left[\frac{\langle\gamma, z\rangle}{\mu(K_n)} \int_{K_n} \langle\gamma, x\rangle\, d\mu(x)\right] \phi(d\gamma) .$$

Using Lemma 1 and $|\mu(K_n)^{-1} \int_{K_n} \langle\gamma, x\rangle\, d\mu(x)| \leqq 1$ we get

$$\sup_{z\in G} \int_{\hat{G}} \left|\frac{\langle\gamma, z\rangle}{\mu(K_n)} \int_{K_n} \langle\gamma, x\rangle\, d\mu(x) - 1_{\{\hat{0}\}}(\gamma)\right|^2 F(d\gamma) \to 0$$

as $n \to \infty$, where $F(d\gamma) = E(|\psi(d\gamma)|^2)$. Thus by the usual isomorphism (see Doob (1967)) we conclude that, uniformly in z,

$$\int_{\hat{G}} \left[\frac{\langle \gamma, z \rangle}{\mu(K_n)} \int_{K_n} \langle \gamma, x \rangle \, d\mu(x) \right] \psi(d\gamma) \to \psi(\{\hat{0}\}) .$$

LEMMA 2. *Let B be a compact symmetric neighborhood of* 0. *If $\{K_n\}$ is regular then so are $\{K_n + B\}$ and $\{(K_n)_B\}$.*

PROOF. That $\{K_n + B\}$ satisfies (i) and (ii) and $\{(K_n)_B\}$ satisfies (i) is clear. Now let U be any open, symmetric, relatively compact neighborhood of 0. Since $B + U$ is also an open, symmetric, relatively compact neighborhood of 0,

$$\lim_{n\to\infty} \frac{\mu((K_n)_{B+U})}{\mu(K_n + B + U)} = 1 . \tag{3}$$

Now, $(K_n + B)_U \subset K_n + B + U$ and $(K_n)_{B+U} \subset (K_n + B)_U$. The inequalities

$$1 \geqq \frac{\mu((K_n + B)_U)}{\mu(K_n + B + U)} \geqq \frac{\mu((K_n)_{B+U})}{\mu(K_n + B + U)}$$

and (3) imply that $\{K_n + B\}$ satisfies (iii).

Let V be a relatively compact, open, symmetric neighborhood of 0 with $B \subset V$. By the regularity of $\{K_n\}$, the relations $\mu((K_n)_B) \geqq \mu(K_n + V)/2 > 0$ hold for n large.

Finally, let U be any relatively compact, symmetric, open neighborhood of 0. From $(K_n)_{B+U} \subset ((K_n)_B)_U$ and the regularity of K_n

$$\frac{\mu(((K_n)_B)_U)}{\mu((K_n)_B + U)} \geqq \frac{\mu((K_n)_{B+U})}{\mu(K_n + B + U)} \to 1$$

as $n \to \infty$. Thus $\{(K_n)_B\}$ satisfies (ii) and (iii).

PROOF OF THEOREM 2. Let B be a compact symmetric neighborhood of 0 with $\mu(B) = 1$. Let $W_z(x) = A(x + z + B)$ for $x, z \in G$. Since A is weakly stationary so is W_z for each z.

By Fubini's theorem it follows that for $K \in \mathscr{C}$

$$\int_K W_z(x) \, d\mu(x) = \int_G \mu(K \cap (y + B)) A(dy + z) .$$

Using this we conclude that

$$\int_{K_n+B} W_z(x) \, d\mu(x) \geqq \int_{K_n} \mu((K_n + B) \cap (y + B)) A(dy + z) \geqq A(K_n + z) .$$

Because $y \notin K_n \Rightarrow (K_n)_B \cap (B + y) = \phi$ we also obtain that

$$\int_{(K_n)_B} W_z(x) \, d\mu(x) \leqq A(K_n + z) .$$

Consequently, for sufficiently large n,

$$\frac{1}{\mu(K_n)} \int_{(K_n)_B} W_z(x) \, d\mu(x) \leqq \frac{A(K_n + z)}{\mu(K_n)} \leqq \frac{1}{\mu(K_n)} \int_{K_n+B} W_z(x) \, d\mu(x) .$$

Theorem 2 now follows.

REFERENCES

[1] DEBES, H., KERSTAN, J., LIEMANT, A. and MATTHES, K. (1970). Verallgemeinerungen eines Satzes von Dobruschin, I. *Math. Nachrichten* **47** 183–244.

[2] DOOB, J. L. (1967). *Stochastic Processes.* Wiley, New York.

[3] LOOMIS, L. H. (1953). *An Introduction to Abstract Harmonic Analysis.* Van Nostrand, Princeton.

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