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MARTINGALES AND BOUNDARY CROSSING PROBABILITIES FOR MARKOV PROCESSES[1]

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Robbins and Siegmund have made use of the martingale

$$\int_0^\infty \exp(yW(t) - \tfrac{1}{2}ty^2)\, dF(y), \quad t \geqq 0,$$

to evaluate the probability that the Wiener process $W(t)$ would ever cross certain boundaries which are moving with time. By making use of martingales of the form $u(X(t), t)$, we apply the Robbins–Siegmund method to find boundary crossing probabilities for other Markov processes $X(t)$. The question of when $u(X(t), t)$ is a martingale is first studied. We generalize a result of Doob based on semigroups of type Γ, and we examine in particular the situations for stochastic integrals and processes with stationary independent increments.

1. Introduction. In [13], Robbins and Siegmund have proved that if $\varepsilon > 0$ and $Z(t)$, $t \geqq h$, is a nonnegative martingale satisfying certain conditions, then

$$(1) \qquad P[Z(t) \geqq \varepsilon \text{ for some } t \geqq h] = P[Z(h) \geqq \varepsilon] + \varepsilon^{-1} \int_{[Z(h)<\varepsilon]} Z(h)\, dP.$$

For the standard Wiener process $W(t)$, they have used this fact and the martingale $\int_0^\infty \exp(yW(t) - ty^2/2)\, dF(y)$, $t \geqq h$, to evaluate the probability that W ever crosses a boundary g for a certain class of boundaries g. These boundary crossing probabilities play an important part in certain sequential statistical procedures such as power-one tests, confidence sequences and selection and ranking problems (cf. [12], [14]).

In Section 5 below, by making use of martingales of the form $u(X(t), t)$, we shall apply the Robbins–Siegmund method to find boundary crossing probabilities for other Markov processes $X(t)$, This procedure first raises the following question: When is $u(X(t), t)$ a martingale? Suppose $X(t)$, $0 \leqq t < \zeta$, is a continuous Markov process with lifetime $\zeta \leqq \infty$ and state space M which is a locally compact, second countable, Hausdorff space. If $u: M \times [0, \infty) \to R$ is a continuous function such that $u(X(t), t + r)$ is a martingale with respect to P_x for all $x \in M$, $r \geqq 0$ (assuming the usual convention that $u(X(t), t) = 0$ if $t \geqq \zeta$), then u is harmonic for the corresponding space-time process $Z(t)$ and consequently $\mathscr{U}u = 0$, where $\mathscr{U}$ is the characteristic operator of $Z(t)$. In particular, if $Y(t)$ is a diffusion on R^d corresponding to the diffusion coefficients $a_{ij}(x)$, $b_i(x)$, $i, j = 1, \cdots, d$ and $c(x) \geqq 0$ such that the matrix a is uniformly positive definite

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and the coefficients a_{ij}, b_i and c are bounded and uniformly Hölder continuous on R^d, then the set of continuous solutions of $\mathscr{U}u = 0$ coincides with the set of solutions of class $C^{2,1}$ of the equation

$$\frac{\partial u}{\partial t} + \sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial^2 u}{\partial x_i\, \partial x_j} + \sum_{i=1}^{d} b_i(x)\frac{\partial u}{\partial x_i} - c(x)u = 0 \tag{2}$$

(cf. [10]), where $C^{2,1}$ stands for the set of all continuous functions u on $R^d \times [0, \infty)$ such that $\partial u/\partial t$, $\partial u/\partial x_i$, $\partial^2 u/\partial x_i\, \partial x_j$ $(i, j = 1, \cdots, d)$ are continuous. The partial differential equation (2) is called a backward parabolic equation (cf. page 172 of [6]); we can reduce it to the usual parabolic equation by a change of variable $t' = -t$.

To find martingales $u(Y(t), t)$, we would therefore consider solutions of (2). However, do such solutions necessarily give us martingales? Now $u(Y(t), t)$, $t \geqq 0$, is martingale if u admits the following integral representation

$$u(x, s) = \int_{R^d} u(y, t)p(t - s, x, y)\, dy\,, \qquad x \in R^d,\ 0 \leqq s < t\,. \tag{3}$$

From the theory of parabolic equations, if $u \in C^{2,1}$ is a solution of (2) and satisfies the growth condition

$$\max_{0\leq t\leq T} |u(x, t)| = o(\exp(\varepsilon|x|^2)) \quad \text{as } |x| \to \infty \text{ for any } \varepsilon > 0,\ T > 0\,,$$

then u has the integral representation (3). Furthermore, the integral representation (3) holds for any nonnegative continuous solution of (2) (cf. [6] page 48). In the case where $Y(t)$ is the standard Wiener process, integral representations of nonnegtive solutions for the corresponding heat equation were first proved by Widder [16]. Recently, Robbins and Siegmund [14] have obtained a probabilistic proof of Widder's theorem. They have also applied a similar probabilistic argument to study the equation

$$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + x^{-1}\frac{\partial u}{\partial x} = 0\,, \qquad x > 0\,, \tag{4}$$

which corresponds to the Bessel process $r(t) = (\beta_1^2(t) + \beta_2^2(t) + \beta_3^2(t))^{\frac{1}{2}}$, i.e., $r(t)$ is the radial part of the 3-dimensional Brownian motion $(\beta_1(t), \beta_2(t), \beta_3(t))$. The result turns out to be strikingly different from that of the Wiener process. In particular, for the positive solution $u(x, t) = x^{-1}\exp(-\sqrt{2}\,x - t)$ of (4), $u(r(t), t)$ is not a martingale.

For a general Markov process $X(t)$, $0 \leqq t < \zeta$, with transition function $P(t, x, \Lambda)$, state space M and lifetime $\zeta \leqq \infty$, clearly $u(X(t), t)$, $t \geqq 0$, is a martingale if u admits the following integral representation:

$$u(x, s) = \int u(y, t)P(t - s, x, dy)\,, \qquad x \in M,\ 0 \leqq s < t\,. \tag{5}$$

We want to find the class of functions u which admit the above integral representation. Doob [4] has proved that when the transition function induces a semigroup of type Γ with infinitesimal generator A, then u admits the integral representation (5) if $u(x, t) = e^{-\lambda t}g(x)$ and $Ag = \lambda g$, or if $u(x, t) = g(x) - \lambda t$ and $Ag = \lambda$. The semigroup formulation requires one to find an appropriate Banach space. For a standard Itô process on the real line with bounded diffusion coefficients $\sigma(x)$ and $\mu(x)$, Doob [4] has shown that if the Banach space is chosen as the space of all continuous functions f on the real line with $||f|| = \sup_\xi |f(\xi)|e^{-\nu|\xi|} < \infty$ $(\nu > 0)$, then the infinitesimal generator of the corresponding semigroup coincides with $D = \frac{1}{2}\sigma^2(x)\, d^2/dx^2 + \mu(x)\, d/dx$ on the space C_2 of all twice continuously differentiable functions f such that outside some finite interval, f has a third continuous derivative with $\sup_\xi e^{-\nu|\xi|}|f'''(\xi)| < \infty$. Thus by introducing the weight function $e^{-\nu|\xi|}$ (which in fact means imposing a growth condition on the functions in the corresponding Banach space), Doob has proved that for a standard Itô process $X(t)$, if $g \in C_2$, then $Dg = \lambda g$ implies that $e^{-\lambda t}g(X(t))$ is a martingale, while $Dg = \lambda$ implies that $g(X(t)) - \lambda t$ is a martingale.

In Section 2 below, we shall extend the above approach of Doob to general Markov processes $X(t)$ and to more general functions $u(x, t)$. We first derive certain properties of semigroups of type Γ and then use them to generalize Doob's result to obtain the integral representation (5) for solutions u of $(A + \partial/\partial t)u = 0$. In Section 3, we examine in particular the situations for processes with stationary independent increments. It is well known that if X_j, $j = 1, 2, \cdots$, are independent random variables such that $EX_j = \mu$ and $\operatorname{Var} X_j = \sigma^2$ for all j, then letting $S_m = X_1 + \cdots + X_m$, $\{S_m - m\mu, m \geqq 1\}$ and $\{(S_m - m\mu)^2 - m\sigma^2, m \geqq 1\}$ are martingales with zero expectations and these martingales are related to the moment identities of Wald for randomly stopped sums in sequential analysis. Generalizations of this to the case where the X_j's are martingale differences and $n > 2$ have been studied by Chow, Robbins and Teicher [3] and Brown [2]. Hall [8] studies Wald's equations for processes with stationary independent increments by making use of certain analogous martingales in continuous time. In Section 3, we shall characterize all the polynomial martingales associated with a process $X(t)$ with stationary independent increments. In Section 4, we shall return to the standard Itô processes and more generally, we shall consider martingales associated with stochastic integrals.

2. Semigroups of type Γ and the associated martingales. Contraction semigroups play a very important part in the study of Markov processes. Since we shall work with a function space induced by a weight function instead of the space of all bounded measurable functions, we have to resort to semigroups which do not have the contraction property. We shall consider semigroups $\{T_t, t \geqq 0\}$ of bounded linear operators mapping a Banach space $\mathscr{B}$ into itself such that

$$\text{(6)} \qquad \text{For every} \quad \delta \in (0, 1)\,, \quad \sup_{\delta \leqq t \leqq 1/\delta} ||T_t|| < \infty\,.$$

Doob [4] has called a semigroup satisfying (6) and certain subsidiary conditions a semigroup of type Γ. In the case of contraction semigroups, and more generally semigroups satisfying

$$\sup_{0 \leqq t \leqq \delta} ||T_t|| < \infty \qquad \text{for any } \delta > 0 , \tag{7}$$

it is well known (cf. [5] pages 22–23) that the set

$$\mathscr{B}_0 = \{f \in \mathscr{B}: \text{s-lim}_{t \downarrow 0} T_t f = f\} \tag{8}$$

is a closed subspace of $\mathscr{B}$ and coincides with the strong closure of the domain of the infinitesimal generator of $\{T_t, t \geqq 0\}$. (The notation s-lim denotes strong limit.) However, for semigroups satisfying (6), $||T_t||$ may not be bounded as $t \downarrow 0$ and the set $\mathscr{B}_0$ may not be closed. Define

$$\mathscr{B}_1 = \{f \in \mathscr{B}: \text{s-lim}_{t \to h} T_t f = T_h f \text{ for all } h > 0\} . \tag{9}$$

From (6), it easily follows that $\mathscr{B}_1$ is a closed subspace of $\mathscr{B}$. Also $\mathscr{B}_0 \subset \mathscr{B}_1$ and $T_t[\mathscr{B}_1] \subset \mathscr{B}_0$ for any $t > 0$. Let A be the infinitesimal generator of the semigroup and let its domain be $\mathscr{D}(A)$. Obviously, $\mathscr{D}(A) \subset \mathscr{B}_0$ and it can be shown by imitating the proof in the contraction semigroup case (cf. Theorem 2.1.3 B of [5]) that $\mathscr{D}(A)$ is dense in $\mathscr{B}_0$. Hence the strong closure of $\mathscr{D}(A)$ coincides with the strong closure of $\mathscr{B}_0$. The following lemma contains some further properties of semigroups satisfying (6) which we shall use in the sequel.

LEMMA 1. *Let $\{T_t, t \geqq 0\}$ be a semigroup of bounded linear operators mapping a Banach space $\mathscr{B}$ into itself such that condition* (6) *is satisfied. Let A be the infinitesimal generator of the semigroup and its domain be $\mathscr{D}(A)$. Let the subspace $\mathscr{B}_0$ be defined by* (8) *and $\mathscr{B}_1$ by* (9).

(a) *If $f_n \in \mathscr{B}_0$ and* $\text{s-lim}_{n\to\infty} f_n = f$, *then there exists sequence $t_n > 0$ such that* $\lim_{n\to\infty} t_n = 0$ *and* $\text{s-lim}_{n\to\infty} T_{t_n} f_n = f$.

(b) *If $\psi:(0, \infty) \to \mathscr{B}$ is a strongly differentiable function and its range lies in $\mathscr{D}(A)$, then $T_t\psi(t)$ is strongly differentiable for all $t > 0$ and*

$$\frac{d}{dt} T_t \psi(t) = T_t(A\psi(t) + \frac{d}{dt}\psi(t)) . \tag{10}$$

PROOF. To prove (a), for each n, we can choose $t_n \in (0, 1/n)$ such that $||T_{t_n} f_n - f_n|| < 1/n$. Therefore $\text{s-lim}_{n\to\infty} T_{t_n} f_n = \text{s-lim}_{n\to\infty} f_n = f$.

To prove (b), since $\psi(s) \in \mathscr{D}(A) \subset \mathscr{B}_1$ for all $s > 0$ and $\mathscr{B}_1$ is a closed subspace of $\mathscr{B}$, it follows that $(d/dt)\psi(t) \in \mathscr{B}_1$ and $A\psi(t) \in \mathscr{B}_1$. Therefore for $t > 0$, $\text{s-lim}_{\varepsilon\to 0} T_{t+\varepsilon}(A\psi(t) + (d/dt)\psi(t)) = T_t(A\psi(t) + (d/dt)\psi(t))$. For $|\varepsilon| < t/2$, we have

$$\begin{aligned}
&\left|\left| T_{t+\varepsilon}\left(\varepsilon^{-1}\{\psi(t+\varepsilon) - \psi(t)\} - \frac{d}{dt}\psi(t)\right)\right|\right| \\
&\qquad \leqq \sup_{t/2 \leqq s \leqq 3t/2} ||T_s|| \left|\left| \varepsilon^{-1}\{\psi(t+\varepsilon) - \psi(t)\} - \frac{d}{dt}\psi(t)\right|\right| \\
&\qquad \to 0 \qquad\qquad \text{as } \varepsilon \to 0 .
\end{aligned}$$

We note that for $t > 0$ and $\varepsilon > 0$,

$$\varepsilon^{-1}(T_t\psi(t) - T_{t-\varepsilon}\psi(t-\varepsilon))$$
$$= T_{t-\varepsilon}\left[\left(\frac{T_\varepsilon - I}{\varepsilon} - A\right)\psi(t)\right] + T_{t-\varepsilon}\left[\varepsilon^{-1}\{\psi(t) - \psi(t-\varepsilon)\} - \frac{d}{dt}\psi(t)\right]$$
$$+ T_{t-\varepsilon}\left(A\psi(t) + \frac{d}{dt}\psi(t)\right).$$

It is now clear that the left derivative $(d^-/dt)T_t\psi(t)$ is equal to the right-hand side of (10). The right derivative $(d^+/dt)T_t\psi(t)$ can be computed similarly. □

As an application of (10), we let $f \in \mathscr{D}(A)$ such that $Af = \lambda f$ for some real number λ. Define $\psi(t) = e^{-\lambda t}f$. Then by (10), for $t > 0$, $(d/dt)T_t\psi(t) = 0$ and so there exists $g \in \mathscr{B}$ such that $T_t\psi(t) = e^{-\lambda t}T_t f = g$ for all $t > 0$. Since $f \in \mathscr{D}(A) \subset \mathscr{B}_0$, s-$\lim_{t\downarrow 0} T_t f = f$. Hence $g = f$ and so $T_t f = e^{\lambda t}f$. If $h^* \in \mathscr{B}$ such that $T_t h^* = h^*$ for all $t > 0$, then we can similary use (10) to prove that $Af = \lambda h^*$ implies that $T_t f = f + \lambda t h^*$ for any $t \geqq 0$. If furthermore $\mathscr{B}$ is a σ-complete Banach lattice and T_t is positivity-preserving, then using (10), we see that $Af \geqq \lambda h^*$ implies that $T_t f \geqq f + \lambda t h^*$, while $Af \geqq \lambda f$ implies $T_t f \geqq e^{\lambda t}f$. Thus (10) gives an alternative proof of Doob's result ([4] Theorem 1.1) in a more general setting.

THEOREM 1. *Let $(X(t), \mathscr{F}_t, P_x)$ be a Markov process with lifetime $\zeta \leqq \infty$, state space M and transition function $P(t, x, \Lambda)$. Suppose $\varphi\colon M \to R$ is a measurable function satisfying*:

(W1) $\inf_{x\in M}\varphi(x) > 0$;
(W2) $\limsup_{t\downarrow 0} \int \varphi(y)P(t, x, dy) < \infty$ *for any* $x \in M$;
(W3) $\sup_{\delta\leqq t\leqq \delta^{-1}} \sup_{x\in M} (\varphi(x))^{-1} \int \varphi(y)P(t, x, dy) < \infty$ *for any* $\delta \in (0, 1)$.

Let $\mathscr{B}$ be the Banach space of all measurable functions f on M such that $\|f\| = \sup_{x\in M} |f(x)|/\varphi(x) < \infty$, and let $\{T_t, t \geqq 0\}$ be the semigroup of bounded linear operators defined by

$$T_t f(x) = \int_M f(y)P(t, x, dy).$$

Denote the infinitesimal generator of this semigroup by A and its domain by $\mathscr{D}(A)$. Suppose $u\colon M \times [0, \infty) \to R$ satisfies for all $t \geqq 0$

$$u(\cdot, t) \in \mathscr{D}(A) \tag{11}$$

$$\lim_{0\leqq s\to t}\left\|\frac{u(\cdot, s) - u(\cdot, t)}{s - t} - \frac{\partial u}{\partial t}(\cdot, t)\right\| = 0. \tag{12}$$

Then if for all $t > 0$,

$$\frac{\partial u}{\partial t}(x, t) + Au(x, t) = (\geqq)\, 0 \tag{13}$$

holds for all $x \in M$, $\{u(X(t), t + r), \mathscr{F}_t, t \geqq 0\}$ is a martingale (submartingale) with respect to P_a for all $a \in M$ and for all $r \geqq 0$, and (13) also holds at $t = 0$. Conversely *if* $\{u(X(t), t), \mathscr{F}_t, t \geqq 0\}$ *is a martingale (submartingale) with respect to* P_a, *then* (13) *holds for all* $t \geqq 0$ *and* $x \in M - N_t$, *where* $P_a[X(t) \in N_t] = 0$.

LEMMA 2. *Let* $\{T_t, t \geqq 0\}$ *be the semigroup defined in Theorem* 1. *Suppose* u *satisfies* (11) *and* (12). *Then given* $t \geqq 0$, *there exists a sequence of positive numbers* $\varepsilon_n \to 0$ *as* $n \to \infty$ *such that for any* $x \in M$,

$$(14) \qquad \lim_{n\to\infty} (T_{\varepsilon_n} u(x, t + \varepsilon_n) - u(x, t))/\varepsilon_n = Au(x, t) + \frac{\partial u}{\partial t}(x, t)\,.$$

PROOF. We note that condition (W3) guarantees that the semigroup satisfies (6). From (11) and (12), $(\partial u/\partial t)(\cdot, t)$ can be written as the strong limit of a sequence $f_n \in \mathscr{B}_0$, i.e., $\text{s-lim}_{n\to\infty} f_n = (\partial u/\partial t)(\cdot, t)$. By Lemma 1(a), we can choose positive numbers $\varepsilon_n \to 0$ such that $\text{s-lim}_{n\to\infty} T_{\varepsilon_n} f_n = (\partial u/\partial t)(\cdot, t)$. Since $u_t \in \mathscr{D}(A)$, $\text{s-lim}_{n\to\infty} \varepsilon_n^{-1}(T_{\varepsilon_n} u_t - u_t) = Au_t$, where $u_t = u(\cdot, t)$. Using (W2), we also have

$$|T_{\varepsilon_n}(\varepsilon_n^{-1}\{u_{t+\varepsilon_n}(x) - u_t(x)\} - f_n(x))|$$
$$\leqq \left(\left\|\varepsilon_n^{-1}(u_{t+\varepsilon_n} - u_t) - \frac{\partial u}{\partial t}(\cdot, t)\right\| + \left\|\frac{\partial u}{\partial t} - f_n\right\|\right) \int \varphi(y) P(\varepsilon_n, x, dy) \to 0$$

as $n \to \infty$. Therefore the desired conclusion follows. □

PROOF OF THEOREM 1. Take any $x \in M$ and $h \geqq 0$. Setting $u_t = u(\cdot, t)$ as before, we define $G_{x,h}(t) = T_t u_{t+h}(x)$ for $t > 0$. By conditions (11) and (12), Lemma 1(b) is applicable and we have for $t > 0$,

$$(15) \qquad G'_{x,h}(t) = T_t\left(Au_{t+h}(x) + \frac{d}{dt} u_{t+h}(x)\right).$$

Since $u_h \in \mathscr{D}(A)$, $\text{s-lim}_{t\downarrow 0} T_t u_h = u_h$. Also by (W2) and (12),

$$|T_t(u_{t+h}(x) - u_h(x))| \leqq \|u_{t+h} - u_h\| \int \varphi(y) P(t, x, dy) \to 0 \qquad \text{as } t \downarrow 0\,.$$

Hence it is easy to see that

$$(16) \qquad \lim_{t\downarrow 0} G_{x,h}(x) = u(x, h)\,.$$

Suppose $(\partial u/\partial t)(\cdot, t) + Au(\cdot, t) \geqq 0$ for all $t > 0$. Then (15) implies that $G_{x,h}(t)$ is non-decreasing for $t > 0$, and using (16), we obtain that $G_{x,h}(t) \geqq u(x, h)$ for all $t > 0$. From this, it follows that $\{u(X(t), t + r), \mathscr{F}_t, t \geqq 0\}$ is a submartingale with respect to P_a for all $a \in M$ and $r \geqq 0$. Also since $T_t u_t(x) \geqq u(x, 0)$ for all $t > 0$, it then follows from Lemma 2 that $Au(x, 0) + (\partial u/\partial t)(x, 0) \geqq 0$. Likewise we can show that if $(\partial u/\partial t)(\cdot, t) + Au(\cdot, t) = 0$ for all $t > 0$, then the integral representation (5) holds and so the desired conclusion follows.

Now suppose that $\{u(X(t), t), \mathscr{F}_t, t \geqq 0\}$ is a martingale (submartingale) with respect to P_a. Take any $t \geqq 0$. Then by Lemma 2, we can choose $\varepsilon_n \downarrow 0$ such that for any $x \in M$, (14) holds. For each n, there exists $N_n \subset M$ such that $P_a[X(t) \in N_n] = 0$ and if $x \in M - N_n$, then

$$(17) \qquad T_{\varepsilon_n} u_{t+\varepsilon_n}(x) = E_a[u(X(t + \varepsilon_n), t + \varepsilon_n) \mid X(t) = x] = (\geqq)\, u(x, t)\,.$$

Let $N_t = \bigcup_1^\infty N_n$. Then $P_a[X(t) \in N_t] = 0$, and if $x \notin N_t$, then (17) holds, and so by (14), $Au(x, t) + (\partial u/\partial t)(x, t) = (\geqq) 0$. □

Condition (12) is obviously satisfied in the case $u(x, t) = f(t)g(x)$ or $u(x, t) = f(t) + g(x)$ where f is a differentiable function and $g \in \mathscr{B}$. In the case when $\partial^2 u/\partial t^2$ exists everywhere and $\sup_{0 \leqq t \leqq T} \sup_{x \in M} |(\partial^2 u/\partial t^2)(x, t)|/\varphi(x) < \infty$ for every $T > 0$, then using the mean value theorem, one can easily see that (12) is again satisfied.

Suppose the weight function φ satisfies (W1) and the following condition:

(W2a) There exists a Borel function $\psi: [0, \infty) \to (0, \infty)$ such that

$$\int_0^\infty e^{-\alpha t}\psi(t)\, dt < \infty$$

for $\alpha > \alpha_0$ ($\geqq 0$); and $\int \varphi(y)P(t, x, dy) \leqq \varphi(x)\psi(t)$ for all $t \geqq 0$, $x \in M$; and $\sup_{0 \leqq t \leqq T} \psi(t) < \infty$ for all $T > 0$.

The corresponding semigroup then satisfies condition (7). We note that (W2a) implies both (W2) and (W3). In this case, for $\alpha > \alpha_0$, we can define $G_\alpha: \mathscr{B}_0 \to \mathscr{B}$ by $G_\alpha g = \int_0^\infty e^{-\alpha t} T_t g\, dt$. Since (7) implies that $\mathscr{B}_0$ is closed, it can be shown by a similar argument as that given in ([5] page 25) for contraction semigroups that for $\alpha > \alpha_0$, $\mathscr{D}(A) = G_\alpha[\mathscr{B}_0]$, G_α is injective and $A = \alpha - G_\alpha^{-1}$. Also for $\alpha > \alpha_0$, $Ag = \alpha g$ implies that $g = 0$, but we may be able to find nontrivial solutions of $Ag = \alpha g$ for $\alpha \leqq \alpha_0$.

3. Polynomial and exponential martingales for processes with stationary independent increments. To illustrate some applications of the results of Section 2, let us consider the Poisson process $N(t)$, $t \geqq 0$, with parameter $\lambda > 0$. Let $\varphi(k) = e^{\nu k}$ ($\nu > 0$) be a weight function, and let $\mathscr{B}$ be the Banach space of all functions g on $\{0, 1, 2, \cdots\}$ such that $\sup_k |g(k)|/\varphi(k) < \infty$. Then

$$\sum_{n=k}^\infty \varphi(n)e^{-\lambda t}(\lambda t)^{n-k}/(n-k)! = \varphi(k)\exp(\lambda t(e^\nu - 1)). \tag{18}$$

Since $\int_0^\infty e^{-\alpha t}\exp(\lambda t(e^\nu - 1))\, dt < \infty$ for $\alpha > \lambda(e^\nu - 1)$, condition (W2a) is clearly satisfied. It is easy to show that the infinitesimal generator A of the corresponding semigroup is given by $Ag(k) = \lambda(g(k+1) - g(k))$ with domain $\mathscr{D}(A) = \mathscr{B}$. Suppose $u: \{0, 1, 2, \cdots\} \times [0, \infty) \to R$ satisfies the following three conditions:

$$u(k, \cdot) \in C^\infty[0, \infty) \qquad \text{for every fixed} \quad k\,; \tag{19}$$

$$\sup_{0 \leqq t \leqq T} \sup_{k \geqq 0} e^{-\nu k}|u(k, t)| < \infty \qquad \text{for every} \quad T > 0\,; \tag{20}$$

$$\lambda(u(k+1, t) - u(k, t)) + \frac{\partial u}{\partial t}(k, t) = 0\,, \qquad k = 0, 1, \cdots, t \geqq 0\,. \tag{21}$$

We note that (20) and (21) imply that $\partial u/\partial t$ also satisfies (20). This in turn implies, upon differentiating (21) with respect to t, that $\partial^2 u/\partial t^2$ also satisfies (20). Hence by Theorem 1, $u(N(t), t + r)$, $t \geqq 0$, is a martingale with respect to P_x for all $r \geqq 0$ and $x = 0, 1, 2, \cdots$.

Let us now consider solutions of (21). Let $u(0, t) = f(t)$. Then $f \in C^\infty[0, \infty)$ and for $k \geqq 1$, $t \geqq 0$,

$$u(k, t) = f(t) - \lambda^{-1}\binom{k}{1}f'(t) + \lambda^{-2}\binom{k}{2}f''(t) - \cdots + (-1)^k\lambda^{-k}f^{(k)}(t)\,. \tag{22}$$

In particular if $f(t) = t^n$ $(n \geqq 1)$, then it follows from (22) that

$$\begin{aligned} u(k, t) = t^n - \binom{n}{1}k\lambda^{-1}t^{n-1} + \binom{n}{2}k(k-1)\lambda^{-2}t^{n-2} - \cdots \\ + (-1)^n k(k-1)\cdots(k-n+1)\lambda^{-n}\,. \end{aligned} \tag{23}$$

Since u defined in (23) is a polynomial of degree n in k and t, it obviously satisfies (20) and so $u(N(t), t + r)$, $t \geqq 0$, is a martingale for any $r \geqq 0$. Now suppose that $u(0, t) = f(t)$ is an analytic function, i.e., $f(t) = a_0 + \sum_1^\infty a_n t^n$, $t \geqq 0$. Then it follows from (22) that

$$u(k, t) = a_0 + \textstyle\sum_1^\infty a_n u_n(k, t)\,, \tag{24}$$

where $u_n(k, t)$ is the expression on the right-hand side of (23). Given $r \geqq 0$, $x = 0, 1, \cdots$, we have seen that $u_n(N(t), t + r)$ is a martingale with respect to P_x for any $n \geqq 1$. It can be shown that

$$\textstyle\sum_1^\infty |a_n| E_x |u_n(N(t), t + r)| < \infty \qquad \text{for all } t \geqq 0\,.$$

Hence by the dominated convergence theorem for conditional expectations, it follows that for u defined by (24), $u(N(t), t + r)$, $t \geqq 0$, is a martingale with respect to P_x. As an example, set $f(t) = \cos \lambda t$. Then from (22),

$$u(N(t), t) = \cos \lambda t + \binom{N(t)}{1} \sin \lambda t + \binom{N(t)}{2} \cos \lambda t + \cdots \tag{25}$$

is a martingale, where we define $\binom{n}{r} = 0$ if $r > n$. As another example, let ξ be any real number and let $\theta = e^\xi - 1$. Setting $f(t) = e^{-\lambda\theta t}$, we obtain from (22) that

$$\begin{aligned} u(N(t), t) &= e^{-\lambda\theta t} + \binom{N(t)}{1}\theta e^{-\lambda\theta t} + \cdots + \theta^{N(t)}e^{-\lambda\theta t} \\ &= \exp(\xi N(t) - \lambda t(e^\xi - 1)) \end{aligned} \tag{26}$$

is a martingale, and this is the familiar exponential martingale involving the moment generating function $\exp(\lambda t(e^\xi - 1))$.

The martingale $u(N(t), t)$ with u defined by (23) is closely related to the moments of the Poisson distribution. It is an example of polynomial martingales defined below. Theorem 2 studies polynomial martingales for processes with stationary independent increments.

DEFINITION. Let $\{X(t), \mathscr{F}_t, t \geqq 0\}$ be a stochastic process. Suppose $\{V(t), \mathscr{F}_t, t \geqq 0\}$ is a martingale such that for $i = 1, \cdots, m$, there exists $b_i: [0, \infty) \to R$ for which with probability 1,

$$V(t) = b_0(t)X^m(t) + b_1(t)X^{m-1}(t) + \cdots + b_m(t)\,, \qquad \text{for all } t \geqq 0\,. \tag{27}$$

Then $\{V(t), \mathscr{F}_t, t \geqq 0\}$ is called a polynomial martingale (of degree $\leqq m$) associated with the process $\{X(t), \mathscr{F}_t, t \geqq 0\}$.

THEOREM 2. *Suppose $X(t)$ is a process with stationary independent increments and is continuous in probability. Assume that $X(0) = 0$ and $E|X(1)|^m < \infty$ for some positive integer m. Let $\mu_n = EX^n(1)$ and for $t \in R$, $j = 1, \cdots, m$, define*

$$p_j(t) = \sum_{Q_j} (-1)^r \frac{j!}{i_1! \cdots i_r!} \mu_{i_1} \cdots \mu_{i_r} \binom{t+r-1}{r} \tag{28}$$

where $Q_j = \{(i_1, \cdots, i_r): 1 \leqq r \leqq j, 1 \leqq i_\nu \leqq j$ and $i_1 + \cdots + i_r = j\}$, i.e., Q_j is the set of all ordered partitions of j into positive integers, and $\binom{t+r-1}{r} = (t+r-1)(t+r-2)\cdots t/(r!)$. Also set $h_0(x, t) = 1$, $h_1(x, t) = x + p_1(t)$ and in general for $2 \leqq n \leqq m$, define

$$h_n(x, t) = x^n + \binom{n}{1} p_1(t) x^{n-1} + \binom{n}{2} p_2(t) x^{n-2} + \cdots + p_n(t) . \tag{29}$$

(i) *Let $\mathscr{F}_t$ be the σ-field generated by $\{X(s), s \leqq t\}$, and let $U_n(t) = h_n(X(t), t)$. Then $\{U_n(t), \mathscr{F}_t, t \geqq 0\}$ is a martingale with zero expectation for $n = 1, \cdots, m$.*

(ii) *Suppose $\{V(t), \mathscr{F}_t, t \geqq 0\}$ is a polynomial martingale of degree $\leqq m$ associated with process $X(t)$. Then with probability 1, $\{V(t), t \geqq 0\}$ is a linear combination of the martingales $\{U_0(t), t \geqq 0\}, \cdots, \{U_m(t), t \geqq 0\}$. Consequently, with probability 1, $V(t) = h(X(t), t)$, $t \geqq 0$, where $h(x, t)$ is some polynomial of degree $\leqq m$ in x, t.*

(iii) *Letting $\mu_n(t) = EX^n(t)$, we can express $\mu_n(t)$ in terms of μ_j, $1 \leqq j \leqq n$, by the following recurrence relations:*

$$\begin{aligned} &\mu_1(t) = \mu_1 t ; \\ &\mu_n(t) + \binom{n}{1} p_1(t) \mu_{n-1}(t) + \cdots \\ &\qquad + \binom{n}{n-1} p_{n-1}(t) \mu_1(t) + p_n(t) = 0 , \qquad n = 2, \cdots, m . \end{aligned} \tag{30}$$

(iv) *For $j = 1, \cdots, m$ and $s, t \in R$, the following combinatorial identity holds:*

$$p_j(t+s) = p_j(t) + \binom{j}{1} p_{j-1}(t) p_1(s) + \binom{j}{2} p_{j-2}(t) p_2(s) + \cdots + p_j(s) . \tag{31}$$

PROOF. The characteristic function of $X(t)$ is $(\psi(u))^t$, where ψ is some infinitely divisible characteristic function and therefore vanishes nowhere. Hence for all $u \in R$, $e^{iuX(t)}(\psi(u))^{-t}$, $t \geqq 0$, is a martingale. Since $E|X(1)|^m < \infty$, it follows that $\psi^{(n)}(u)$ is continuous for all real u and $\psi^{(n)}(0) = i^n \mu_n$, and $\{(\partial^n/\partial u^n)(e^{iuX(t)}(\psi(u))^{-t})|_{u=0}, \mathscr{F}_t, t \geqq 0\}$ is a martingale with zero expectation for $1 \leqq n \leqq m$. We now proceed to show

$$\frac{\partial^n}{\partial u^n} (e^{iux}(\psi(u))^{-t}) \Big|_{u=0} = i^n h_n(x, t) . \tag{32}$$

To prove (32), using the Leibnitz rule of successive differentiation, we have

$$\begin{aligned} \frac{\partial^n}{\partial u^n} (e^{iux}(\psi(u))^{-t}) &= (ix)^n e^{iux}(\psi(u))^{-t} + \binom{n}{1} (ix)^{n-1} e^{iux} \frac{\partial}{\partial u} (\psi(u))^{-t} \\ &\quad + \binom{n}{2} (ix)^{n-2} e^{iux} \frac{\partial^2}{\partial u^2} (\psi(u))^{-t} + \cdots \\ &\quad + e^{iux} \frac{\partial^n}{\partial u^n} (\psi(u))^{-t} . \end{aligned} \tag{33}$$

By induction, we can prove that for any real number t and $j = 1, \cdots, m$,

$$\frac{\partial^j}{\partial u^j}(\psi(u))^{-t} = \sum_{Q_j}(-1)^r \frac{j!}{i_1! \cdots i_r!} \psi^{(i_1)}(u) \cdots \psi^{(i_r)}(u)(\psi(u))^{-(t+r)}\binom{t+r-1}{r} \tag{34}$$

where Q_j, $\binom{t+r-1}{r}$ are as defined in (28). Form (34), we see that for $j = 1, \cdots, m$ and t real,

$$\frac{\partial^j}{\partial u^j}(\psi(u))^{-t}\Big|_{u=0} = i^j p_j(t) . \tag{35}$$

Hence we have proved (i). To prove (ii), we note that since the characteristic function $\psi(u)$ is infinitely divisible, either $P[X(t) = 0 \ \forall t \geqq 0] = 1$ or the support of the distribution of $X(t)$ for any $t > 0$ is not reducible to a finite set of points. Let $\{V(t), \mathscr{F}_t, t \geqq 0\}$ be a polynomial martingale of degree $\leqq m$ associated with the process $X(t)$, i.e., with probability 1, $V(t)$ can be represented by (27). In the case where $P[X(t) = 0 \ \forall t \geqq 0] = 1$, there exists a constant c such that $P[V(t) = c \ \forall t \geqq 0] = 1$. In the other case, the fact that $\{V(t), \mathscr{F}_t, t \geqq 0\}$ is a martingale implies that $b_0(t) = b_0(1)$ for $t > 0$, and it can be proved by induction on m that with probability 1, $\{V(t), t \geqq 0\}$ is a linear combination of the martingales $\{U_0(t), t \geqq 0\}, \cdots, \{U_m(t), t \geqq 0\}$.

(iii) is an immediate corollary of (i). We now give a probabilistic proof of (iv). Suppose $h(x, t)$ is a polynomial in x, t such that $\{h(X(t), t), \mathscr{F}_t, t \geqq 0\}$ is a martingale. Then it is easy to check that for any real numbers x and s, $\{h(X(t) + x, t + s), \mathscr{F}_t, t \geqq 0\}$ is also a martingale. In particular, consider

$$h(X(t), t + s) = X^n(t) + \binom{n}{1}p_1(t + s)X^{n-1}(t) + \cdots + p_n(t + s) . \tag{36}$$

It is easy to see that (31) holds for $j = 1$. Now assume that (31) holds for $j \leqq n - 1$. Then the martingale defined by (36) can be written as

$$\begin{aligned} h(X(t), t + s) = {} & U_n(t) + \binom{n}{1}p_1(s)U_{n-1}(t) + \cdots + \binom{n}{n-1}p_{n-1}(s)U_1(t) \\ & + \{p_n(t + s) - [p_n(t) + \binom{n}{1}p_1(s)p_{n-1}(t) + \cdots \\ & + \binom{n}{n-1}p_{n-1}(s)p_1(t)]\} . \end{aligned} \tag{37}$$

Now $Eh(X(t), t + s) = h(0, s) = p_n(s)$ and $EU_1(t) = \cdots = EU_n(t) = 0$. Therefore it follows from (37) that (31) holds for $j = n$. □

Theorem 2(i) can be applied to derive higher-order Wald's equations as continuous-time analogs of the results of Brown [2] who first introduced sets like Q_j to describe moment identities for randomly stopped discrete-time martingales. The exponential martingale $e^{iuX(t)}(\psi(u))^{-t}$ used in the proof of Theorem 2 completely characterize the process $X(t)$ itself. More specifically, we have the following lemma:

LEMMA 3. *Suppose ψ is a complex-valued function on the real line such that $\psi(u) \neq 0$ for all u. Let $\{X(t), \mathscr{F}_t, t \geqq 0\}$ be a stochastic process. If $\{e^{iuX(t)}(\psi(u))^{-t}, \mathscr{F}_t, t \geqq 0\}$ is a martingale for all u, then $X(t)$, $t \geqq 0$, has stationary independent and is continuous in probability, and*

$$Ee^{iuX(t)} = (\psi(u))^t Ee^{iuX(0)} .$$

4. Standard Itô processes and stochastic integrals. Suppose $X(t)$, $t \geqq 0$, is a standard Itô process (as defined in [4]) with bounded diffusion coefficients $\sigma(x) > 0$ and $\mu(x)$. Let $e^{\nu|x|}$ $(\nu > 0)$ be a weight function and let $\mathscr{B}$ be the Banach space of all measurable functions such that $||f|| = \sup_x e^{-\nu|x|}|f(x)| < \infty$. Doob [4] has shown that the infinitesimal generator of the corresponding semigroup coincides with the differential operator D when restricted to the space C_2 referred to in Section 1. It follows from (2.9) in [4] that $\int_{-\infty}^{\infty} e^{\nu|y|}P(t, x, dy) \leqq 2L(t)e^{\nu|x|}$, where $L(t) = \exp(k + k\nu t + k\nu^2 t)$, k being a bound for both $\sigma(x)$ and $\mu(x)$. Hence $\int_0^\infty e^{-\alpha t}L(t)\,dt < \infty$ if $\alpha > \alpha_0 = k\nu + k\nu^2$, and so condition (W2a) is satisfied. By Theorem 1, $u(X(t), t + r)$, $t \geqq 0$, is a martingate with respect to P_x for all $x \in R$ and $r \geqq 0$ if the following four conditions are satisfied:

$$u(\cdot, t) \in C^3(-\infty, \infty) \qquad \text{for any fixed} \quad t \geqq 0\,; \tag{38}$$

$$\sup_x e^{-\nu|x|}\left(|u(x, t)| + \left|\frac{\partial^3 u}{\partial x^3}(x, t)\right|\right) < \infty \qquad \text{for any fixed} \quad t \geqq 0\,; \tag{39}$$

$$\sup_{0 \leq t \leq T} \sup_x e^{-\nu|x|}\left|\frac{\partial^2 u}{\partial x^2}(x, t)\right| < \infty \qquad \text{for any} \quad T > 0\,; \tag{40}$$

$$\tfrac{1}{2}\sigma^2(x)\frac{\partial^2 u}{\partial x^2} + \mu(x)\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = 0\,. \tag{41}$$

Let us restrict our attention to those solutions $u(x, t)$ of (41) such that $\partial u/\partial t$, $\partial u/\partial x$, $\partial^2 u/\partial x^2$ are jointly continuous in x, t. Since $X(t)$ is a stochastic integral, Itô's lemma (cf. [11] Section 2.6) is applicable. The following theorem shows how Itô's lemma together with a suitable growth condition on the solution of (41) yields martingales $u(X(t), t)$ associated with $X(t)$. We shall in fact treat stochastic integrals in a more general setting and refer the reader to [11] for the definitions and notations.

THEOREM 3. *Let S_d denote the class of all nonnegative definite symmetric $(d \times d)$ matrices with real entries, and let $\beta(t) = (\beta_1(t), \cdots, \beta_d(t))$ denote the d-dimensional Brownian motion. Let $a\colon R^d \times [0, \infty) \to S_d$ and $b\colon R^d \times [0, \infty) \to R^d$ be bounded Borel functions. Let γ be the nonnegative definite symmetric square root of a. Suppose $Y(t) \in R^d$ satisfies*

$$Y(t) = x + \int_0^t \gamma(Y(s), s)\,d\beta(s) + \int_0^t b(Y(s), s)\,ds\,, \qquad t \geqq 0\,.$$

Let u be a solution of

$$\tfrac{1}{2}\textstyle\sum_{i,j=1}^d a_{ij}(x, t)\dfrac{\partial^2 u}{\partial x_i\,\partial x_j} + \sum_{i=1}^d b_i(x, t)\dfrac{\partial u}{\partial x_i} + \dfrac{\partial u}{\partial t} = 0 \tag{42}$$

such that for $i, j = 1, \cdots, d$, $\partial u/\partial x_i$, $\partial^2 u/\partial x_i\,\partial x_j$ and $\partial u/\partial t$ are continuous on $R^d \times [0, \infty)$ and

$$\left|\gamma_{ij}(x, t)\frac{\partial u}{\partial x_i}(x, t)\right| \leqq \exp\{(1 + |x|)\nu(t)\}\,, \qquad x \in R^d,\ t \geqq 0\,, \tag{43}$$

where $\nu(t)$ is bounded on every compact subset of $[0, \infty)$. Then $u(Y(t), t)$, $t \geqq 0$, is a martingale.

PROOF. Let $|\gamma_{ij}(x, t)| + |b_i(x, t)| \leqq c$ for all $x \in R^d$, $t \geqq 0$ and $i, j = 1, \cdots, d$. Then $|\int_0^t b(Y(s), s)\, ds| \leqq d^{\frac{1}{2}} ct$. Also for any real α,

$$E \exp(\alpha |\textstyle\int_0^t \gamma(Y(s), s)\, d\beta(s)|) \leqq E \exp(|\alpha| \sum_{i,j=1}^d |\int_0^t \gamma_{ij}(Y(s), s)\, d\beta_j(s)|) \leqq 2^d \exp(\alpha^2 c^2 d^3 t/2) . \tag{44}$$

The last inequality above can be easily checked when each $\gamma_{ij}(Y(s), s)$ is simple non-anticipating, and the general case then follows by an application of Fatou's lemma. Without loss of generality, we can assume the function ν in (43) to be measurable. Then it follows from (43) and (44) that for all $t \geqq 0$,

$$\begin{aligned} E \int_0^t \left| \gamma_{ij}(Y(s), s) \frac{\partial u}{\partial x_i}(Y(s), s) \right|^2 ds & \\ \leqq \int_0^t E \exp\{2\nu(s)(1 + |Y(s)|)\}\, ds & \\ \leqq \int_0^t E \exp\{2\nu(s)(1 + |x| + |\textstyle\int_0^s \gamma(Y(\sigma), \sigma)\, d\beta(\sigma)| + d^{\frac{1}{2}} cs)\}\, ds < \infty . & \end{aligned} \tag{45}$$

Now from Itô's lemma and (42), we have

$$u(Y(t), t) = u(x, 0) + \sum_{i,j=1}^d \int_0^t \gamma_{ij}(Y(s), s) \frac{\partial u}{\partial x_i}(Y(s), s)\, d\beta_j(s) . \tag{46}$$

Since (45) holds, it then follows that $u(Y(t), t)$, $t \geqq 0$, is a martingale. □

We point out that without regularity conditions like (45), a stochastic integral $\int_0^t \xi(s)\, dW(s)$ may fail to be a martingale. To give an example, consider the Bessel process $R(t)$ of order m (where m is any real number $\geqq 2$), i.e., the transition function of $R(t)$ is given by

$$p(t, x, y) = t^{-1}(xy)^{1-m/2} I_{m/2-1}(xy/t) y^{m-1} \exp(-(x^2 + y^2)/2t) , \qquad x > 0 ,$$

where $I_r(z)$ is the modified Bessel function of the first kind of order r. Now $R(t)$ can be written as a stochastic integral:

$$R(t) = R(0) + W(t) + \int_0^t (m-1)/(2R(s))\, ds .$$

Hence if u is a continuous solution of

$$\frac{1}{2}\left(\frac{\partial^2 u}{\partial x^2} + \frac{m-1}{x}\frac{\partial u}{\partial x}\right) + \frac{\partial u}{\partial t} = 0 , \tag{47}$$

then u is of class C^∞, and by Itô's lemma,

$$u(R(t), t) = u(R(0), 0) + \int_0^t \frac{\partial u}{\partial x}(R(s), s)\, dW(s) .$$

Although $u(R(t), t)$ has this stochastic integral representation, it does not necessarily follow that $u(R(t), t)$, $t \geqq 0$, is a martingale. In fact, consider the C^∞ solution $u(x, t) = e^{-\lambda t} x^{1-m/2} K_{m/2-1}((2\lambda)^{\frac{1}{2}} x)$ of (47), where $\lambda > 0$ and $K_r(z)$ is the modified Bessel function of the second kind of order r. It is easy to check that $u(R(t), t)$, $t \geqq 0$, is not a martingale with respect to P_x for any $x > 0$.

The function $g_1(x) = x^{1-m/2}K_{m/2}((2\lambda)^{\frac{1}{2}}x)$ is the positive decreasing solution of

$$\frac{1}{2}\left(\frac{d^2g}{dx^2} + \frac{m-1}{x}\frac{dg}{dx}\right) = \lambda g\,. \tag{48}$$

We have pointed out that $e^{-\lambda t}g_1(R(t))$, $t \geqq 0$, is not a martingale. On the other hand, for the positive increasing solution $g_2(x) = x^{1-m/2}I_{m/2-1}((2\lambda)^{\frac{1}{2}}x)$, it is easy to check that $e^{-\lambda t}g_2(R(t))$, $t \geqq 0$, is a martingale. This phenomenon is in fact common to all regular one-dimensional diffusions on (r_0, r_1) where r_0 is an entrance boundary and r_1 is a natural boundary. In [10], we study the asymptotic behavior of the first passage probabilities near the boundary points and use this to establish the above phenomenon for general regular one-dimensional diffusions.

5. Evaluation of boundary crossing probabilities. We begin by evaluating certain boundary crossing probabilities of one-sided stable processes and obtaining the upper half of the law of the iterated logarithm as an immediate corollary. Let $X(t)$, $t \geqq 0$, be a one-sided stable process with right continuous increasing sample paths and $X(0) = 0$. The Laplace transform of such a process is of the form $Ee^{-\lambda X(t)} = \exp(-\xi\lambda^\nu t)$, $0 < \nu \leqq 1$, $\xi > 0$ (see pages 31–33 of [9]). The case $\nu = 1$ corresponds to $X(t) = \xi t$ and is trivial; so we shall only consider $0 < \nu < 1$. Let us write $\eta = \nu^{-1}$, $c = \xi^{-\eta}$. Let F be any measure on $(0, \infty)$ such that $0 < f(0, 1) < \infty$, where we define

$$f(x, t) = \textstyle\int_0^\infty \exp(-cxy^\eta + ty)\, dF(y)\,.$$

Then $f(X(t) + x, t)$, $t \geqq 0$, is a martingale for any $x \geqq 0$. For $\varepsilon > 0$, define $A(t, \varepsilon) = \sup\{x \geqq 0 : f(x, t) \geqq \varepsilon\}$. Given $\varepsilon > 0$ and $\delta > 0$, since $\lim_{t\to\infty} f(x + \delta t^\eta, t) = 0$, we have for all large t

$$P[f(x + X(t), t) \geqq \varepsilon] \leqq P[X(t) \leqq \delta t^\eta] = P[X(1) \leqq \delta]\,.$$

Since $\lim_{\delta\downarrow 0} P[X(1) \leqq \delta] = 0$, it follows that $f(x + X(t), t) \to_P 0$, and so by the martingale convergence theorem, $f(x + X(t), t) \to 0$ a.e. From the path properties of $X(t)$, it is easy to see that the sample paths of $f(x + X(t), t)$, $t \geqq 0$, have the property (*) of Remark (d) on page 1417 of [13]. Hence by [13] (page 1415), for any $x \geqq 0$, $\tau \geqq 0$,

$$\begin{aligned} &P[X(t) + x \leqq A(t, \varepsilon) \text{ for some } t \geqq \tau] \\ &\quad = P[X(\tau) + x \leqq A(\tau, \varepsilon)] + \varepsilon^{-1} \textstyle\int_{[X(\tau)+x > A(\tau,\varepsilon)]} f(X(\tau) + x, \tau)\, dP\,. \end{aligned} \tag{49}$$

In the case where F is the degenerate measure putting unit mass at the point $a > 0$, we have

$$A(t, \varepsilon) = (\xi/a^{1-\nu})^{1/\nu} t - (\xi/a)^{1/\nu} \log \varepsilon\,. \tag{50}$$

Now write $e^e = e_2$, $\log\log y = \log_2 y$, etc. Suppose $n \geqq 2$, $\delta > 0$ and

$$dF(y) = \varphi(y)\, dy \quad \text{for} \quad 0 < y < 1/e_n \quad \text{and} \quad = 0 \quad \text{elsewhere, where} \tag{51}$$
$$\varphi(y) = \{y(\log y^{-1}) \cdots (\log_{n-1} y^{-1})(\log_n y^{-1})^{1+\delta}\}^{-1}\,.$$

Then as $t \to \infty$,

(52) $$A(t, \varepsilon) = \nu(\xi t)^{1/\nu}\{(1-\nu)^{-1}[\log_2 t + \tfrac{3}{2}\log_3 t + \textstyle\sum_{k=4}^{n+1}\log_k t + \delta \log_{n+1} t + \log(\varepsilon(2\pi\nu)^{-\frac{1}{2}}) + o(1)]\}^{-(1-\nu)/\nu}.$$

We shall show (52) in the case $n = 2$. Letting B be defined by the equation $A(t, \varepsilon) = t^\eta/(c\eta B^{\eta-1})$, it is easy to see that $B \to \infty$ and $B = o(t)$ as $t \to \infty$. Take $\gamma > 1$. Then for all large t,

(53) $$\begin{aligned}\varepsilon &\geqq \varphi(\gamma B/t)\textstyle\int_0^{\gamma B/t} \exp\{yt(1-(t^{\eta-1}y^{\eta-1})/(\eta B^{\eta-1})\}\,dy \\ &= (B/t)\varphi(\gamma B/t)\textstyle\int_0^\gamma \exp\{Bz(1-z^{\eta-1}/\eta)\}\,dz \\ &\sim (B/t)\varphi(\gamma B/t)\{2\pi/(B(\eta-1))\}^{\frac{1}{2}}\exp\{B(1-\eta^{-1})\} \qquad \text{as } t \to \infty.\end{aligned}$$

The last expression above follows from Laplace's asymptotic formula (cf. [15] Section 7.2). Letting $\gamma \uparrow 1$, we obtain

(54) $$\limsup_{t\to\infty} \frac{\exp\{B(1-\eta^{-1})\}}{(\log tB^{-1})(\log_2 tB^{-1})^{1+\delta}B^{\frac{1}{2}}}\{2\pi/(B(\eta-1))\}^{\frac{1}{2}} \leqq \varepsilon.$$

From (54), it follows that $B = O(\log_2 t)$ as $t \to \infty$. Let $\alpha_t = (\log_2 t)^{-1}$. Take $\gamma \in (0, 1)$ and let $k > \eta^{1/(\eta-1)}$ (> 1). Then for all large t,

(55) $$\begin{aligned}\varepsilon &= \textstyle\int_0^{e^{-e}} \exp\{ty - (t^\eta y^\eta)/(\eta B^{\eta-1})\}\varphi(y)\,dy \\ &= \textstyle\int_0^{\alpha_t B/t} + \int_{\alpha_t B/t}^{\gamma B/t} + \int_{\gamma B/t}^{kB/t} + \int_{kB/t}^{e^{-e}}.\end{aligned}$$

(The integrand $\exp\{ty - (t^\eta y^\eta)/(\eta B^{\eta-1})\}\varphi(y)$ is omitted hereafter for the simplicity of notation.)

Since $B = O(\log_2 t)$, it can be proved that $\int_0^{\alpha_t B/t} = o(1)$. Also, by the dominated convergence theorem, $\int_{kB/t}^{e^{-e}} = o(1)$ as $t \to \infty$. We also note that

$$\begin{aligned}\textstyle\int_{\alpha_t B/t}^{\gamma B/t} &\leqq \varphi(\alpha_t B/t)\textstyle\int_0^\gamma (B/t)\exp\{B(z - z^\eta/\eta)\}\,dz \\ &\leqq \frac{\gamma \log_2 t}{\{\log(tB^{-1}\log_2 t)\}\{\log_2(tB^{-1}\log_2 t)\}^{1+\delta}}\exp\{B(\gamma - \gamma^\eta/\eta)\}.\end{aligned}$$

Using Laplace's asymptotic formula as in (53), it can be shown that

$$\textstyle\int_{\gamma B/t}^{kB/t} \leqq \frac{\exp\{B(1-\eta^{-1})\}}{\gamma(\log tB^{-1})(\log_2 tB^{-1})^{1+\delta}B^{\frac{1}{2}}}\{2\pi/(B(\eta-1))\}^{\frac{1}{2}}(1+o(1)).$$

From these results and (55), it follows that upon letting $\gamma \uparrow 1$,

(56) $$\varepsilon \leqq \liminf_{t\to\infty} \frac{\exp\{B(1-\eta^{-1})\}}{(\log tB^{-1})(\log_2 tB^{-1})B^{\frac{1}{2}}}\{2\pi/(B(\eta-1))\}^{\frac{1}{2}}.$$

From (54) and (56), we obtain (52) in the case $n = 2$. A similar argument gives (52) for $n > 2$.

The relation (52) has been established by Robbins and Siegmund [13] in the case $\xi = 2^{\frac{1}{2}}$ and $\nu = \frac{1}{2}$ using the "dual" results for Brownian motion, and their argument obviously does not carry over for a genenral ν. Since $f(X(t) + x, t) \to 0$ a.e. as $t \to \infty$, it follows that $P[X(t) \geqq A(t, \varepsilon)$ for all large $t] = 1$ for any $\varepsilon > 0$, and so (52) implies that with probability one,

$$\liminf_{t\to\infty} \frac{X(t)}{\nu(\xi t)^{1/\nu}} \left(\frac{\log_2 t}{1-\nu}\right)^{(1-\nu)/\nu} \geqq 1 . \tag{57}$$

Thus we have obtained one half of the law of the iterated logarithm for nondecreasing stable processes with exponent $\nu < 1$, which says that in fact equality holds in (57). This law of the iterated logarithm has been proved by Fristedt [7] and Breiman [1] using different methods.

We now consider the Bessel process $R(t)$ of order $m \geqq 2$. Let $p = m/2 - 1$, $h \geqq 0$, and define

$$f_1(x, t) = \int_0^\infty (xy)^{-p} I_p(xy) \exp\{-\tfrac{1}{2}y^2(t+h)\}\, dF(y) , \qquad x > 0,\ t \geqq 0 ,$$

where F is any measure on $(0, \infty)$ such that $0 < f_1(x, t) < \infty$ for all $x > 0$, $t \geqq 0$. Then $f_1(R(t), t)$, $t \geqq 0$, is a martingale with respect to P_a for any $a > 0$. Given $\varepsilon > 0$, let $A_1(t, \varepsilon) = \inf\{x \geqq 0 : f_1(x, t) \geqq \varepsilon\}$. It can be shown that $f_1(R(t), t) \to 0$ a.e., and so by [13], if $\varepsilon > 0$, $\tau > 0$, then

$$\begin{aligned} &P_a[R(t) \geqq A_1(t, \varepsilon) \text{ for some } t \geqq \tau] \\ &\quad = P_a[R(\tau) \geqq A_1(\tau, \varepsilon)] + \varepsilon^{-1} \int_0^\infty \int_{[R(\tau) < A_1(\tau,\varepsilon)]} (yR(\tau))^{-p} I_p(yR(\tau)) \\ &\qquad \times \exp\{-\tfrac{1}{2}y^2(\tau + h)\}\, dP_a\, dF(y) . \end{aligned} \tag{58}$$

In the case where $dF(y) = y^{m-1}\, dy$, $y > 0$, we have $f_1(x, t) = (t+h)^{-(p+1)} \times \exp\{x^2/(2(t+h))\}$, and so

$$A_1(t, \varepsilon) = \{2(t+h)[\log \varepsilon + \tfrac{1}{2}m \log (t+h)]\}^{\frac{1}{2}} . \tag{59}$$

Suppose that the measure F is given by (51). Then it can be shown that as $t \to \infty$,

$$\begin{aligned} A_1(t, \varepsilon) = \{2t[&\log_2 t + (1 + \tfrac{1}{2}m) \log_3 t + \textstyle\sum_{k=4}^{n+1} \log_k t + \delta \log_{n+1} t \\ &+ \log \varepsilon + (\tfrac{1}{2}m - 1) \log 2 + o(1)]\}^{\frac{1}{2}} , \end{aligned} \tag{60}$$

which is an upper-class boundary of the iterated-logarithm type for the Bessel process $R(t)$.

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REFERENCES

[1] Breiman, L. (1968). A delicate law of the iterated logarithm for nondecreasing stable processes. *Ann. Math. Statist.* **39** 1818–1824.

[2] Brown, B. M. (1969). Moments of a stopping rule related to the central limit theorem. *Ann. Math. Statist.* **40** 1236–1249.

[3] Chow, Y. S., Robbins, H. and Teicher, H. (1965). Moments of randomly stopped sums. *Ann. Math. Statist.* **36** 789–799.

[4] Doob, J. L. (1955). Martingales and one-dimensional diffusion. *Trans. Amer. Math. Soc.* **78** 168–208.

[5] Dynkin, E. B. (1965). *Markov Processes* **1**. Springer-Verlag, Berlin.

[6] FRIEDMAN, A. (1964). *Partial Differential Equations of Parabolic Type*. Prentice-Hall, Englewood Cliffs.
[7] FRISTEDT, B. (1964). The behavior of increasing stable processes for both small and large times. *J. Math. Mech.* **13** 849–856.
[8] HALL, W. J. (1970). On Wald's equations in continuous time. *J. Appl. Probability*. **7** 59–68.
[9] ITÔ, K. and MCKEAN, H. P. (1965). *Diffusion Processes and their Sample Paths*. Springer-Verlag, Berlin.
[10] LAI, T. L. (1973). Space-time processes, parabolic functions and one-dimensional diffusions. *Trans. Amer. Math. Soc.* **175** 409–438.
[11] MCKEAN, H. P. (1969). *Stochastic Integrals*. Academic Press, New York.
[12] ROBBINS, H. (1970). Statistical methods related to the law of the iterated logarithm. *Ann. Math. Statist.* **41** 1397–1409.
[13] ROBBINS, H. and SIEGMUND, D. (1970). Boundary crossing probabilities for the Wiener process and sample sums. *Ann. Math. Statist.* **41** 1410–1429.
[14] ROBBINS, H. and SIEGMUND, D. (1973). Statistical tests of power one and the integral representation of solutions of certain partial differential equations. *Bull. Inst. Math. (Acad. Sinica)* **1** 93–120.
[15] WIDDER, D. V. (1941). *The Laplace Transform*. Princeton Univ. Press.
[16] WIDDER, D. V. (1944). Positive temperatures on an infinite rod. *Trans. Amer. Math. Soc.* **55** 85–95.

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