# SOME ITERATED LOGARITHM RESULTS FOR SUMS OF INDEPENDENT TWO-DIMENSIONAL RANDOM VARIABLES[1]

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Let $Z_i = (X_i, Y_i)$, $i \geq 1$, be independent two-dimensional random variables, defined on a probability triple $(\Omega, \mathscr{A}, P)$, such that $E(X_i) = E(Y_i) = E(X_i Y_i) = 0$, $E(X_i^2) < \infty$, $E(Y_i^2) < \infty$ for all $i$. The purpose of this paper is to investigate the limit points of $\{(S_n(\omega)/L(n), T_n(\omega)/M(n)), n = 1, 2, \cdots\}$, where $\omega \in \Omega$, $S_n = \sum_{i=1}^{n} X_i$, $T_n = \sum_{i=1}^{n} Y_i$, $L(n) = [2E(S_n^2) \log \log E(S_n^2)]^{\frac{1}{2}}$, $M(n) = [2E(T_n^2) \log \log E(T_n^2)]^{\frac{1}{2}}$. The author will show the limit sets are the closed unit disk almost surely under some general conditions. An example with all limit points lying on the two axes with probability one will be constructed.

1. **Introduction.** Let $R^k$ be the $k$-dimensional Euclidean space. Denote the *one-point compactification* of $R^k$ by $\bar{R}^k = R^k \cup \{\infty\}$ with the usual topology. Let $\{a_n, n \geq 1\}$ be a sequence of points in $R^k$. A point $x_0$ in $\bar{R}^k$ is a *limit point* (*accumulation point*) of $\{a_n\}$ if either $x_0 \in R^k$ and $\forall \varepsilon > 0$, $\forall n$, $\exists m \ni m \geq n$ and $|a_m - x_0| < \varepsilon$, where $|\ |$ is the Euclidean norm, i.e. $|(y_1, \cdots, y_k)| = (y_1^2 + \cdots + y_k^2)^{\frac{1}{2}}$, or $x_0 = \infty$ and $\{a_n\}$ is unbounded. The collection of all such $x_0$ is called the *limit set* of $\{\alpha_n\}$. Now Hartman–Wintner's law of the iterated logarithm [6] can be stated in the following manner: suppose $X_1, X_2, \cdots$ are i.i.d. real-valued random variables, defined on a probability triple $(\Omega, \mathscr{A}, P)$, with means zero and variance one. Then, with probability 1, the limit set of $\{[X_1(\omega) + \cdots + X_n(\omega)]/(2n \log \log n)^{\frac{1}{2}}, n \geq 3\}$ is $[-1, 1]$. The author is interested in the analogue of this result for the multidimensional variables. Namely, suppose we have, say, 2-dimensional random variables $Z_n = (X_n, Y_n)$, $n \geq 1$; let us assume $E(X_n) = E(Y_n) = E(X_n Y_n) = 0$ for all $n$; let $S_n = \sum_{i=1}^{n} X_i$, $T_n = \sum_{i=1}^{n} Y_i$, $L(n) = [2E(S_n^2) \log \log E(S_n^2)]^{\frac{1}{2}}$, $M(n) = [2E(T_n^2) \log \log E(T_n^2)]^{\frac{1}{2}}$. What can be said about the limit sets of $\{(S_n/L(n), T_n/M(n))\}$ under reasonable conditions? The main results of this paper are Theorem (2.10) and Theorem (2.14).

The following notations and conventions will be used throughout the article.

(1.1 a)     $(\Omega, \mathscr{A}, P)$ is the probability triple on which the random variables considered in each statement are defined.

(1.1 b)    $Z_n = (X_n, Y_n)$, $n \geq 1$, is a sequence of *independent* random variables in $R^2$.

(1.1 c)                          $\log_2 = \log \log$ .

**2. Statement of results.** The limits sets related to the law of the iterated logarithm are not random. In fact,

(2.1)    PROPOSITION. *Recall* (1.1 b). *Let* $\{\phi(n)\}$ *and* $\{\psi(n)\}$ *be two sequences of numbers such that*

(2.2)              $\lim_{n \to \infty} \phi(n) = \infty$ ,      $\lim_{n \to \infty} \psi(n) = \infty$ .

*Let*

$$S_n = \sum_{i=1}^n X_i , \qquad T_n = \sum_{i=1}^n Y_i .$$

*Then there exists a unique nonempty closed subset $H$ of $\bar{R}^2$ such that*

(2.3)    $P\{\omega \,|\, \text{the limit set of } \{(S_n(\omega)/\phi(n), T_n(\omega)/\psi(n))\} \text{ is } H\} = 1$ .

*In fact,*

(2.4)    $H = \{x_0 \in \bar{R}^2 \,|\, x_0 \text{ is a limit point of } \{(S_n(\omega)/\phi(n), T_n(\omega)/\psi(n))\}$
                      *for almost all* $\omega \in \Omega\}$ .

The proof is based on the Kolmogorov 0–1 law and the fact that $\bar{R}^2$ is a separable metric space. Following (2.1), we make the

(2.5)    DEFINITION. In Proposition (2.1), $H$ is called the a.s. limit set of $\{(S_n/\phi(n), T_n/\psi(n))\}$.

Thus, the goal is to determine $H$ with given $\phi$ and $\psi$.

For the case when $Z_n$'s are i.i.d., it has been known in the literature that the a.s. limit set of $\{\sum_{i=1}^n Z_i/(2n \log_2 n)^{\frac{1}{2}}\}$ is the closed unit disk if $Z_1$ has mean 0 and covariance matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. We use a result due to Strassen [13] to prove that the converse is also true. Hence,

(2.6)    PROPOSITION. *Suppose* $Z_n = (X_n, Y_n)$, $n \geq 1$, *are i.i.d. Let*

$$W_n = \sum_{i=1}^n Z_i .$$

*Then the* a.s. *limit set $H$ of* $\{W_n/(2n \log_2 n)^{\frac{1}{2}}\}$ *is the closed unit disk if and only if*

$$E(Z_1) = (0, 0) , \qquad \text{Cov}\,(Z_1) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) .$$

Hence, whenever $E(Z_1) = (0, 0)$ and Cov$(Z_1)$ exists, then a.s. limit set of $\{W_n/(2n \log_2 n)^{\frac{1}{2}}\}$ is an ellipsoid.

When the variables do not have the same distribution but the two coordinates for each variable have equal variances, Kolmogorov's result [8] can be easily extended. We remind you of (1.1 b).

(2.7)    PROPOSITION. *Suppose* $Z_n$ *satisfies*

(2.8 a)              $E(X_i) = E(Y_i) = E(X_i Y_i) = 0$ ,

(2.8 b)              $E(X_i^2) = E(Y_i^2) = \sigma_i^2 < \infty$ ,                      $\forall i$ .

*Let*

(2.8 c) $$s_n{}^2 = \sum_{i=1}^n \sigma_i{}^2 , \qquad W_n = \sum_{i=1}^n Z_i .$$

*Let*

$$|X_k| = \text{ess sup}_{\omega \in \Omega} |X_k(\omega)| ,$$
$$|Y_k| = \text{ess sup}_{\omega \in \Omega} |Y_k(\omega)| .$$

*Then the* a.s. *limit set H of* $\{W_n/(2s_n{}^2 \log_2 s_n{}^2)^{\frac{1}{2}}\}$ *is the closed unit disk if*

(2.9) $$s_n{}^2 \to \infty , \qquad \sup_{k \leq n} (|X_k| + |Y_k|) = o((s_n{}^2/\log_2 s_n{}^2)^{\frac{1}{2}}) .$$

Proposition (2.7) is proved by using linear functionals on $R^2$ then applying the one-dimensional result.

For the case when two coordinates have unequal variances, we construct an example in Section 4 which amounts to saying the following:

(2.10)    THEOREM. *There exists a sequence of independent random variables,* $Z_n = (X_n, Y_n)$, $n \geq 1$ *such that* $X_n$ *and* $Y_n$ *are independent for each n and*

$$E(X_i) = E(Y_i) = 0 , \qquad E(X_i^2) < \infty , \qquad E(Y_i^2) < \infty , \qquad \forall i ,$$

*but the* a.s. *limit set of*

$$\{(S_n/(2s_n{}^2 \log_2 s_n{}^2)^{\frac{1}{2}}, T_n/(2t_n{}^2 \log_2 t_n{}^2)^{\frac{1}{2}}\} \quad is \quad \{(a, b) \,|\, ab = 0, |a| \leq 1, |b| \leq 1\} ,$$

*where*

$$s_n{}^2 = \sum_{i=1}^n E(X_i^2) , \qquad t_n{}^2 = \sum_{i=1}^n E(Y_i^2) , \qquad S_n = \sum_{i=1}^n X_i , \qquad T_n = \sum_{i=1}^n Y_i .$$

It is well known that the law of the iterated logarithm is strongly related to the Central Limit Theorem, and conditions stronger than Lindeberg's are needed. We present in Theorem (2.14) a condition on the rate of convergence to the normal law.

To state (2.14), assume (1.1 b) and

$$E(X_i) = 0 , \qquad E(Y_i) = 0 , \qquad E(X_i Y_i) = 0 ,$$
$$E(X_i^2) = \sigma_i{}^2 < \infty , \qquad E(Y_i^2) = \tau_i{}^2 < \infty .$$

Define

$$s^2(n) = s_n{}^2 = \sum_{i=1}^n \sigma_i{}^2 , \qquad t^2(n) = t_n{}^2 = \sum_{i=1}^n \tau_i{}^2 ,$$
$$S(n) = S_n = \sum_{i=1}^n X_i , \qquad T(n) = T_n = \sum_{i=1}^n Y_i .$$

To avoid triviality, assume that $s_n{}^2 \to \infty$ and $t_n{}^2 \to \infty$. Let

(2.11) $$\rho(n) = \rho(F_n, \Phi) = \sup_{-\infty < x, y < \infty} |F_n(x, y) - \Phi(x, y)| ,$$

where $\Phi$ is the standard 2-dim normal distribution function and $F_n$ the distribution function of $(S_n/s_n, T_n/t_n)$.

Define two subsequences $\{n_k\}$ and $\{m_k\}$ by induction. Take any $\gamma_0, \gamma > 1, \gamma_0$ fixed. Let

(2.12)    $n_0$ be the first $n$ such that $s^2(n) \geq 3$ and $t^2(n) \geq 3$, when $n_0, \cdots, n_{k-1}$

are defined, $n_k$ is the first $n$ such that

$$s^2(n) \geqq \gamma_0 s^2(n_{k-1}) \qquad or \qquad t^2(n) \geqq \gamma_0 t^2(n_{k-1}) \,.$$

(2.13)    Let $m_0 = n_0$; when $m_0, \cdots, m_{k-1}$ are defined, $m_k$ is the first $n$ such that $s^2(n) \geqq \gamma_0 s^2(m_{k-1})$ and $t^2(n) \geqq \gamma_0 t^2(m_{k-1})$.

(2.14)    THEOREM. *Suppose* $\exists \beta > 0 \ni$

$$(2.15) \qquad \rho(n) = O(\min \{1/(\log s_n^2)^{1+\beta}, 1/(\log t_n^2)^{1+\beta}\}) \,.$$

*Let H be the* a.s. *limit set of*

$$\{(S_n/(2s_n^2 \log_2 s_n^2)^{\frac{1}{2}}, \ T_n/(2t_n^2 \log_2 t_n^2)^{\frac{1}{2}})\} \qquad and \qquad (a, b) \in R^2 \,.$$

*Then* $(a, b) \notin H$ *if* $\exists \delta > 0 \ni$

$$(2.16) \qquad \sum_{k=0}^{\infty} 1/(\log s^2(n_k))^{a^2-\delta}(\log t^2(n_k))^{b^2-\delta} < \infty \,,$$

*and* $(a, b) \in H$ *if* $\exists \delta > 0 \ni$

$$(2.17) \qquad \sum_{k=0}^{\infty} 1/(\log s^2(m_k))^{a^2+\delta}(\log t^2(m_k))^{b^2+\delta} = \infty \,.$$

(2.18)    REMARK. In (2.16), when $a = 0$ or $b = 0$, $(\log s^2(n_k))^{a^2-\delta}$ or $(\log t^2(n_k))^{b^2-\delta}$ should be considered equal to 1 respectively.

As applications of (2.14), we show

(2.19)    COROLLARY. *Assume* (1.1b), (2.8a), (2.8b), (2.8c), *and* $s_n^2 \to \infty$. *Suppose, furthermore, all* $Z_n$'s *are normal. Then the* a.s. *limit set H of* $\{W_n/(2s_n^2 \log_2 s_n^2)^{\frac{1}{2}}\}$ *is the closed disk with center origin and radius d if and only if*

$$(2.20\,\mathrm{a}) \qquad \sum_{k=0}^{\infty} 1/(\log s^2(n_k))^{d^2-\delta} = \infty \,, \qquad\qquad \forall \delta > 0 \,,$$

$$(2.20\,\mathrm{b}) \qquad \sum_{k=0}^{\infty} 1/(\log s^2(n_k))^{d^2+\delta} < \infty \,, \qquad\qquad \forall \delta > 0 \,.$$

(2.21)    COROLLARY. *Under* (1.1b), (2.8a), (2.8b), (2.8c), *and* $s_n^2 \to \infty$, *the* a.s. *limit set of* $\{W_n/(2s_n^2 \log_2 s_n^2)^{\frac{1}{2}}\}$ *is the closed unit disk if* $\exists \beta > 0, \lambda < \infty \ni$

$$(2.22) \qquad E[(X_i^2 + Y_i^2)(\log (X_i^2 + Y_i^2))^{1+\beta}] \leqq \lambda \sigma_i^2 \,, \qquad\qquad \forall i \,.$$

## 3. The proofs.

(3.1)    LEMMA. *Let* $\{U_n, n \geq 1\}$ *be a sequence of random variables in* $R^2$. *Then for any* $x_0 \in \bar{R}^2$, *the set*

$$(3.2) \qquad A(x_0) = \{\omega \in \Omega \,|\, x_0 \ is \ a \ limit \ point \ of \ \{U_n(\omega)\}\} \quad is \ measurable.$$

PROOF. Since $\bar{R}^2$ satisfies the first axiom of countability,

$$A(x_0) = \bigcap_{V \in \mathscr{V}} \{U_n \in V \ \text{i.o.}\} \,,$$

where $\mathscr{V}$ is a countable family of open neighborhoods at $x_0$. □

(3.3)    THE PROOF OF (2.1). Define $A(x_0)$ by (3.2) for each $x_0 \in \bar{R}^2$ by taking $U_n(\omega) = (S_n(\omega)/\phi(n), T_n(\omega)/\phi(n))$. By (2.2), $A(x_0)$ is a tail event. Applying the Kolmogorov 0–1 law, $P(A(x_0)) = 0$ or 1. Let $H = \{x_0 \in \bar{R}^2 \,|\, P(A(x_0)) = 1\}$.

(a) Claim: *H* is closed.

Indeed, take any $\{x_n\}$ in *H* such that

$$x_n \to \bar{x}, \qquad\qquad \text{some } \bar{x} \in \bar{R}^2.$$

Because any limit set is closed,

$$\bigcap_n A(x_n) \subset A(\bar{x}).$$

Hence,

$$P(\bigcap_n A(x_n)) = 1 \Rightarrow P(A(\bar{x})) = 1 \Rightarrow \bar{x} \in H.$$

(b) Claim: $P\{\omega \mid H$ is the limit set of $\{(S_n(\omega)/\phi(n), T_n(\omega)/\phi(n))\}\} = 1$.

Note that $y \in H$ if for all open nbd *V* of *y* we have $P((S_n/\phi(n), T_n/\phi(n)) \in V$, i.o.) = 1. Since $H^c$ is open, $\forall y \in H^c$ ∃ open nbd $D(y)$ of $y \ni D(y) \subset H^c$ and

$$P((S_n/\phi(n), T_n/\phi(n)) \in D(y), \text{ i.o.}) < 1.$$

Again, by (2.2), the event $\{(S_n/\phi(n), T_n/\phi(n)) \in D(y)$, i.o.$\}$ is a tail event. Hence, by the Kolmogorov 0–1 law,

$$(3.4) \qquad P((S_n/\phi(n), T_n/\phi(n)) \in D(y), \text{ i.o.}) = 0.$$

Since $\bigcup_{y \in H^c} D(y) = H^c$ and $H^c$ is Lindelöf, there exists a countable subcovering $D(y_1), D(y_2), \cdots$, i.e.

$$\bigcup_n D(y_n) = H^c.$$

Let $B_m = \{\omega \mid S_n(\omega)/\phi(n), T_n(\omega)/\phi(n)) \in \bigcup_{k=1}^m D(y_k), \text{ i.o.}\}^c$ for each *m*. Then (3.4) implies $P(B_m) = 1$, $\forall m$. Let *G* be a countable dense subset of *H*. Let $C_m = B_m \cap (\bigcap_{x \in G} A(x))$. Then $P(C_m) = 1$, $\forall m$. Observe that $\forall \omega \in C_m$, the limit set of $\{(S_n(\omega)/\phi(n), T_n(\omega)/\phi(n))\}$ contains *H* but is contained in $(\bigcup_{k=1}^m D(y_k))^c = \bigcap_{k=1}^m D(y_k)^c$. Let $C = \bigcap_m C_m$. Note $\bigcap_{k=1}^\infty (D(y_k))^c = H$. Hence, $P(C) = 1$ and $\forall \omega \in C$, the limit set of $\{(S_n(\omega)/\phi(n), T_n(\omega)/\phi(n))\}$ is exactly *H*. Moreover, if we choose $D(y)$ in such a way that $\overline{D(y)} \subset H^c$, then by the compactness of $\overline{D(y)}$ (in $\bar{R}^2$) we see

$$C = \{\omega \mid \text{the limit set of } \{(S_n(\omega)/\phi(n), T_n(\omega)/\phi(n))\} \text{ is exactly } H\}.$$

Since for each $\omega$, the limit set of $\{U_n(\omega)\}$ is unique and nonempty, *H* is unique and nonempty. □

(3.5)   REMARK. From the above proof, one sees that if $E \subset \bar{R}^2$, *E* is closed, $E \cap H = \phi$, then $P((S_n/\phi(n), T_n/\phi(n)) \in E$, i.o.) = 0 by the compactness of *E*. To prove (2.6), observe a simple fact:

(3.6)   LEMMA. *Let $\{a_n\}$ be an indexed set in $R^2$. Suppose $T: R^2 \to R^2$ is a continuous map and $x_0$, $x_0 \in R^2$, is a limit point of $\{a_n\}$. Then $T(x_0)$ is a limit point of $\{T(a_n)\}$.*

(3.7)   THE PROOF OF (2.6). The proof of the "if" part can be found in a paper by Finkelstein [4]. For the "only if" part, suppose the a.s. limit set *H* of $\{W_n/(2n \log_2 n)^{\frac{1}{2}}\}$ is the closed unit disk. Let $S_n = \sum_{i=1}^n X_i$. Since $\infty \notin H$, $\{W_n(\omega)/(2n \log_2 n)^{\frac{1}{2}}\}$ is bounded almost surely. By (3.6) and Remark (3.5), the

a.s. limit set of $\{S_n/(2n \log_2 n)^{\frac{1}{2}}\}$ is $[-1, 1]$. Applying a result by Strassen [13], $E(X_1) = 0$, $E(X_1^2) = 1$. Similarly, $E(Y_1) = 0$, $E(Y_1^2) = 1$. We know $\exists \lambda_1, \lambda_2 > 0$ and an orthogonal transformation $A$ such that the cavariance matrix of $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} A Z_1$ is $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Let $Z_i' = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} A Z_i$, $W_n' = \sum_{i=1}^{n} Z_i'$. By "if" part, we see the a.s. limit set of $\{W_n'/(2n \log_2 n)^{\frac{1}{2}}\}$ is the closed unit disk. But $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} A$ is a 1–1 onto linear map, the a.s. limit set of $\{W_n'/(2n \log_2 n)^{\frac{1}{2}}\}$ is the image of the a.s. limit set of $\{W_n/(2n \log_2 n)^{\frac{1}{2}}\}$ under the map $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} A$. Hence, thh unit disk is invariant under the map $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} A$, which cannot hold unless $\lambda_1 = \lambda_2 = 1$. Therefore,

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \text{Cov}(Z_1') = A \, \text{Cov}(Z_1) A^t = \text{Cov}(Z_1) . \qquad \square$$

(3.8)    THE PROOF OF (2.7). Take a point $(a, b)$ on the unit circle. Consider $V_n = a X_n + b Y_n$, $n \geq 1$. Let

$$U_n = \sum_{i=1}^{n} V_i .$$

By (2.8a, b, c), and (2.9), one sees $E(V_i) = 0$, $E(V_i^2) = \sigma_i^2$, and $\sup_{k \leq n} |V_k| = o((s_n^2/\log_2 s_n^2)^{\frac{1}{2}})$, $s_n^2 \to \infty$. By Kolmogorov,

$$(3.9) \qquad \limsup U_n/(2 s_n^2 \log_2 s_n^2)^{\frac{1}{2}} = 1 \quad \text{a.s.}$$

This is true as long as $a^2 + b^2 = 1$. Using (3.6), we deduce

$$(3.10) \qquad H \subset \{(x, y) \mid x^2 + y^2 \leq 1\} .$$

(3.11)    Claim: $H \supset \{(x, y) \mid x^2 + y^2 = 1\}$.

Suppose the above claim is not true. Then $\exists (a_0, b_0)$ and $B_\delta(a_0, b_0)$, $\delta > 0$, such that

$$(3.12) \qquad P(W_n/(2 s_n^2 \log_2 s_n^2)^{\frac{1}{2}} \in B_\delta(a_0, b_0), \text{ i.o.}) = 0$$

by Proposition (2.1), where $a_0^2 + b_0^2 = 1$ and $B_\delta(a_0, b_0) = \{(a, b) \mid |(a, b) - (a_0, b_0)| < \delta\}$. Let $V_n^0 = a_0 X_n + b_0 Y_n$, $n \geq 1$, and $U_n^0 = \sum_{i=1}^{n} V_i^0$. Using elementary geometry and (3.10), (3.12), we have

$$\limsup U_n^0/(2 s_n^2 \log_2 s_n^2)^{\frac{1}{2}} \leq 1 - \delta^2/2 < 1 \quad \text{a.s.}$$

which contradicts (3.9).

To probe $H \supset \{(x, y) \mid x^2 + y^2 \leq 1\}$, take another independent copy $\{Z_n' = (X_n', Y_n')\}$ or $\{Z_n\}$. Consider $\Lambda_n = \sum_{i=1}^{n} (X_i, Y_i, X_i', Y_i')$. Corrying out the same arguments as before, we see the a.s. limit of $\{\Lambda_n/(2 s_n^2 \log_2 s_n^2)^{\frac{1}{2}}\}$ contains $\{(a, b, c, d) \mid a^2 + b^2 + c^2 + d^2 = 1\}$. Then taking the projection, we have

$$(3.13) \qquad H \supset \{(x, y) \mid x^2 + y^2 \leq 1\} .$$

The theorem follows from (3.10) and (3.13). $\square$

We shall work on Theorem (2.14). The following lemma (3.14) is known.

(3.14)    LEMMA. *Let $Y$ be $N(0, 1)$.*

(a) $P(Y \geq t) \leq (2\pi)^{-\frac{1}{2}} t^{-1} e^{-t^2/2}$, $t \geq 0$. *In particular, when $t \geq 1$, $P(Y \geq t) \leq e^{-t^2/2}$.*

(b) *Let $-\infty < a < b > \infty$, $m = \max\{a^2, b^2\}$.*

*Then* $P(a \leqq Y \leqq b) \geqq (2\pi)^{-\frac{1}{2}}(b - a)e^{-m/2}$. *In particular, when* $b - a \geqq (2\pi)^{\frac{1}{2}}$, $P(a \leqq Y \leqq b) \geqq e^{-m/2}$.

(3.15)　LEMMA. *Let* $Z_i = (X_i, Y_i), 1 \leqq i \leqq n$, *be independent such that* $E(X_i) = E(Y_i) = 0, E(X_i^2) = \sigma_i^2 < \infty, E(Y_i^2) = \tau_i^2 < \infty, 1 \leqq i \leqq n$. *Let* $s_n^2 = \sum_{i=1}^n \sigma_i^2$, $t_n^2 = \sum_{i=1}^n \tau_i^2, S_k = \sum_{i=1}^k X_i, T_k = \sum_{i=1}^k Y_i, 1 \leqq k \leqq n$. *Then for any* $x, y, -\infty < x < \infty, -\infty < y < \infty$,

$$P(\textstyle\bigcup_{k=1}^n [S_k \geqq xs_n, T_k \geqq yt_n]) \leqq 3P(S_n \geqq (x - 3^{\frac{1}{2}})s_n, T_n \geqq (y - 3^{\frac{1}{2}})t_n).$$

PROOF.　Let

$$
\begin{aligned}
A_k &= [S_k \geqq xs_n, T_k \geqq yt_n], & 1 \leqq k \leqq n, \\
B_k &= A_1^c \cap \cdots \cap A_{k-1}^c \cap A_k, & 2 \leqq k \leqq n, \\
&= A_1, & k = 1, \\
C_k &= [\textstyle\sum_{i=k+1}^n X_i \geqq -3^{\frac{1}{2}}s_n, \sum_{i=k+1}^n Y_i \geqq -3^{\frac{1}{2}}t_n], & 1 \leqq k \leqq n - 1, \\
&= \Omega, & k = n,
\end{aligned}
$$

and also, let $D = [S_n \geqq (x - 3^{\frac{1}{2}})s_n, T_n \geqq (y - 3^{\frac{1}{2}})t_n]$. Then

$$
\begin{aligned}
P(D) &\geqq P(D \cap (\textstyle\bigcup_{k=1}^n A_k)) = P(D \cap (\textstyle\bigcup_{k=1}^n B_k)) \\
&= \textstyle\sum_{k=1}^n P(D \cap B_k) \geqq \sum_{k=1}^n P(B_k \cap C_k) \\
&= \textstyle\sum_{k=1}^n P(B_k)P(C_k) \quad \text{by independence.}
\end{aligned}
$$

But

$$
\begin{aligned}
P(C_k) &= P(\textstyle\sum_{i=k+1}^n X_i \geqq -3^{\frac{1}{2}}s_n, \sum_{i=k+1}^n Y_i \geqq -3^{\frac{1}{2}}t_n) \\
&\geqq 1 - P(\textstyle\sum_{i=k+1}^n X_i < -3^{\frac{1}{2}}s_n) - P(\sum_{i=k+1}^n Y_i < -3^{\frac{1}{2}}t_n) \\
&\geqq 1 - \frac{\sum_{i=k+1}^n \sigma_i^2}{3s_n^2} - \frac{\sum_{i=k+1}^n \tau_i^2}{3t_n^2} \qquad \text{by Chebyshev} \\
&\geqq 1 - \tfrac{1}{3} - \tfrac{1}{3} = \tfrac{1}{3}, & 1 \leqq k \leqq n - 1.
\end{aligned}
$$

Of course,

$$P(C_n) = P(\Omega) > \tfrac{1}{3}.$$

Hence,

$$P(D) \geqq \tfrac{1}{3} \textstyle\sum_{k=1}^n P(B_k) = \tfrac{1}{3}P(\bigcup_{k=1}^n B_k) = \tfrac{1}{3}P(\bigcup_{k=1}^n A_k).$$

(3.16)　THE PROOF OF (2.14). Remember (2.12) and (2.13). First note that the convergence or divergence of the series in (2.16) does not really depend on $\gamma_0$ in the sense that given any other $\gamma, \gamma > 1$, define a subsequence $\{v_k\}$ by the procedure described in (2.12) but using $\gamma$ instead of $\gamma_0$, then

(3.17)　$\sum_k 1/(\log s^2(v_k))^{a^2-\delta}(\log t^2(v_k))^{b^2-\delta} < \infty$ if and only if (2.16) holds. This is true because if $\gamma > \gamma_0$, thed $v_k \geqq n_k, \forall k$; therefore, (2.16) $\Rightarrow$ (3.17); if $\gamma < \gamma_0$, then $\exists$ integer $l \ni \gamma^l \geqq \gamma_0$, hence, $v_{2lk} \geqq n_k, \forall k$; therefore, (2.16) $\Rightarrow$ $\sum_k 1/(\log s^2(v_{2lk}))^{a^2-\delta}(\log t^2(v_{2lk}))^{b^2-\delta} < \infty \Rightarrow$ (3.17) by the fact that both $s^2(n)$ and $t^2(n)$ are non-decreasing. Reversing the roles of $\gamma_0$ and $\gamma$, one sees (2.16 $\Leftrightarrow$ (3.17). The same argument can be applied to (2.17), too. Hence, we may take $\gamma_0$ to

be any value we like as long as $\gamma_0 > 1$. Without loss of generality, assume that $a > 0$, $b > 0$. Given $\varepsilon > 0$, $\varepsilon$ is small. Let $L(j) = (2s_j^2 \log_2 s_j^2)^{\frac{1}{2}}$, $M(j) = (2t_j^2 \log_2 t_j^2)^{\frac{1}{2}}, j \geqq 1$. Define the event $A_j = [S_j \geqq (a - \varepsilon)L(j), T_j \geqq (b - \varepsilon)M(j)]$. Let $C_k = \bigcup_{n_{k-1} \leqq j \leqq n_k - 1} A_j$. Then

$$P(C_k) \leqq P(\bigcup_{n_{k-1} \leqq j \leqq n_k - 1}[S_j \geqq (a - \varepsilon)L(n_{k-1}), T_j \geqq (b - \varepsilon)M(n_{k-1})]) \,.$$

Since

$$s^2(n_{k-1}) \leqq s^2(n_k - 1) \leqq \gamma_0 s^2(n_{k-1}) \,, \qquad t^2(n_{k-1}) \leqq t^2(n_k - 1) \leqq \gamma_0 t^2(n_{k-1}) \,,$$

we can choose $\gamma_0$ so close to 1 that $(a - \varepsilon)L(n_{k-1}) \geqq (a - 2\varepsilon)L(n_k - 1)$ and $(b - \varepsilon)M(n_{k-1}) \geqq (b - 2\varepsilon)M(n_k - 1)$.
Thus,

$$\begin{aligned} P(C_k) &\leqq P(\bigcup_{n_{k-1} \leqq j \leqq n_k - 1}[S_j \geqq (a - 2\varepsilon)L(n_k - 1), T_j \geqq (b - 2\varepsilon)M(n_k - 1)]) \\ &\leqq 3P(S(n_{k-1}) \geqq (a - 2\varepsilon)L(n_k - 1) - 3^{\frac{1}{2}}s(n_k - 1), \\ &\qquad T(n_k - 1) \geqq (a - 2\varepsilon)M(n_k - 1) - 3^{\frac{1}{2}}t(n_k - 1)) \end{aligned}$$

by (3.15).
If $k$ is sufficiently large, we have

$$(a - 2\varepsilon)L(n_k - 1) - 3^{\frac{1}{2}}s(n_k - 1) \geqq (a - 3\varepsilon)L(n_k - 1) \,,$$
$$(b - 2\varepsilon)M(n_k - 1) - 3^{\frac{1}{2}}t(n_k - 1) \geqq (b - 3\varepsilon)M(n_k - 1) \,.$$

Hence,

$$\begin{aligned} (3.18) \quad P(C_k) &\leqq 3P(S(n_{k-1}) \geqq (a - 3\varepsilon)L(n_k - 1), \\ &\qquad T(n_k - 1) \geqq (b - 3\varepsilon)M(n_k - 1)) \\ &\leqq 3[3\rho(n_k - 1) + P(X \geqq (a - 3\varepsilon)(2 \log_2 s^2(n_k - 1))^{\frac{1}{2}}, \\ &\qquad Y \geqq (b - 3\varepsilon)(2 \log_2 t^2(n_k - 1))^{\frac{1}{2}})] \end{aligned}$$

by (2.11), where $X$, $Y$ are two independent $N(0, 1)$. Now by (2.15),

$$\begin{aligned} \rho(n_k - 1) &\leqq K_0 \min \{1/(\log s^2(n_k - 1))^{1+\beta}, 1/(\log t^2(n_k - 1))^{1+\beta}\} \\ &\leqq K_0 \min \{1/(\log s^2(n_{k-1}))^{1+\beta}, 1/(\log t^2(n_{k-1}))^{1+\beta}\} \,, \end{aligned}$$

where $K_0$ is an absolute constant. Since for each $i$, $s^2(n_i) \geqq \gamma_0 s^2(n_{i-1})$ and $t^2(n_i) \geqq \gamma_0 t^2(n_{i-1})$, we have $s^2(n_{k-1}) \geqq \gamma_0^{\frac{1}{2}(k-1)}s^2(n_0)$ and $t^2(n_{k-1}) \geqq \gamma_0^{\frac{1}{2}(k-1)}t^2(n_0)$. Hence,

$$(3.19) \qquad\qquad \rho(n_{k-1}) \leqq K_1/[\tfrac{1}{2}(k - 1)]^{1+\beta} \,, \qquad\qquad \text{some } K_1 > 0 \,.$$

Next,

$$\begin{aligned} &P(X \geqq (a - 3\varepsilon)(2 \log_2 s^2(n_k - 1))^{\frac{1}{2}}, Y \geqq (b - 3\varepsilon)(2 \log_2 t^2(n_k - 1))^{\frac{1}{2}}) \\ (3.20) \quad &\leqq 1/(\log s^2(n_k - 1))^{(a-3\varepsilon)^2}(\log t^2(n_k - 1))^{(b-3\varepsilon)^2} \qquad \text{by (3.14a)} \\ &\leqq 1/(\log s^2(n_{k-1}))^{(a-3\varepsilon)^2}(\log t^2(n_{k-1}))^{(b-3\varepsilon)^2} \\ &\leqq 1/(\log s^2(n_{k-1}))^{a^2-\delta}(\log t^2(n_{k-1}))^{b^2-\delta} \end{aligned}$$

when $\varepsilon$ is sufficiently small. With (3.18), (3.19), (3.20) and (2.16), we obtain $\sum_k P(C_k) < \infty$.

By Borel–Cantelli,

$$P(C_k, \text{ i.o.}) = 0 \Rightarrow P(A_j, \text{ i.o.}) = 0 \Rightarrow (a, b) \notin H.$$

To prove the second part of the theorem, let $\| \ \|$ be the max norm; i.e. $\|(a, b)\| = \max(|a|, |b|)$. Given $\varepsilon > 0$. Clearly,

$$\left[\left\|\left(\frac{S(m_k) - S(m_{k-1})}{L(m_k)}, \frac{T(m_k) - T(m_{k-1})}{M(m_k)}\right) - (a, b)\right\| > 2\varepsilon\right]$$

$$\subset \left[\left\|\left(\frac{S(m_k)}{L(m_k)}, \frac{T(m_k)}{M(m_k)}\right) - (a, b)\right\| > \varepsilon\right] \cup \left[\left\|\left(\frac{S(m_{k-1})}{L(m_k)}, \frac{T(m_{k-1})}{M(m_k)}\right)\right\| > \varepsilon\right].$$

Therefore,

$$P\left(\left\|\left(\frac{S(m_k) - S(m_{k-1})}{L(m_k)}, \frac{T(m_k) - T(m_{k-1})}{M(m_k)}\right) - (a, b)\right\| > 2\varepsilon\right)$$

$$\leq P\left(\left\|\left(\frac{S(m_k)}{L(m_k)}, \frac{T(m_k)}{M(m_k)}\right) - (a, b)\right\| > \varepsilon\right)$$

$$+ P\left(\left\|\left(\frac{S(m_{k-1})}{L(m_k)}, \frac{T(m_{k-1})}{M(m_k)}\right)\right\| > \varepsilon\right).$$

Subtracting both sides from 1,

$$P\left(\left\|\left(\frac{S(m_k) - S(m_{k-1})}{L(m_k)}, \frac{T(m_k) - T(m_{k-1})}{M(m_k)}\right) - (a, b)\right\| \leq 2\varepsilon\right)$$

$$(3.21) \qquad \geq P\left(\left\|\left(\frac{S(m_k)}{L(m_k)}, \frac{T(m_k)}{M(m_k)}\right) - (a, b)\right\| \leq \varepsilon\right)$$

$$- P\left(\left\|\left(\frac{S(m_{k-1})}{L(m_k)}, \frac{T(m_{k-1})}{M(m_k)}\right)\right\| \geq \varepsilon\right).$$

But

$$(3.22) \qquad P(\|(S(m_{k-1})/L(m_k), T(m_{k-1})/M(m_k))\| \geq \varepsilon)$$

$$\leq P(|S(m_{k-1})/L(m_k)| \geq \varepsilon) + P(|T(m_{k-1})/M(m_k)| \geq \varepsilon).$$

By (2.11) and (3.14a),

$$(3.23) \qquad P(|S(m_{k-1})/L(m_k)| \geq \varepsilon)$$

$$\leq 2\rho(m_{k-1}) + \exp(-[\varepsilon^2 s^2(m_k) \log_2 s^2(m_k)]/s^2(m_{k-1})),$$

where

$$(3.24) \qquad \rho(m_{k-1}) \leq K_0 \min\{1/(\log s^2(m_{k-1}))^{1+\beta}, 1/(\log t^2(m_{k-1}))^{1+\beta}\}$$

$$\leq K_2/(k-1)^{1+\beta}, \qquad \text{some } K_2 > 0,$$

by the definition of $\{m_k\}$, and

$$(3.25) \qquad \exp(-[\varepsilon^2 s^2(m_k) \log_2 s^2(m_k)]/s^2(m_{k-1})) \leq 1/(\log s^2(m_k))^{r_0 \varepsilon^2}$$

$$\leq K_3/k^{r_0 \varepsilon^2}, \qquad \text{some } K_3 > 0.$$

If we choose $\gamma_0$ so large that $\gamma_0 \varepsilon^2 \geqq 2$, then by (3.23), (3.24) and (3.25),

(3.26)                    $\sum_k P(|S(m_{k-1})/L(m_k)| \geqq \varepsilon) < \infty$ .

Similarly,

(3.27)                    $\sum_k P(|T(m_{k-1})/M(m_k)| \geqq \varepsilon) < \infty$ .

With (3.22), (3.26) and (3.27), one sees that

(3.28)          $\sum_k P(\|(S(m_{k-1})/L(m_k), T(m_{k-1})/M(m_k))\| \geqq \varepsilon) < \infty$ .

Again, applying (2.11), (2.15) and (3.14b),

(3.29)     $P(\|(S(m_k)/L(m_k), T(m_k)/M(m_k)) - (a, b)\| \leqq \varepsilon)$

$$\geqq -4\rho(m_k) + 1/(\log s^2(m_k))^{a^2+2\varepsilon|a|+\varepsilon^2}(\log t^2(m_k))^{b^2+2\varepsilon|b|+\varepsilon^2}$$

$$\geqq -K_4/k^{1+\beta} + 1/(\log s^2(m_k))^{a^2+\delta}(\log t^2(m_k))^{b^2+\delta}$$

if $\varepsilon$ is sufficiently small.  By (2.17), (3.21), (3.28) and (3.29), we conclude that

$$\sum_k P\left(\left\|\left(\frac{S(m_k) - S(m_{k-1})}{L(m_k)} , \frac{T(m_k) - T(m_{k-1})}{M(m_k)}\right) - (a, b)\right\| \leqq 2\varepsilon\right) = \infty .$$

By independence and Borel–Cantelli,

(3.30)   $P\left(\left\|\left(\dfrac{S(m_k) - S(m_{k-1})}{L(m_k)} , \dfrac{T(m_k) - T(m_{k-1})}{M(m_k)}\right) - (a, b)\right\| \leqq 2\varepsilon, \text{ i.o.}\right) = 1$ .

Using (2.15) and a result of Petrov [11] in the one-dimensional case, we see that for almost all $\omega$, $\exists k_0(\omega) \ni \forall k \geqq k_0(\omega)$,

$$S(m_{k-1}, \omega) \leqq 2L(m_{k-1}) \leqq 2/\gamma_0^{\frac{1}{2}} L(m_k)$$

$$T(m_{k-1}, \omega) \leqq 2M(m_{k-1}) \leqq 2/\gamma_0^{\frac{1}{2}} M(m_k) .$$

Thus, if $\gamma_0$ is sufficiently large, then (3.30) implies

$$P(\|(S(m_k)/L(M_k), T(m_k)/M(m_k)) - (a, b)\| \leqq 3\varepsilon, \text{ i.o.}) = 1 .$$

Since $\varepsilon$ is arbitrary, $(a, b) \in H$. ☐

(3.31)     THE PROOF OF (2.19).  In this case, $Z_n$'s are normal; therefore, (2.15) is trivially satisfied.  Since $s_n^2 = t_n^2$, $n_k = m_k$.  Suppose (2.20a, b).  With (2.20a) and (2.17), we see that $(a, b) \in H$ if $a^2 + b^2 < d^2$.  With (2.20b) and (2.16), we see that $(a, b) \notin H$ if $a^2 + b^2 > d^2$.
Hence,

$$H = \{(a, b) \mid a^2 + b^2 \leqq d^2\} .$$

Conversely, suppose $H = \{(a, b) \mid a^2 + b^2 \leqq d^2\}$.  Then $(2.16) \Rightarrow (2.20a)$ and $(2.17) \Rightarrow (2.20b)$. ☐

(3.32)     THE PROOF OF (2.21).  By a multidimensional CLT, viz, Bhattacharya [1], we know

(3.33)                    $\rho(F_n, \Phi) \leqq K(\Lambda + \Delta + \Gamma)$ ,          for some $K > 0$ .

where

$$\Gamma = \max_{i \le n} \sigma_i / s_n \,,$$

$$\Delta = \sum_{i=1}^{n} \sup_{(a,b):a^2+b^2=1} \int_{[|aX_i+bY_i| \le s_n]} \left| \frac{aX_i + bY_i}{s_n} \right|^3 dP \,,$$

$$\Lambda = \sum_{i=1}^{n} \int_{[X_i^2+Y_i^2 \ge s_n^2]} \frac{X_i^2 + Y_i^2}{s_n^2} \, dP \,.$$

By (2.22) and Jensen's inequality,

$$E(X_i^2)(\log E(X_i^2))^{1+\beta} \le E[X_i^2(\log X_i^2)^{1+\beta}] \le \lambda \sigma_i^2 \,, \qquad \forall i \,.$$

Hence,

$$\sigma_i^2 \le \exp(\lambda^{1/(1+\beta)}) \,, \qquad \forall i \,,$$

(3.34)
$$\Gamma \le \frac{1}{s_n} \exp(\tfrac{1}{2}\lambda^{1/(1+\beta)}) \,.$$

Next,

$$\Lambda = \sum_{i=1}^{n} \int_{[X_i^2+Y_i^2 \ge s_n^2]} \frac{X_i^2 + Y_i^2}{s_n^2} \, dP$$

(3.35)
$$\le \frac{1}{s_n^2 (\log s_n^2)^{1+\beta}} \sum_{i=1}^{n} \int (X_i^2 + Y_i^2)(\log (X_i^2 + Y_i^2))^{1+\beta} \, dP$$

$$\le \lambda / (\log s_n^2)^{1+\beta}$$

by (2.22).

And,

$$\Delta = \sum_{i=1}^{n} \sup_{(a,b)} \int_{[aX_i+bY_i| \le s_n]} \left| \frac{aX_i + bY_i}{s_n} \right|^3 dP$$

(3.36)
$$\le \frac{1}{s_n^2 (\log s_n^2)^{1+\beta}} \sum_{i=1}^{n} \int (X_i^2 + Y_i^2)(\log (X_i^2 + Y_i^2))^{1+\beta} \, dP$$

$$\le \lambda / (\log s_n^2)^{1+\beta} \,.$$

Put (3.34), (3.35), (3.36) in (3.33), to see that

$$\rho(F_n, \Phi) = O(1/(\log s_n^2)^{1+\beta}) \,.$$

Note that $s_n^2 = t_n^2$, $n_k = m_k$, and $\log s^2(n_k)$ grows linearly. Hence,

$$\sum_k 1/(\log s^2(n_k))^{a^2+b^2} = \infty \quad \text{if} \quad a^2 + b^2 < 1 \,,$$
$$< \infty \quad \text{if} \quad a^2 + b^2 > 1 \,.$$

By Theorem (2.14), we conclude that the a.s. limit set is

$$\{(a, b) \,|\, a^2 + b^2 \le 1\} \,. \qquad \square$$

**4. An example.** We present here an example for Theorem (2.10). The idea is to construct $Z_n$, $n \ge 1$, so that when $\{S_n/(2s_n^2 \log_2 s_n^2)^{\frac{1}{2}}\}$ clusters around any point, $\{T_n/(2t_n^2 \log_2 t_n^2)^{\frac{1}{2}}\}$ will stay around 0; and vice versa. To achieve

this, let us take two independent copies of fair-coin-tossing variables; i.e. $V_1, V_2, \cdots, U_1, U_2, \cdots$ i.i.d. $\pm 1$ with probability $\frac{1}{2}$ each. We know

$$\sum_{i=1}^{n} V_i/(2n \log_2 n)^{\frac{1}{2}} \to 0 \quad \text{in probability.}$$

Therefore, $\exists \{n_k\} \ni$

(4.1) $$\sum_{i=1}^{n_k} V_i/(2n_k \log_2 n_k)^{\frac{1}{2}} \to 0 \quad \text{a.s.}$$

Define $Z_1, Z_2, \cdots$:

$$\begin{pmatrix} V_1 \\ 0 \end{pmatrix}, \begin{pmatrix} V_2 \\ 0 \end{pmatrix}, \cdots, \begin{pmatrix} V_{n_1} \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ U_1 \end{pmatrix}, \begin{pmatrix} 0 \\ U_2 \end{pmatrix}, \cdots, \begin{pmatrix} 0 \\ U_{n_1} \end{pmatrix},$$

$$\begin{pmatrix} V_{n_1+1} \\ 0 \end{pmatrix}, \cdots, \begin{pmatrix} V_{n_2} \\ 0 \end{pmatrix}, \cdots.$$

Write $Z_n = (X_n, Y_n)$, $S_n = \sum_{i=1}^{n} X_i$, $T_n = \sum_{i=1}^{n} Y_i$,

$$s_n^2 = E(S_n^2), \qquad t_n^2 = E(T_n^2), \qquad\qquad n \geq 1.$$

When $S_n/(2s_n^2 \log_2 s_n^2)^{\frac{1}{2}}$ visits around $a$, $-\infty < a < \infty$, $T_n/(2t_n^2 \log_2 t_n^2)^{\frac{1}{2}}$ is equal to $\sum_{i=1}^{n_k} U_i/(2n_k \log_2 n_k)^{\frac{1}{2}}$, for some $n_k$, which tends to zero almost surely. Similarly, if we look at those $n$'s when $T_n/(2t_n^2 \log_2 t_n^2)^{\frac{1}{2}}$ visits around $b$, $-\infty < b < \infty$, $S_n(2s_n^2 \log_2 s_n^2)^{\frac{1}{2}}$ is equal to $\sum_{i=1}^{n_k} V_i/(2n_k \log_2 n_k)^{\frac{1}{2}}$, for some $n_k$, which also tends to zero almost surely. Therefore, the a.s. limit set sits in the two axes. Applying the one-dimensional result, one argues that the a.s. limit set is exactly $\{(a, b) \mid ab = 0, |a| \leq 1, |b| \leq 1\}$. Formally,

(4.2)     Claim: the a.s. limit set $H$ of $\{(S_n/L(n), T_n/M(n))\}$ is $\{(a, b) \mid ab = 0, |a| \leq 1, |b| \leq 1\}$. To show this claim, let $\varepsilon > 0$ and let

$$A_k = [|S_k/L(k)| \geq \varepsilon, |T_k/M(k)| \geq \varepsilon],$$

$$B_k = \bigcup_{2n_{k-1} < i \leq 2n_{k-1}+m_k} A_i,$$

$$C_k = \bigcup_{2n_{k-1}+m_k < i \leq 2n_k} A_i.$$

where $m_k = n_k - n_{k-1}$ and $n_0 = 0$. Since $\{A_k, \text{ i.o.}\} \subset \{B_k, \text{ i.o.}\} \cup \{C_k, \text{ i.o.}\}$, we have

$$P(A_k, \text{ i.o.}) \leq P(B_k, \text{ i.o.}) + P(C_k, \text{ i.o.}).$$

But

$$P(B_k, \text{ i.o.}) \leq P(|T_{n_{k-1}}/M(n_{k-1})| \geq \varepsilon, \text{ i.o.}) = 0,$$

$$P(C_k, \text{ i.o.}) \leq P(|S_{n_k}/L(n_k)| \geq \varepsilon, \text{ i.o.}) = 0$$

by (4.1). Hence, $P(A_k, \text{ i.o.}) = 0$. Since $\varepsilon$ is arbitrary, $H \subset \{(a, b) \mid ab = 0\}$. Furthermore, all $X_n$ and $Y_n$ are bounded by 1 and $s_n^2 \to \infty$, $t_n^2 \to \infty$. Using Kolmogorov's result and Lemma (3.6), one sees

$$H \subset \{(a, b) \mid |a| \leq 1, |b| \leq 1\}.$$

Hence,

(4.3) $$H \subset \{(a, b) \mid ab = 0, |a| \leq 1, |b| \leq 1\}.$$

To show the other direction of inclusion, suppose

(4.4)     $(a_0, b_0) \notin H$ for some $(a_0, b_0)$ such that $a_0 b_0 = 0$, $|a_0| \leq 1$, $|b_0| \leq 1$.

Without loss of generality, assume $b_0 = 0$. Remember $H$ is closed. By (4.3) and (4.4), $\exists \epsilon > 0 \ni$ if $E = \{(x, y) \,|\, |x - a_0| \leq \epsilon\}$, then $E \cap H = \varnothing$. Applying Remark (3.5),

$$P((S_n/L(n), T_n/M(n)) \in E, \ \text{i.o.}) = 0 .$$

Hence,

$$P(|[S_n/L(n)] - a_0| \leq \epsilon, \ \text{i.o.}) = 0 ,$$

which contradicts the one-dimensional result. $\square$

(4.5)     REMARK. It is obvious that we can make $E(X_n{}^2) > 0$, $E(Y_n{}^2) > 0$ for all $n$ by adding "small" variables to each $Z_n$ and the result remains valid.

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