The Annals of Probability
1974, Vol. 2, No. 6, 1090–1107

SOME FUNCTIONAL LIMIT THEOREMS FOR DEPENDENT RANDOM VARIABLES[1]

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We prove theorems on weak convergence of random elements $X_m(\omega, t)$, $0 \leqq t \leqq 1$, to a Gaussian process. In Part I, these random elements are constructed on the basis of the linear means of a lacunary trigonometric series $\sum a_j \cos n_j \omega$. In Part II, the lacunarity hypothesis is dropped and replaced by the hypothesis of linear independence of the real numbers n_j.

1. Introduction. In this paper we generalize, in several directions, a theorem of Billingsley (cf. [1] page 410, Theorem 23.1) on weak convergence of random elements of $(D, \mathscr{D})$ to W, the Wiener process. Let D be the set of real-valued functions defined on $[0, 1]$ that are right-continuous and have left-hand limits; let $\mathscr{D}$ be the Skorohod σ-field in D (cf. [2] page 111); let C be the set of real-valued continuous functions defined on $[0, 1]$; and let $\mathscr{C}$ be the σ-field that gives the uniform topology in C. The technique we use to prove that a sequence X_m of random elements converges weakly (cf. [2] page 23) to the random element X (we shall write $X_m \Rightarrow X$) is a well-known one: namely, we prove that X_m is tight (cf. [2] page 37) in $(D, \mathscr{D})$—or in $(C, \mathscr{C})$—and that all the finite-dimensional distributions of X_m tend respectively to those of X (cf. [2] page 124, Theorem 15.1 and page 54, Theorem 8.1).

In Part I, the random elements $\{X_m(\omega, t): 0 \leqq t \leqq 1\}$ which appear in Theorems (1.2.1), (1.3.1), (1.4.1), (1.5.1), (1.5.2), (1.6.1) and (1.6.2) are constructed on the basis of the linear means of a lacunary trigonometric series $\sum_j a_j \cos n_j \omega$, $\omega \in [0, 2\pi]$. (See also (1.2.42).)

In Part II, the lacunarity hypothesis of the sequence $\{n_j\}$ has been dropped and replaced by the hypothesis of linear independence of the real numbers $\{\lambda_j\}$ (cf. Theorem (2.2.2).)

I would like to express my gratitude to my teacher, Professor P. Billingsley, who proposed the theme of this paper.

PART I. Lacunary sequences.

1. Let us consider a lacunary sequence of integers $\{n_j\}, j = 1, 2, \cdots$, $(n_{j+1}/n_j > q > 1$ for all $j)$ and a double array of real numbers $\{b_{m,j}\}$; $m, j = 1, 2, \cdots$.

Received March 13, 1973; revised January 15, 1974.

[1] The first section of this paper is a part of a doctoral dissertation as submitted to the Department of Mathematics, The University of Chicago, in March 1972.

AMS 1970 *subject classifications*. Primary 60F05; Secondary 60G15.

Key words and phrases. Functional limit theorems for dependent random variables, Gaussian processes, lacunary series, linearly independent numbers.

(1.1.1) THEOREM. *Let* $B_m = (\frac{1}{2}\sum_{j=1}^{\infty} b_{m,j}^2)^{\frac{1}{2}} < \infty$ *for each* $m = 1, 2, \cdots$, *and define* $\sigma_m(x) = \sum_{j=1}^{\infty} b_{m,j} \cos n_j x$ *for* $x \in [0, 2\pi]$. *Assume that the following conditions hold*:

(i) $B_m \rightarrow_{m\to\infty} \infty$;
(ii) $B_m^{-1} \sup_j |b_{m,j}| \rightarrow_{m\to\infty} 0$.

Then, for each set $E \subset [0, 2\pi]$ *of positive Lebesgue measure*,[2]

$$\lim_{m\to\infty} |E|^{-1} |\{x \in E: B_m^{-1}\sigma_m(x) \leqq \lambda\}| = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\lambda} \exp(-t^2/2)\, dt \quad \textit{for} \quad \lambda \in \mathbb{R}.$$

Theorem (1.1.1), which plays an essential role in the proof of subsequent theorems, is a generalization of a theorem of Salem and Zygmund (cf. [11] page 57, Theorem 8) and its proof runs along similar lines as those of this same theorem. Theorem (1.1.1) contains, as particular cases, several of the important theorems proved in [10] and [11]. On the other hand, Theorem (1.1.1) for the particular case $E = [0, 2\pi]$, is a special case of a theorem of Gaposhkin (cf. [5] page 423, Theorem 1).

(1.1.2) THEOREM. *Let* $\{a_j\}$, $j = 1, 2, \cdots$, *be a sequence of real numbers and define* $A_m = (\frac{1}{2}\sum_{j=1}^{m} a_j^2)^{\frac{1}{2}}$ *and* $B_m = (\frac{1}{2}\sum_{j=1}^{m} a_j^2(1 - j/(m+1))^2)^{\frac{1}{2}}$. *Let* $\sigma_m(x) = \sum_{j=1}^{m} a_j(1 - j/(m+1)) \cos n_j x$ *be the Cesàro-1 means of* $\sum_{j=1}^{\infty} a_j \cos n_j x$. *If* $A_m \rightarrow \infty$ *and* $A_m^{-1} \max_{1\leqq j\leqq m} |a_j| \rightarrow 0$, *then for each set* $E \subset [0, 2\pi]$ *of positive Lebesgue measure*,

$$\lim_{m\to\infty} |E|^{-1} |\{x \in E: B_m^{-1}\sigma_m(x) \leqq \lambda\}| = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\lambda} \exp(-t^2/2)\, dt \quad \textit{for} \quad \lambda \in \mathbb{R}.$$

For a proof, it is enough to observe that the double array $b_{m,j} = a_j(1 - j/(m+1))$ satisfies conditions (i) and (ii) of Theorem (1.1.1). This theorem is also valid, *mutatis mutandis*, for the Abel means and for the Cesàro-α means of $\sum a_j \cos n_j x$, provided $\alpha \geqq 0$ in the latter.

2. Let us consider a lacunary trigonometric series $S(\omega) = \sum_{j=1}^{\infty} a_j \cos n_j \omega$, where $n_{j+1}/n_j > q > 1$, $\omega \in [0, 2\pi]$ and $a_j \in \mathbb{R}$. Set $A_m^2 = \frac{1}{2}\sum_{j=1}^{m} a_j^2$. Let $\{\alpha_{m,j}\}$, $m, j = 1, 2, \cdots$, be a double array of nonnegative real numbers such that $\alpha_{m,j} = 0$ for $j > m$ (later on, in Section 6, this last assumption will be dropped). Let $\alpha_{m,j} \uparrow 1$ as $m \uparrow \infty$ and assume that $\alpha_{m,j} \downarrow$ as j increases (in Theorem (1.2.1), this last assumption, together with (ii), can be dropped if we assume $a_j = 0(1)$). The linear means of the series $S(\omega)$ will be denoted by $\sigma_m(\omega) = \sum_{j=1}^{m} \alpha_{m,j} a_j \cos n_j \omega$ for $m = 1, 2, \cdots$, and $\sigma_0(\omega) \equiv 0$. The following theorem, and also Theorems (1.2.42), (1.3.1) and (1.6.1), are generalizations of theorem of Billingsley (cf. [1] page 410, Theorem 23.1).

(1.2.1) THEOREM. *Let* $A_m^* = (\frac{1}{2}\sum_{j=1}^{m} \alpha_{m,j}^2 a_j^2)^{\frac{1}{2}}$, $A_0^* = 0$, *and define* $X_m(\omega, t) = \sigma_k(\omega)/A_m^*$ *if* $t \in [A_k^{*2}/A_m^{*2}, A_{k+1}^{*2}/A_m^{*2})$, *where* $k = 0, \cdots, m$, $t \in [0, 1]$ *and* $\omega \in [0, 2\pi]$. *Assume that the following conditions hold*:

[2] By $|B|$ we indicate the Lebesgue measure of the set B.

(i) $A_m \to_{m\to\infty} \infty$;

(ii) $A_m^{-1} \max_{1\leq j\leq m} |a_j| \to_{m\to\infty} 0$;

(iii) $\lim_{m\to\infty} (2\pi)^{-1} \int_0^{2\pi} X_m(\omega, t_1)X_m(\omega, t_2)\, d\omega = \Gamma(t_1, t_2) \in \mathbb{R}$ *for each* $t_1, t_2 \in [0, 1]$ *and* $\Gamma(t, t) = t$ *for* $t \in [0, 1]$.

Consider in $([0, 2\pi], \mathscr{B})^3$ *a probability measure* P *which is absolutely continuous with respect to Lebesgue measure. Then, there exists a Gaussian process* $\{X(\cdot, t): 0 \leqq t \leqq 1\}$ *such that*

$$X_m \Rightarrow X \qquad in \quad (D, \mathscr{D}) .$$

PROOF. For each m, $X_m: [0, 2\pi] \to D$ is a random element of D. We have

$$\int_0^{2\pi} |\sigma_m(\omega) - \sigma_n(\omega)|^4\, d\omega \leqq K_1(A_m^{*2} - A_n^{*2})^2 \tag{1.2.2}$$

for all q-lacunary series and all n and m, where K_1 is a constant depending only on q (cf. [13] page 215, Theorem 8.20). Let us assume that

$$P(B) = |B \cap B_0|/|B_0| \qquad \text{for all} \quad B \in \mathscr{B}, \tag{1.2.3}$$

B_0 being a fixed subset of $[0, 2\pi]$, $|B_0| > 0$. Hence, by Chebychev's inequality,

$$P(|\sigma_m - \sigma_n| \geqq \lambda) \leqq \lambda^{-3}E(|\sigma_m - \sigma_n|^3) \leqq Q\lambda^{-3}(A_m^{*2} - A_n^{*2})^{\frac{3}{2}} \tag{1.2.4}$$

for all $\lambda > 0$, where $Q = Q(q, |B_0|)$ is a constant for all $0 \leqq n \leqq m$ and for all q-lacunary series. If $\lambda > 0$ and $0 \leqq i \leqq j \leqq k$, the preceding formula implies

$$\begin{aligned} P(|\sigma_j - \sigma_i| \geqq \lambda, |\sigma_k - \sigma_j| \geqq \lambda) & \\ \leqq [P(|\sigma_j - \sigma_i| \geqq \lambda)]^{\frac{1}{2}}[P(|\sigma_k - \sigma_j| \geqq \lambda)]^{\frac{1}{2}} & \\ \leqq Q\lambda^{-3}[(A_j^{*2} - A_i^{*2})(A_k^{*2} - A_j^{*2})]^{\frac{3}{4}} & \\ \leqq Q\lambda^{-3}(A_k^{*2} - A_i^{*2})^{\frac{3}{2}} , & \end{aligned} \tag{1.2.5}$$

because the A_m^{*2} are increasing.

We now prove, following the proof of a theorem of Billingsley (cf. [2] page 89, Theorem 12.1), that for all $\lambda > 0$, $0 \leqq i_1 \leqq i_2$ and for all q-lacunary series, there exists a constant $K \geqq 1$ such that

$$P(\max_{i_1\leqq k\leqq i_2} \min [|\sigma_{i_2} - \sigma_k|, |\sigma_k - \sigma_{i_1}|] \geqq \lambda) \leqq Q\lambda^{-3}K(A_{i_2}^{*2} - A_{i_1}^{*2})^{\frac{3}{2}} , \tag{1.2.6}$$

where Q is the constant appearing in (1.2.5). Let us proceed by induction on $m = i_2 - i_1$. Denote by $M(\omega)$ the number

$$M(\omega) = \max_{i_1\leqq k\leqq i_2} \min [|\sigma_{i_2}(\omega) - \sigma_k(\omega)|, |\sigma_k(\omega) - \sigma_{i_1}(\omega)|] .$$

For $m = 0$ and $m = 1$ the result is immediate. If $m = 2$, by (1.2.5) we have

$$P(M(\omega) \geqq \lambda) = P(|\sigma_{i_2} - \sigma_{i_1+1}| \geqq \lambda, |\sigma_{i_1+1} - \sigma_{i_1}| \geqq \lambda) \leqq Q\lambda^{-3}(A_{i_2}^{*2} - A_{i_1}^{*2})^{\frac{3}{2}} .$$

Therefore, if $m = 2$, the result holds for every $K \geqq 1$. Suppose now that (1.2.6) holds for $i_2 - i_1 \leqq m - 1$ (and $m \geqq 3$) and let us prove it for m. Define for $1 \leqq h \leqq m$,

$$H_1(\omega) = \max_{i_1\leqq k\leqq i_1+(h-1)} \min [|\sigma_{i_1+m} - \sigma_k|, |\sigma_k - \sigma_{i_1}|]$$

[3] $\mathscr{B}$ denotes the Borel sets.

and

$$H_2(\omega) = \max_{i_1+h\leqq k\leqq i_1+m} \min\,[|\sigma_{i_1+m} - \sigma_k|, |\sigma_k - \sigma_{i_1}|]\,.$$

Then, $M(\omega) = \max\,(H_1(\omega), H_2(\omega))$ and so

$$P(M(\omega) \geqq \lambda) \leqq P(H_1(\omega) \geqq \lambda) + P(H_2(\omega) \geqq \lambda)\,. \tag{1.2.7}$$

Using the inequality

$$\begin{aligned} H_1(\omega) &= \max_{i_1\leqq k\leqq i_1+(h-1)} \min\,[|\sigma_{i_1+h-1} - \sigma_k|, |\sigma_k - \sigma_{i_1}|] \\ &\quad + \min\,(|\sigma_{i_1+m} - \sigma_{i_1+h-1}|, |\sigma_{i_1+h-1} - \sigma_{i_1}|) + |\sigma_{i_1+m} - \sigma_{i_1}| \\ &\equiv C_1(\omega) + C_2(\omega) + C_3(\omega)\,, \end{aligned}$$

we have

$$\begin{aligned} &P(H_1(\omega) \geqq \lambda) \\ &\quad\leqq P(C_1(\omega) + C_2(\omega) + C_3(\omega) \geqq \lambda) \\ &\quad\leqq \min_{\lambda_1,\lambda_2,\lambda_3>0,\lambda_1+\lambda_2+\lambda_3=\lambda} [P(C_1 \geqq \lambda_1) + P(C_2 \geqq \lambda_2) + P(C_3 \geqq \lambda_3)]\,. \end{aligned} \tag{1.2.8}$$

Since $m \geqq 3$ and the A_n^{*2} are increasing, we can choose $i \leqq h \leqq m$ such that

$$(A_{i_1+h-1}^{*2} - A_{i_1}^{*2}) \leqq \tfrac{1}{2}(A_{i_2}^{*2} - A_{i_1}^{*2})\,, \tag{1.2.9}$$

$$(A_{i_2}^{*2} - A_{i_1+h}^{*2}) \leqq \tfrac{1}{2}(A_{i_2}^{*2} - A_{i_1}^{*2})\,. \tag{1.2.10}$$

Since $h - 1 \leqq m - 1$, it follows from the inductive hypothesis and (1.2.9) that

$$P(C_1 \geqq \lambda_1) \leqq Q\lambda_1^{-3}K(A_{i_1+h-1}^{*2} - A_{i_1}^{*2})^{\frac{3}{2}} \leqq QK\lambda_1^{-3}(A_{i_2}^{*2} - A_{i_1}^{*2})^{\frac{3}{2}} \cdot 2^{-\frac{3}{2}}\,. \tag{1.2.11}$$

From (1.2.5) and (1.2.4),

$$\begin{aligned} P(C_2 \geqq \lambda_2) &= P(|\sigma_{i_2} - \sigma_{i_1+h-1}| \geqq \lambda_2, |\sigma_{i_1+h-1} - \sigma_{i_1}| \geqq \lambda_2) \\ &\leqq Q\lambda_2^{-3}(A_{i_2}^{*2} - A_{i_1}^{*2})^{\frac{3}{2}} \end{aligned} \tag{1.2.12}$$

and

$$P(C_3 \geqq \lambda_3) \leqq Q\lambda_3^{-3}(A_{i_2}^{*2} - A_{i_1}^{*2})^{\frac{3}{2}}\,. \tag{1.2.13}$$

Combining (1.2.8), (1.2.11) and the two preceding formulas, we obtain

$$\begin{aligned} P(H_1(\omega) \geqq \lambda) &\leqq Q(A_{i_2}^{*2} - A_{i_1}^{*2})^{\frac{3}{2}} \\ &\quad\times \inf_{\lambda_1,\lambda_2,\lambda_3>0,\lambda_1+\lambda_2+\lambda_3=\lambda} (\lambda_1^{-3}K2^{-\frac{3}{2}} + \lambda_2^{-3} + \lambda_3^{-3})\,. \end{aligned} \tag{1.2.14}$$

By the method of Lagrange multipliers, we conclude that the above infimum is attained at $\lambda_1 = 2^{-\frac{3}{8}}\lambda K^{\frac{1}{4}}(2 + 2^{-\frac{3}{8}}K^{\frac{1}{4}})^{-1}$ and $\lambda_2 = \lambda_3 = \lambda(2 + 2^{-\frac{3}{8}}K^{\frac{1}{4}})^{-1}$. Consequently,

$$P(H_1(\omega) \geqq \lambda) \leqq Q\lambda^{-3}(A_{i_2}^{*2} - A_{i_1}^{*2})^{\frac{3}{2}}(2 + 2^{-\frac{3}{8}}K^{\frac{1}{4}})^4\,. \tag{1.2.15}$$

For $K \geqq K^* = [2(2^{-\frac{1}{4}} - 2^{-\frac{3}{8}})^{-1}]^4$,

$$(2 + 2^{-\frac{3}{8}}K^{\frac{1}{4}})^4 \leqq \tfrac{1}{2}K\,. \tag{1.2.16}$$

We shall prove (1.2.6) for $K = K^*$. By the two preceding formulas,

$$P(H_1(\omega) \geqq \lambda) \leqq \tfrac{1}{2}QK\lambda^{-3}(A_{i_2}^{*2} - A_{i_1}^{*2})^{\frac{3}{2}}\,. \tag{1.2.17}$$

By using the inequality

$$\begin{aligned} H_2(\omega) &\leqq \max_{i_1+h\leqq k\leqq i_2} \min(|\sigma_k - \sigma_{i_1+h}|, |\sigma_k - \sigma_{i_2}|) \\ &\qquad + \min(|\sigma_{i_2} - \sigma_{i_1+h}|, |\sigma_{i_1+h} - \sigma_{i_1}|) + |\sigma_{i_2} - \sigma_{i_1}| \\ &\equiv B_1(\omega) + B_2(\omega) + C_3(\omega) \end{aligned}$$

and an argument similar to the one leading to formula (1.2.17), we find

$$P(H_2 \geqq \lambda) \leqq \tfrac{1}{2} QK\lambda^{-3}(A_{i_2}^{*2} - A_{i_1}^{*2})^{\frac{3}{2}} . \tag{1.2.18}$$

Finally, by (1.2.17), (1.2.18) and (1.2.7), formula (1.2.6) is established.

Next, we show that there is a constant $K_2 = K_2(q, |B_0|)$ for which

$$P(\max_{i_1\leqq k\leqq i_2} |\sigma_k - \sigma_{i_1}| \geqq \lambda) \leqq K_2\lambda^{-3}(A_{i_2}^{*2} - A_{i_1}^{*2})^{\frac{3}{2}} \tag{1.2.19}$$

for all $\lambda > 0$, $0 \leqq i_1 \leqq i_2$, and for all q-lacunary series. The left-hand side of the previous formula is bounded by

$$P(|\sigma_{i_2} - \sigma_{i_1}| \geqq \tfrac{1}{2}\lambda) + P(\max_{i_1\leqq k\leqq i_2} \min[|\sigma_k - \sigma_{i_2}|, |\sigma_k - \sigma_{i_1}|] \geqq \tfrac{1}{2}\lambda) ;$$

and so, combining (1.2.4) and (1.2.6), we obtain (1.2.19) with $K_2 = 2^3Q(1 + K)$. On the other hand, it follows from (i), (ii) and the assumptions on the $\alpha_{m,j}$ that

$$\text{(i*)}\quad A_m{}^* \to_{m\to\infty} \infty , \qquad \text{(ii*)}\quad \max_{1\leqq j\leqq m} (A_m{}^*)^{-1}|a_j\,\alpha_{m,j}| \to_{m\to\infty} 0 . \tag{1.2.20}$$

We shall need this formula once we prove the following two lemmas, which show that $\{X_m\}$ is tight in $(D, \mathscr{D})$.

LEMMA 1. *For $\varepsilon > 0$ and $\eta > 0$, there exists an $N_0 = N_0(\varepsilon, \eta)$ such that for all $m \geqq N_0$ there exist $r = r(m)$, $0 < t_1 < \cdots < t_r = 1$, $\delta = \delta(\varepsilon, \eta, q, |B_0|)$, $0 < \delta < 1$, and $(t_k - t_{k-1}) \geqq \delta$ such that*

$$\sum_{k=1}^{r} P\left(\sup_{t_{k-1}\leqq t\leqq t_k} |X_m(\omega, t) - X_m(\omega, t_{k-1})| \geqq \frac{\varepsilon}{3}\right) < \eta . \tag{1.2.21}$$

PROOF. For fixed m, suppose $1 \leqq i_1 < i_2 \leqq m$, and let $t_1 = A_{i_1}^{*2}/A_m{}^{*2}$ and $t_2 = A_{i_2}^{*2}/A_m{}^{*2}$. By formula (1.2.19),

$$\begin{aligned} &P\left(\sup_{t_1\leqq t\leqq t_2} |X_m(\omega, t) - X_m(\omega, t_1)| \geqq \frac{\varepsilon}{3}\right) \\ &\quad = P\left(\max_{i_1\leqq k\leqq i_2} |\sigma_k(\omega) - \sigma_{i_1}(\omega)| \geqq \frac{\varepsilon}{3} A_m{}^*\right) \\ &\quad \leqq \left(\frac{3}{\varepsilon}\right)^3 K_2(t_2 - t_1)^{\frac{3}{2}} . \end{aligned} \tag{1.2.22}$$

Taking the points t_k of the form $A_{i_k}^{*2}/A_m{}^{*2}$, by (iii) we have

$$\max_{1\leqq k\leqq m} |t_k - t_{k-1}| \to_{m\to\infty} 0 .$$

Therefore, if we choose $0 < \delta < 1$ so that

$$\delta < 3^{-9}\varepsilon^6\eta^2K_2{}^{-2} , \tag{1.2.23}$$

there exists an $N_0 = N_0(\delta)$ such that

$$t_k - t_{k-1} \leqq 3\delta \tag{1.2.24}$$

for all $m \geqq N_0$ and $k = 1, \cdots, m$. For each $m \geqq N_0$, there exist $r = r(m)$ and $0 = t_0 < t_1 < \cdots < t_r = 1$ for which

$$\delta \leqq t_k - t_{k-1} \quad \text{whenever} \quad k = 1, \cdots, r. \tag{1.2.25}$$

Since $r\delta \leqq 1$, by combining the last three formulas, we see that the left-hand side of (1.2.21) is bounded by

$$3^3\varepsilon^{-3}K_2 \sum_{k=1}^{r} (t_k - t_{k-1})^{\frac{3}{2}} \leqq \eta ,$$

and so Lemma 1 is established.

LEMMA 2. *The sequence $\{X_m\}$ is tight in $(D, \mathscr{D})$.*

PROOF. If $x \in D$, $0 < \delta \leqq 1$, and $\omega_x(\delta) = \sup_{|s-t|<\delta} |x(s) - x(t)|$, then

$$\begin{aligned} &P_m(x \in D\colon \omega_x(\delta) \geqq \varepsilon) \\ &\leqq \sum_{k=1}^{r} P\left(\omega\colon \sup_{t_{k-1} \leqq t \leqq t_k} |X_m(\omega, t) - X_m(\omega, t_{k-1})| \geqq \frac{\varepsilon}{3}\right) \end{aligned} \tag{1.2.26}$$

(cf. [2] page 56 and page 128). By Lemma 1, the right-hand side of (1.2.26) is bounded by η for $m \geqq N_0(\varepsilon, \eta)$ and so

$$P_m(x \in D\colon \omega_x(\delta) \geqq \varepsilon) \leqq \eta \quad \text{for} \quad m \geqq N_0(\varepsilon, \eta) . \tag{1.2.27}$$

Moreover, by the definition of the random elements X_m, if $a > 0$,

$$P_m(x \in D\colon |x(0)| > a) \leqq \eta \quad \text{for} \quad m \geqq 1 . \tag{1.2.28}$$

The two previous formulas imply that the sequence $\{P_m\}$ is tight (cf. [2] page 127, Theorem 15.5). Consequently, Lemma 2 is established. We shall now prove the convergence of the finite-dimensional distributions of $\{X_m\}$.

If we define $b_{m,j} = a_j \alpha_{m,j}$ for $j \leqq m$ and $b_{m,j} = 0$ for $j > m$, (1.2.20) implies that $b_{m,j}$ is a double array of real numbers satisfying conditions (i) and (ii) of Theorem (1.1.1). Since

$$P(X_m(\cdot, t) \leqq x) = P\left(\frac{\sigma_k(\omega)}{A_k^*} \leqq x \frac{A_m^*}{A_k^*}\right) \quad \text{and} \quad \frac{A_k^{*2}}{A_m^{*2}} \rightarrow_{m\to\infty} t ,$$

Theorem (1.1.1) implies that

$$P(X_m(\cdot, t) \leqq x) \rightarrow_{m\to\infty} N(0, t)[x]^{4} \tag{1.2.29}$$

for each $t \in [0, 1]$.

Let us write $\Gamma_m(t_1, t_2) = (1/2\pi) \int_0^{2\pi} X_m(\omega, t_1)X_m(\omega, t_2)\, d\omega$ for $t_1, t_2 \in [0, 1]$. For fixed m, $\{\Gamma_m(t_1, t_2)\}_{t_1, t_2 \in [0,1]}$ is the covariance matrix of the process $\{X_m(\omega, t)\colon 0 \leqq t \leqq 1\}$ with respect to the normalized Lebesgue measure in $[0, 2\pi]$, i.e.,

[4] $N(E, V)[x]$ denotes the normal distribution with expected value E and variance V at the point $x \in \mathbb{R}$.

$\Gamma_m(t_1, t_2) = \mathrm{Cov}\,(X_m(\cdot, t_1), X_m(\cdot, t_2))$. Hence, $\{\Gamma_m(t_1, t_2)\}$ and its limit $\Gamma(t_1, t_2)$ are positive definite matrices. Consequently, by Kolmogorov's theorem, there is a Gaussian process $\{X(\cdot, t)\colon 0 \leqq t \leqq 1\}$, $X(\cdot, t)\colon (\Omega^*, \mathscr{B}^*, P^*) \to (\mathbb{R}, \mathscr{B})$, such that $E(X(\cdot, t)) = 0$ and $\mathrm{Cov}\,(X(\cdot, t_1), X(\cdot, t_2)) = \Gamma(t_1, t_2)$. Furthermore, $V(X(\cdot, t)) = \Gamma(t, t) = \lim_{m\to\infty} A_k^{*2}/A_m^{*2} = t$. Thus, we can rewrite (1.2.29) as follows:

$$X_m(\cdot, t) \Rightarrow X(\cdot, t)\,. \tag{1.2.30}$$

Now we are going to see that this formula is also valid for p points.

LEMMA 3. *For* $t_1, \cdots, t_p \in [0, 1]$,

$$(X_m(\cdot, t_1), \cdots, X_m(\cdot, t_p)) \Rightarrow_{m\to\infty} (X(\cdot, t_1), \cdots, X(\cdot, t_p))\,. \tag{1.2.31}$$

PROOF. We may consider $t_1 < \cdots < t_p$. By the Cramér–Wold theorem (cf. [4]), it suffices to prove that

$$\textstyle\sum_{i=1}^p \xi_i X_m(\cdot, t_i) \Rightarrow_{m\to\infty} \sum_{i=1}^p \xi_i X(\cdot, t_i) \tag{1.2.32}$$

for $\xi_1, \cdots, \xi_p \in \mathbb{R}$. Since $X(\cdot, t)$ is a Gaussian process, we have

$$\textstyle\sum_{i=1}^p \xi_i X_m(\cdot, t_i) \Rightarrow_{m\to\infty} N(0, V[\sum_{i=1}^p \xi_i X(\cdot, t_i)])\,. \tag{1.2.33}$$

If $X_m(\omega, t_i) = A_m^{*-1}\sigma_{k_i}(\omega)$ for $i = 1, \cdots, p$, we can write the left-hand side of the previous formula in the form

$$\textstyle\sum_{i=1}^p \xi_i X_m(\omega, t_i) = \frac{1}{A_m^*} \sum_{j=1}^m b_{m,j} \cos n_j \omega\,, \tag{1.2.34}$$

with the definition

$$b_{m,j} = a_j \textstyle\sum_{i=1}^p \xi_i \alpha_{k_i,j} \quad \text{for } j \leqq m\,, \qquad b_{m,j} = 0 \quad \text{for } j > m\,. \tag{1.2.35}$$

Write $B_m = (\frac{1}{2}\sum_{j=1}^m b_{m,j}^2)^{\frac{1}{2}}$. Since for $t_i < t_h$,

$$\Gamma_m(t_i, t_h) = \frac{1}{2A_m^{*2}} \textstyle\sum_{j=1}^{k_i} a_j^2 \alpha_{k_i,j}\alpha_{k_h,j}\,,$$

$$\frac{B_m^2}{A_m^{*2}} = \textstyle\sum_{i=1}^p \xi_i^2 \frac{A_{k_i}^{*2}}{A_m^{*2}} + 2\sum_{1\leq i<h\leq p} \xi_i \xi_h \Gamma_m(t_i, t_h)\,. \tag{1.2.36}$$

Thus, by (iii), it follows that

$$\begin{aligned}\lim_{m\to\infty} \frac{B_m^2}{A_m^{*2}} &= \textstyle\sum_{i=1}^p \xi_i^2 t_i + 2\sum_{1\leq i<h\leq p} \xi_i\xi_h \Gamma(t_i, t_h) \\ &= V(\textstyle\sum_{i=1}^p \xi_i X(\cdot, t_i))\end{aligned} \tag{1.2.37}$$

and so $B_m \to_{m\to\infty} \infty$. Moreover, by (1.2.20) and (1.2.36)

$$\max_{1\leq j\leq m} \frac{b_{m,j}^2}{B_m^2} \leqq 2^p(\textstyle\sum_{i=1}^p \xi_i^2)\frac{A_m^{*2}}{B_m^2} \max_{1\leq j\leq m} \frac{a_j^2\alpha_{m,j}^2}{A_m^{*2}} \to_{m\to\infty} 0\,.$$

Therefore, the array $b_{m,j}$, defined in (1.2.35), satisfies conditions (i) and (ii) of

Theorem (1.1.1). Combining (1.2.34), Theorem (1.1.1) and (1.2.37), we obtain (1.2.33) and so Lemma 3 is established.

The process $\{X(\cdot, t): 0 \leqq t \leqq 1\}$ may be chosen so that $X \in D$. In fact, let us consider two points t and $t + h$ in the interval $[0, 1)$, $h > 0$. If $\varepsilon > 0$, by (1.2.32) and (1.2.4),

$$(1.2.38) \qquad P^*(|X(\cdot, t+h) - X(\cdot, t)| \geqq \varepsilon) \leqq \frac{Q}{\varepsilon^3} h^{\frac{3}{2}} \rightarrow_{h \downarrow 0} 0 .$$

Furthermore, by (1.2.32) and (1.2.5),

$$(1.2.39) \qquad P^*(|X(\cdot, t_2) - X(\cdot, t)| \geqq \lambda, |X(\cdot, t) - X(\cdot, t_1)| \geqq \lambda) \leqq \frac{Q}{\lambda^3} (t_2 - t_1)^{\frac{3}{2}}$$

for $0 \leqq t_1 \leqq t \leqq t_2$. From the two formulas above, it follows that the Gaussian process $\{X(\cdot, t): 0 \leqq t \leqq 1\}$ may be chosen such that X is a random element of D (cf. [2] page 130, Theorem 15.7).

From Lemmas 2 and 3 we conclude that, under the assumption (1.2.3),

$$(1.2.40) \qquad X_m \Rightarrow X \quad \text{in} \quad (D, \mathscr{D}) .$$

Therefore,

$$(1.2.41) \qquad |B_0 \cap \{X_m \in A\}| \rightarrow_{m\rightarrow\infty} P^*(X \in A)|B_0|$$

for every $B_0 \in \mathscr{B}_0$ and $A \in \mathscr{D}$. Hence, for a fixed $A \in \mathscr{D}$, the sequence $\{X_m^{-1}(A)\}$ is strongly mixing with density $P^*(X \in A)$ (cf. [9]). Since P is absolutely continuous with respect to Lebesgue measure, formula (1.2.41) implies that $P(X_m \in A) \rightarrow_{m\rightarrow\infty} P^*(X \in A)$ (cf. [9] page 216, Theorem 1). Thus, Theorem (1.2.1) is established. □

The previous theorem admits (with an analogous proof) the following immediate generalization:

(1.2.42) THEOREM. *Let us consider a lacunary sequence of integers* $\{n_j\}$, $j = 1, 2, \cdots$, *and a double array of real numbers* $\{b_{m,j}\}$, $m, j = 1, 2, \cdots$. *Let* $B_m = (\frac{1}{2} \sum_{j=1}^{m} b_{m,j}^2)^{\frac{1}{2}}$, $B_0 = 0$, *and define* $\sigma_m(\omega) = \sum_{j=1}^{m} b_{m,j} \cos n_j \omega$ *for* $m = 1, 2, \cdots$; $\sigma_0(\omega) \equiv 0$, *where* $\omega \in [0, 2\pi]$. *Let* $X_m(\omega, t) = \sigma_k(\omega)/B_m$ *if* $t \in [B_k^2/B_m^2, B_{k+1}^2/B_m^2)$, *where* $k = 0, 1, \cdots, m$; $t \in [0, 1]$ *and* $\omega \in [0, 2\pi]$. *Assume that the following conditions hold*:

(a) $b_{m,j} = 0$ *for* $j > m$;
(b) $B_m \rightarrow_{m\rightarrow\infty} \infty$;
(c) $B_m^{-1} \max_{1 \leqq j \leqq m} |b_{m,j}| \rightarrow_{m\rightarrow\infty} 0$;
(d) *sign* $b_{m,j}$ = *constant for fixed* j;
(e) $b_{m,j}^2$ *increases with* m, *for fixed* j;
(f) $\lim_{m\rightarrow\infty} (2\pi)^{-1} \int_0^{2\pi} X_m(\omega, t_1) X_m(\omega, t_2)\, d\omega = \Gamma(t_1, t_2) \in \mathbb{R}$, *for each* t_1; $t_2 \in [0, 1]$ *and* $\Gamma(t, t) = t$ *for* $t \in [0, 1]$.

Consider in $([0, 2\pi], \mathscr{B})$ *a probability measure* P *which is absolutely continuous with* *respect to Lebesgue measure. Then, there exists a Gaussian process* $\{X(\cdot, t): 0 \leqq t \leqq 1\}$ *such that*

$$X_m \Rightarrow X \quad \text{in} \quad (D, \mathscr{D}) .$$

3. Since the Wiener process in $(D, \mathscr{D})$, $W \equiv \{W_t(\cdot): 0 \leqq t \leqq 1\}$, is characterized by the properties $E(W_t) = 0$ and $\operatorname{Cov}(W_t, W_u) = \min\{t, u\}$, an immediate corollary of Theorem (1.2.1) is the following

(1.3.1) THEOREM. *Under the conditions of Theorem* (1.2.1) *and the additional condition* $\Gamma(t_1, t_2) = \min\{t_1, t_2\}$, *we have* $X_m \Rightarrow W$ *in* $(D, \mathscr{D})$.

4. The next theorem gives a closed expression for the limit process appearing in Theorem (1.2.1) for the case of Cesàro-α means, $\alpha > 0$, and constant coefficients. Observe that these limit processes $X_\alpha(\cdot, t)$ are such that $X_\alpha(\cdot, t) \to_{\alpha \downarrow 0} W_t(\cdot)$ (cf. Theorem (1.5.1)).

Observe also that for $\alpha \geqq 0$, if $\sigma_m(\omega) = \sum_{j=1}^m \alpha_{m,j} a_j \cos n_j \omega$ are the Cesàro-α means of $S(\omega) = \sum_{j=1}^\infty a_j \cos n_j \omega$, then the double array $\alpha_{m,j} = [(1 + \alpha/(m - j + 1)) \cdots (1 + \alpha/m)]^{-1}$ satisfies the conditions required in Theorem (1.2.1), excluding (iii).

(1.4.1) THEOREM. *Using the notation of Theorem* (1.2.1), *let* $\sigma_m(\omega)$ *be the Cesàro-α means of* $S(\omega)$ *with* $\alpha > 0$ *and* $|a_j| \equiv M \neq 0$. *Then,*

$$X(\cdot, t) = X_\alpha(\cdot, t) = \alpha(1 + 2\alpha)^{\frac{1}{2}} \frac{1}{t^\alpha} \int_0^t W_x(\cdot)(t - x)^{\alpha-1}\, dx$$

for $t > 0$, $X_\alpha(\cdot, 0) \equiv 0$, *and so* $X_\alpha(\cdot, t) \to_{\alpha \downarrow 0} W_t(\cdot)$.

PROOF. By definition, $X_\alpha(\cdot, t)$ is continuous for $t > 0$. Furthermore, since $W_x(\cdot)$ is continuous and $W_0(\cdot) \equiv 0$, $|X_\alpha(\cdot, t)| \to_{t \downarrow 0} 0$.

Hence, $X_\alpha(\cdot, t)$ is continuous for $t \in [0, 1]$. Consequently, X_α is a random element of C because $X_\alpha(\cdot, t)$ is Borel measurable for each t. By definition, $\{X_\alpha(\cdot, t): 0 < t \leqq 1\}$ is a Gaussian process. It follows from the continuity of the Wiener paths that

$$\frac{1}{\Gamma(\alpha)} \int_0^t W_x(\cdot)(t - x)^{\alpha-1}\, dx \to_{\alpha \downarrow 0} W_t(\cdot) ,$$

where Γ denotes Euler's gamma function. Consequently,

$$\lim_{\alpha \downarrow 0} X_\alpha(\cdot, t) = \lim_{\alpha \downarrow 0} \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha)} \int_0^t W_x(\cdot)(t - x)^{\alpha-1}\, dx = W_t(\cdot) .$$

Since $E(W_x(\cdot)) = 0$ and $E(|W_x(\cdot)|) = ((2/\pi)x)^{\frac{1}{2}}$, Fubini's theorem implies that

$$E(X_\alpha(\cdot, t)) = 0 \qquad \text{for all } t . \tag{1.4.2}$$

Since $E(|W_x(\cdot)||W_y(\cdot)|) \leqq (xy)^{\frac{1}{2}}$ and $E(W_x(\cdot)W_y(\cdot)) = \min\{x, y\}$, Fubini's theorem implies again that

$$E(X_\alpha(\cdot, t_1)X_\alpha(\cdot, t_2)) = (1 + 2\alpha) \frac{1}{t_1^\alpha t_2^\alpha} \int_0^{t_1} x^\alpha[(t_2 - t_1) + x]^\alpha\, dx \tag{1.4.3}$$

for $t_1 \leqq t_2$, and, in particular,

$$E(X_\alpha^2(\cdot, t)) = t\,. \tag{1.4.4}$$

By definition,

$$\sigma_m(\omega) = \sum_{j=1}^m \frac{H_{m-j}^\alpha}{H_m{}^\alpha} a_j \cos n_j \omega\,,$$

where $H_m{}^\alpha = \binom{m+\alpha}{m} = (1/m!)(\alpha+1)\cdots(\alpha+m)$ (cf. [13] page 77, (1.15)). Since $H_m{}^\alpha = (m^\alpha/(\Gamma(\alpha+1)))(1+O(1/m))$ (cf. [13] (1.18)), it follows that

$$\alpha_{m,j} = \frac{H_{m-j}^\alpha}{H_m{}^\alpha} = \left(1-\frac{j}{m}\right)^\alpha + O\left(\frac{1}{m}\right). \tag{1.4.5}$$

Therefore,

$$2A_m{}^{*2} \simeq M^2 \sum_{j=0}^m \left(1-\frac{j}{m}\right)^{2\alpha} \tag{1.4.6}$$

and similarly

$$\sum_{j=1}^k a_j^2 \alpha_{k,j}\alpha_{l,j} \simeq M^2 \sum_{j=0}^k \left(1-\frac{j}{k}\right)^\alpha \left(1-\frac{j}{l}\right)^\alpha, \tag{1.4.7}$$

where k and l are such that $X_m(\omega, t_1) = A_m{}^{*-1}\sigma_k(\omega)$, $X_m(\omega, t_2) = A_m{}^{*-1}\sigma_l(\omega)$ and $t_1 \leqq t_2$. By (1.4.6),

$$2A_m{}^{*2} \simeq \frac{M^2}{(1+2\alpha)} \cdot m \tag{1.4.8}$$

and so

$$\frac{k}{m} \to_{m\to\infty} t_1\,, \qquad \frac{l}{m} \to_{m\to\infty} t_2\,. \tag{1.4.9}$$

By (1.4.7),

$$\sum_{j=1}^k a_j^2 \alpha_{k,j}\alpha_{l,j} \simeq M^2 \frac{(l-k)^{2\alpha+1}}{(kl)^\alpha} \int_0^{k/l-k} t^\alpha(1+t)^\alpha\,dt\,. \tag{1.4.10}$$

Combining (1.4.10) and (1.4.8), we have

$$\Gamma_m(t_1, t_2) \simeq \frac{(l-k)^{2\alpha+1}}{(kl)^\alpha} \frac{(1+2\alpha)}{m} \int_0^{k/l-k} t^\alpha(1+t)^\alpha\,dt\,;$$

and so, by (1.4.9)

$$\Gamma(t_1, t_2) = \frac{(1+2\alpha)}{t_1{}^\alpha t_2{}^\alpha} \int_0^{t_1} x^\alpha[(t_2-t_1)+x]^\alpha\,dx\,. \tag{1.4.11}$$

Consequently, conditions (i), (ii) and (iii) of Theorem (1.2.1) are satisfied. Moreover, by (1.4.3) and (1.4.11),

$$E(X_\alpha(\cdot, t_1)X_\alpha(\cdot, t_2)) = \Gamma(t_1, t_2) \tag{1.4.12}$$

for $t_1 \leqq t_2$. By (1.4.2), (1.4.4) and (1.4.12), Theorem (1.4.1) is established. □

5. The limit process $X(\cdot, t)$ appearing in Theorem (1.2.1) is not necessarily a Wiener process (see Theorem (1.4.1)). To ensure convergence to the Wiener process, additional conditions have to be added. A variety of conditions which ensure convergence of X_m to the Wiener process appear in the next theorems, which are an immediate consequence of Theorem (1.3.1). Theorem (1.5.1) is a theorem of Billingsley (cf. [1] page 410, Theorem 23.1).

(1.5.1) THEOREM. *Under the conditions of Theorem* (1.2.1) *with assumption* (iii) *excluded, if* σ_m *are the partial sums of* $S(\omega) = \sum_{j=1}^{\infty} a_j \cos n_j \omega$, *then* $X_m \Rightarrow W$ *in* $(D, \mathscr{D})$.

(1.5.2) THEOREM. *Under the conditions of Theorem* (1.2.1) *with assumption* (iii) *excluded, if* $\sigma_m(\omega)$ *are the Cesàro-*1 *means of* $S(\omega)$ *and if* $(n+1)^{-1} \sum_{j=1}^{n} j a_j^2 = o(A_n^2)$, *then* $X_m \Rightarrow W$ *in* $(D, \mathscr{D})$.

Observe that the conclusion of the previous theorem continues to hold if $a_n^2 = O(1/n)$ or if $a_n^2 = O(1/n \lg_1 n \cdots (\lg_p n)^\alpha)$, where $0 \leqq \alpha \leqq 1$, $p = 1, 2, \cdots$, and $\lg_p$ denotes the natural logarithm iterated p times.

6. In the following two theorems, which are generalizations of Theorem (1.2.1), the triangular matrix $\{\alpha_{m,j}\}$ is replaced by a doubly infinite one. Theorem (1.6.2) refers to the particular case of Abel means.

The proof of the next theorem follows the same lines as that of Theorem (1.2.1).

(1.6.1) THEOREM. *In Theorem* (1.2.1), *if the condition* $\alpha_{m,j} = 0$ *for* $j > m$ *is replaced by* $\sum_{j=1}^{\infty} \alpha_{m,j}|a_j| < \infty$, *the conclusion continues to hold with the following new definitions of* $\sigma_m(\omega)$ *and* A_m^*: $\sigma_m(\omega) = \sum_{j=1}^{\infty} \alpha_{m,j} a_j \cos n_j \omega$ *and* $A_m^* = (\frac{1}{2} \sum_{j=1}^{\infty} \alpha_{m,j}^2 a_j^2)^{\frac{1}{2}}$.

In particular, we may set $\alpha_{m,j} = r_m^{n_j}$, where r_m is a sequence such that $0 \leqq r_m < 1$ and $r_m \uparrow 1$ as $m \to \infty$. In this case, the $\sigma_m(\omega)$ are the Abel means (with a discrete parameter) of $S(\omega) = \sum_{j=1}^{\infty} a_j \cos n_j \omega$.

The next theorem deals with the Abel means of $S(\omega)$ with a continuous parameter $r \in [0, 1)$.

(1.6.2) THEOREM. *Let* $\sum_{j=1}^{\infty} r^{n_j}|a_j| < \infty$ *for every* $r \in [0, 1)$. *Define*

$$A_r^* = (\tfrac{1}{2} \textstyle\sum_{j=1}^{\infty} r^{2n_j} a_j^2)^{\frac{1}{2}}, \qquad \sigma_r(\omega) = \sum_{j=1}^{\infty} a_j r^{n_j} \cos n_j \omega \qquad \text{and}$$

$X_r(\omega, t) = \sigma_s(\omega)/A_r^*$ *if* $A_s^{*2}/A_r^{*2} = t$. *Assume conditions* (i) *and* (ii) *of Theorem* (1.2.1) *and assume that* $\lim_{r \to 1} (1/2\pi) \int_0^{2\pi} X_r(\omega, t_1) X_r(\omega, t_2)\, d\omega = \Gamma(t_1, t_2) \in \mathbb{R}$ *whenever* $t_1, t_2 \in [0, 1]$. *Let* P *be a probability measure on* $([0, 2\pi], \mathscr{B})$, *absolutely continuous with respect to Lebesgue measure. Then there exists a Gaussian process* $\{X(\cdot, t) : 0 \leqq t \leqq 1\}$ *such that* $X_r \Rightarrow_{r \to 1} X$ *in* $(C, \mathscr{C})$.

PROOF. The X_r are random elements of $(C, \mathscr{C})$. By Fubini's theorem, all the arguments leading to (1.2.4) in the proof of Theorem (1.2.1) are valid for the random elements $X_r(\omega, t)$. Therefore, there is a constant Q such that

$$P(|\sigma_s - \sigma_u| \geqq \lambda) \leqq \frac{Q}{\lambda^3} (A_s^{*2} - A_u^{*2})^{\frac{3}{2}} \tag{1.6.3}$$

for all $0 \leqq u \leqq s \leqq 1$ and every $\lambda > 0$, provided P is the probability measure defined by (1.2.3). Therefore, by the definition of X_r,

$$P(|X_r(\cdot, t_2) - X_r(\cdot, t_1)| \geqq \lambda) \leqq \frac{Q}{\lambda^3}(t_2 - t_1)^{\frac{3}{2}} \tag{1.6.4}$$

for $0 \leqq t_1 \leqq t_2 \leqq 1$ and $r \in [0, 1)$. Since $X_r(\cdot, 0) \equiv 0$, the previous formula implies that the family $\{X_r\}_{r\in[0,1)}$ is tight in $(C, \mathscr{C})$ (cf. [2] page 95, Theorem 12.3). Furthermore, because of Theorem (1.1.1), Lemma 3 which is used in the proof of Theorem (1.2.1), is valid for the random elements X_r. Therefore, there is a Gaussian process $\{X(\cdot, t): 0 \leqq t \leqq 1\}$ such that

$$(X_r(\cdot, t_1), \cdots, X_r(\cdot, t_p)) \Rightarrow_{r\to 1} (X(\cdot, t_1), \cdots, X(\cdot, t_p)) \tag{1.6.5}$$

for $t_1, \cdots, t_p \in [0, 1]$.

By the last two formulas and the Cramér–Wold theorem, we conclude that

$$P^*(|X(\cdot, t_2) - X(\cdot, t_1)| \geqq \lambda) \leqq \frac{Q}{\lambda^3}(t_2 - t_1)^{\frac{3}{2}}$$

It follows from this formula that we may choose X such that X is a random element of C (cf. [2] page 96, Theorem 12.4). Therefore, $X_r \Rightarrow_{r\to 1} X$ in $(C, \mathscr{C})$, if the probability measure P is defined by formula (1.2.3) of Theorem (1.2.1). If P is absolutely continuous with respect to Lebesgue measure, we proceed as in Theorem (1.2.1). □

7. Remark. In Theorem (1.2.1), because of (1.2.32) and (1.2.2), we have

$$E(|X_m(\cdot, t+h) - X_m(\cdot, t)|^3) \to_{m\to\infty} E(|X(\cdot, t+h) - X(\cdot, t)|^3) < \infty$$

for t and $t + h$ in [0, 1]. Since we also have

$$E(|X_m(\cdot, t+h) - X_m(\cdot, t)|^3) \leqq Q\left(\frac{A_r^{*2}}{A_m^{*2}} - \frac{A_l^{*2}}{A_m^{*2}}\right)^{\frac{3}{2}}$$

for $X_m(\omega, t+h) = A_m^{*-1}\sigma_r(\omega)$ and $X_m(\omega, t) = A_m^{*-1}\sigma_l(\omega)$, it follows that

$$E(|X(\cdot, t+h) - X(\cdot, t)|^3) \leqq Q|h|^{\frac{3}{2}} .$$

Consequently, for a given $0 < \xi < \frac{1}{6}$, there exists a stochastic process $\{Z(\cdot, t): 0 \leqq t \leqq 1\}$ with the same finite-dimensional distributions as those of $\{X(\cdot, t): 0 \leqq t \leqq 1\}$, such that, with probability one, it satisfies a Lipschitz condition of order ξ (cf., e.g., [3] page 74). For analogous reasons, the same conclusion holds for the limit process $\{X(\cdot, t): 0 \leqq t \leqq 1\}$ appearing in Theorem (1.6.2). If $X = W$, it is known that we may take $0 < \xi < \frac{1}{2}$ (for this theorem of Wiener see, e.g., [3] page 75).

PART 2. Sequences of linearly independent numbers.

1. The lacunarity hypothesis used in the previous functional limit theorems has been dropped in Theorem (2.2.2). A theorem of Kac–Steinhaus (cf. [7]

page 47, and [8] page 11, Theorem 5) and its generalization by Salem and Zygmund (cf. [11] page 60, Theorem xii) suggested that a theorem like Theorem (2.2.2) might be true. An essential tool for the proof of Theorem (2.2.2) is Theorem (2.1.1), which deals with triangular arrays, and generalizes Theorem xii of Salem and Zygmund.

Real numbers $\lambda_1, \lambda_2, \cdots$ are called linearly independent if the only solution in integers $(k_1, k_2, \cdots)$ of the equation $k_1\lambda_1 + k_2\lambda_2 + \cdots = 0$ is $k_1 = k_2 = \cdots = 0$. In what follows, $\{\lambda_\nu\}$, $\nu = 1, 2, \cdots$, will be a sequence of linearly independent real numbers.

(2.1.1) THEOREM. *Let* $\{b_{m,j}\}_{j=1,2,\cdots,m;m=1,2,\cdots}$, *be a triangular array of real numbers. Let us write* $B_m = (\frac{1}{2}\sum_{j=1}^m b_{m,j}^2)^{\frac{1}{2}}$ *and* $S_m(\omega) = \sum_{j=1}^m b_{m,j}\cos\lambda_j\omega$ *for all* $\omega \in \mathbb{R}$. *Assume that the following two conditions hold*:

(i) $B_m \to_{m\to\infty} \infty$;
(ii) $\max_{1\leqq j\leqq m} B_m^{-1}|b_{m,j}| \to_{m\to\infty} 0$.

Then,

$$\lim_{m\to\infty}\lim_{T\to\infty}\frac{1}{2T}|\{\omega\in[-T,T]: B_m^{-1}S_m(\omega)\leqq y\}| = (2\pi)^{-\frac{1}{2}}\int_{-\infty}^{y}\exp(-\tfrac{1}{2}t^2)\,dt$$

for all $y \in \mathbb{R}$, *with the possible exception of a countable set.*

PROOF. Since $B_m^{-1}S_m(\omega)$ is an almost periodic function in the sense of Bohr, by Wintner's theorem (cf. [12] and [6] page 249),

$$F_m^*(y) = \lim_{T\to\infty}|\{\omega\in[-T,T]: B_m^{-1}S_m(\omega)\leqq y\}|/2T$$

exists for all $y \in \mathbb{R}$ with the possible exception of a countable set.

If we write $F_m(y) = F_m^*(y^+)$ for all $y \in \mathbb{R}$, then the theorem will be established if we show that

(2.1.2)
$$\begin{aligned}\phi_m(x) &= \int_{-\infty}^{\infty}\exp(ixy)\,dF_m(y)\\ &= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\exp(ixB_m^{-1}S_m(y))\,dy \to_{m\to\infty}\exp(-\tfrac{1}{2}x^2).\end{aligned}$$

As in the case of Theorem (1.1.1), if $\xi_m(y) = \frac{1}{2}B_m^{-2}\sum_{j=1}^m b_{m,j}^2\cos 2\lambda_j y$,

(2.1.3)
$$\begin{aligned}\phi_m(x) = o(1) + \exp(-\tfrac{1}{2}x^2)\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}&\exp(-\tfrac{1}{2}x^2\xi_m(y))\\ &\times\prod_{j=1}^m(1+ixB_m^{-1}b_{m,j}\cos\lambda_j y)\,dy.\end{aligned}$$

Set $E_{m,\eta,T} = \{y\in[-T,T]: |\xi_m(y)|\geqq\eta\}$; then

(2.1.4)
$$\begin{aligned}\frac{1}{2T}|E_{m,\eta,T}| &\leqq \frac{1}{2T}\eta^{-2}\int_{-T}^{T}\xi_m^2(y)\,dy\\ &= \tfrac{1}{4}\eta^{-2}B_m^{-4}\sum_{j=1}^m b_{m,j}^4\frac{1}{4T}\int_{-2T}^{2T}\cos^2\lambda_j y\,dy\\ &\quad + \tfrac{1}{4}\eta^{-2}B_m^{-4}\sum_{i\neq j} b_{m,j}^2 b_{m,i}^2\frac{1}{4T}\int_{-2T}^{2T}\cos\lambda_j y\cos\lambda_i y\,dy.\end{aligned}$$

Since

(2.1.5) $$\frac{1}{2T}\int_{-T}^{T}\cos\lambda_j y\,dy \to_{T\to\infty} 0 \quad\text{and}\quad \frac{1}{2T}\int_{-T}^{T}\cos^2\lambda_j y\,dy \to_{T\to\infty} \tfrac{1}{2},$$

by (2.1.4) we obtain

(2.1.6) $$\limsup_{T\to\infty}\frac{1}{2T}|E_{m,\eta,T}| \leqq 2^{-3}\eta^{-2}B_m^{-4}\sum_{j=1}^{m} b_{m,j}^4 .$$

By (ii), the right-hand side of (2.1.6) tends to 0 as $m\to\infty$ and so

(2.1.7) $$\limsup_{T\to\infty}\frac{1}{2T}|E_{m,\eta,T}| \to_{m\to\infty} 0 .$$

Formulas (2.1.7) and (2.1.3) imply that

(2.1.8) $$\phi_m(x) = o(1) + \exp(-\tfrac{1}{2}x^2) \times \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\prod_{j=1}^{m}(1 + ixB_m^{-1}b_{m,j}\cos\lambda_j y)\,dy .$$

By (2.1.5) and the independence of the system $\{\cos\lambda_j y\}$ with respect to the relative measure (cf. [8] page 10, Theorem 3), the limit which appears in the right-hand side of (2.1.8) equals 1 and, therefore, (2.1.2) holds. □

2. In the proof of Theorem (2.2.2) we shall use the following lemma, which has a simple proof.

(2.2.1) LEMMA. *Under the hypotheses of Theorem* (2.1.1) *and with the notation* $S_k^*(\omega) = B_k^{-1}S_k(\omega)$, *we have that*

$$\lim_{T\to\infty}\frac{1}{2T}|\{\omega\in[-T,T]: y_1 < S_{m_1}^*(\omega) \leqq y_1', \cdots, y_p < S_{m_p}^*(\omega) \leqq y_p'\}|$$

exists for all except, at most, a countable set of values of y_i, y_i'.

(2.2.2) THEOREM. *Let* $\{a_j\}$, $j = 1, 2, \cdots$, *be a bounded sequence of real numbers and set* $A_m = (\frac{1}{2}\sum_{j=1}^{m} a_j^2)^{\frac{1}{2}}$. *For each* m, $T = 1, 2, \cdots, \omega\in[-T,T]$ *and* $t\in[0,1)$ *define* $X_T^m(\omega,t) = A_m^{-1}\sum_{j=1}^{[mt]} a_j\cos\lambda_j\omega$[5] *and* $X_T^m(\omega,1) = A_m^{-1}\times\sum_{j=1}^{m-1} a_j\cos\lambda_j\omega$; *and consider the normalized Lebesgue measure in* $[-T,T]$. *If* $(2A_m^2/m)\to_{m\to\infty} c\neq 0$, *then, there exists a family of stochastic processes* $\{Z^m(\cdot,t): 0\leqq t\leqq 1\}$, $m = 1, 2, \cdots$, *such that* $X_T^m \Rightarrow_{T\to\infty} Z^m \Rightarrow_{m\to\infty} W$ *in* $(D, \mathscr{D})$.

PROOF. It is clear that the X_T^m are random elements of $(D,\mathscr{D})$. Let us write $S_k(\omega) = \sum_{j=1}^{k} a_j\cos\lambda_j\omega$, and

(2.2.3) $$F_m^*(y_1,\cdots,y_k; t_1,\cdots,t_k) = \lim_{T\to\infty}\frac{1}{2T}|\{\omega\in[-T,T]: X_T^m(\omega,t_i)\leqq y_i \text{ for all } i\}|$$

[5] $[mt]$ indicates the largest integer less than or equal to mt.

for $y_1, \cdots, y_k \in \mathbb{R}$ and $t_1, \cdots, t_k \in [0, 1]$. By Lemma (2.2.1), F_m^* is well defined for all except, at most, a countable number of $(y_1, \cdots, y_k) \in \mathbb{R}^k$.

We now consider the k-dimensional distribution function F^m defined by

$$F^m(y_1, \cdots, y_k; t_1, \cdots, t_k) = F_m^*(y_1^+, \cdots, y_k^+; t_1, \cdots, t_k) .$$

By Kolmogorov's existence theorem, there is a family of stochastic processes $\{Y^m(\cdot, t): 0 \leqq t \leqq 1\}$, $m = 1, 2, \cdots$, such that

$$\text{(2.2.4)} \qquad F^m(y_1, \cdots, y_k; t_1, \cdots, t_k) = P_m(\omega' \in \Omega_m': Y^m(\omega', t_i) \leqq y_i \text{ for all } i = 1, \cdots, k) ;$$

therefore, it follows from (2.2.3) that

$$\text{(2.2.5)} \qquad (X_T{}^m(\cdot, t_1), \cdots, X_T{}^m(\cdot, t_k)) \Rightarrow_{T\to\infty} (Y^m(\cdot, t_1), \cdots, Y^m(\cdot, t_k)) ;$$

and also

$$\text{(2.2.6)} \qquad (|X_T{}^m(\cdot, t) - X_T{}^m(\cdot, t_1)|, |X_T{}^m(\cdot, t_2) - X_T{}^m(\cdot, t)|) \Rightarrow_{T\to\infty} (|Y^m(\cdot, t) - Y^m(\cdot, t_1)|, |Y^m(\cdot, t_2) - Y^m(\cdot, t)|)$$

for $t_1, t, t_2 \in [0, 1]$. Let $\alpha_1, \cdots, \alpha_k \in \mathbb{R}$ and $0 = t_0 \leqq t_1 < \cdots < t_k \leqq 1$. It follows from (2.2.5) that

$$\text{(2.2.7)} \qquad \begin{aligned} &P_m(\textstyle\sum_{j=1}^k \alpha_j Y^m(\cdot, t_j) \leqq x) \\ &\quad = \lim_{T\to\infty} (2T)^{-1}|\{\omega \in [-T, T]: \textstyle\sum_{j=1}^k \alpha_j X_T{}^m(\omega, t_j) \leqq x\}| \\ &\quad = \lim_{T\to\infty} (2T)^{-1}|\{\omega \in [-T, T]: \textstyle\sum_{j=1}^m b_{m,j} \cos \lambda_j \omega \leqq x \cdot A_m\}| ; \end{aligned}$$

where $b_{m,j} = a_j \sum_{i=1,\cdots,k,[mt_i]\geqq j} \alpha_i$. If $t_k = 1$, substitute $m - 1$ for $[mt_k]$. Set $B_m = (\frac{1}{2} \sum_{j=1}^m b_{m,j}^2)^{\frac{1}{2}}$. Since $\lim_{m\to\infty} A_m^{-2}A_{[mt]}^2 = t$ for $0 \leqq t < 1$ and $A_m^{-2}A_{m-1}^2 \to 1$, we have

$$\lim_{m\to\infty} B_m^2 A_m^{-2} = t_1(\alpha_1 + \cdots + \alpha_k)^2 + (t_2 - t_1)(\alpha_2 + \cdots + \alpha_k)^2 + \cdots + (t_k - t_{k-1})\alpha_k^2 .$$

Therefore the double array $b_{m,j}$ satisfies the hypotheses of Theorem (2.1.1), and consequently (2.2.7) can be written

$$\begin{aligned} &\lim_{m\to\infty} P_m(\textstyle\sum_{j=1}^k \alpha_j Y^m(\cdot, t_j) \leqq x) \\ &\quad = N(0, t_1(\alpha_1 + \cdots + \alpha_k)^2 + (t_2 - t_1)(\alpha_2 + \cdots + \alpha_k)^2 + \cdots + (t_k - t_{k-1})\alpha_k^2)[x] , \end{aligned}$$

that is,

$$\text{(2.2.8)} \qquad \textstyle\sum_{j=1}^k \alpha_j Y^m(\cdot, t_j) \Rightarrow_{m\to\infty} \sum_{j=1}^k \alpha_j W_{t_j}(\cdot) .$$

By using the Cramér–Wold theorem, we conclude that

$$\text{(2.2.9)} \qquad (Y^m(\cdot, t_1), \cdots, Y^m(\cdot, t_k)) \Rightarrow_{m\to\infty} (W_{t_1}(\cdot), \cdots, W_{t_k}(\cdot)) .$$

If $0 \leqq t_1 < t < t_2 \leqq 1$ and $y > 0$, then

$$\begin{aligned} &\frac{1}{2T}|\{\omega \in [-T, T]: |X_T{}^m(\omega, t_2) - X_T{}^m(\omega, t)| \geqq y, \\ &\qquad |X_T{}^m(\omega, t) - X_T{}^m(\omega, t_1)| \geqq y\}| \\ &\qquad \leqq A_m{}^{-3}y^{-3}\left(\left(\frac{1}{2T}\right)\int_{-T}^{T}|S_{[mt_2]} - S_{[mt]}|^4\,d\omega \right. \\ &\qquad \times \left.\left(\frac{1}{2T}\right)\int_{-T}^{T}|S_{[mt]} - S_{[mt_1]}|^4\,d\omega\right)^{\frac{3}{8}} \\ &\qquad \leqq H\cdot y^{-3}([mt_2] - [mt])^{\frac{3}{4}}([mt] - [mt_1])^{\frac{3}{4}}, \end{aligned} \tag{2.2.10}$$

where H is a constant. Since $(2T)^{-1}\int_{-T}^{T}\cos\mu\omega\,d\omega \to_{T\to\infty} 0$ for $\mu \neq 0$, and since the sequence $\{\lambda_j\}$ is linearly independent, it is easy to see that there is a constant K, independent of m, t_1 and t_2, such that

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}(S_{[mt_2]}(\omega) - S_{[mt_1]}(\omega))^4\,d\omega \leqq K(A^2_{[mt_2]} - A^2_{[mt_1]})^2. \tag{2.2.11}$$

It follows from (2.2.11) and from the inequalities of Chebychev and Hölder that for $0 \leqq t_1 \leqq t_2 \leqq 1$ there is a constant K' independent of m, t_1 and t_2, such that

$$\begin{aligned} &\lim_{T\to\infty}\frac{1}{2T}|\{\omega \in [-T, T]: |S_{[mt_2]}(\omega) - S_{[mt_1]}(\omega)| \geqq y\}| \\ &\qquad \leqq K'\cdot y^{-3}(A^2_{[mt_2]} - A^2_{[mt_1]})^{\frac{3}{2}} \end{aligned} \tag{2.2.12}$$

for all values of $y > 0$, with the possible exception of a denumerable set. The limit which appears in the left-hand side of the previous formula exists because $S_{[mt_2]}(\omega) - S_{[mt_1]}(\omega)$ is an almost periodic function. By (2.2.12) and Lemma (2.2.1),

$$\begin{aligned} &\lim_{T\to\infty}\frac{1}{2T}|\{\omega \in [-T, T]: |S_{[mt_2]} - S_{[mt]}| \geqq y, |S_{[mt]} - S_{[mt_1]}| \geqq y\}| \\ &\qquad \leqq K'y^{-3}(A^2_{[mt_2]} - A^2_{[mt_1]})^{\frac{3}{2}} \end{aligned} \tag{2.2.13}$$

for all $0 \leqq t_1 \leqq t \leqq t_2 \leqq 1$ and all $y > 0$.

We now show that there is a constant K^* independent of t_1, t, t_2 and m such that

$$\begin{aligned} &\lim_{T\to\infty}\frac{1}{2T}|\{\omega \in [-T, T]: |X_T{}^m(\omega, t_2) - X_T{}^m(\omega, t)| \geqq y, \\ &\qquad |X_T{}^m(\omega, t) - X_T{}^m(\omega, t_1)| \geqq y\}| \\ &\qquad \leqq \frac{K^*}{y^3}(t_2 - t_1)^{\frac{3}{2}}. \end{aligned} \tag{2.2.14}$$

In fact, if $[mt_2] - [mt_1] \leqq 1$, the left-hand side of the formula above vanishes; therefore it is enough to consider the case $|t_2 - t_1| \geqq 1/m$. By (2.2.13), the left-hand side of (2.2.14) is bounded by $K^*y^{-3}(A_m{}^{-2}A^2_{[mt_2]} - A_m{}^{-2}A^2_{[mt_1]})^{\frac{3}{2}}$. Hence, it is enough to show that $A_m{}^{-2}A^2_{[mu]} - A_m{}^{-2}A^2_{[mv]} \leqq \hat{K}(u - v)$, where $0 \leqq u < v < 1$, $v - u \geqq 1/2m$, $[mu] = [mv] + 1$, and $\hat{K}$ is a constant independent of m, u and

v; this is in fact true because $m/2A_m^2$ is bounded, and so (2.2.14) holds. Combining formulas (2.2.14) and (2.2.6), we obtain

$$(2.2.15) \qquad P_m(|Y^m(\cdot, t_2) - Y^m(\cdot, t)| \geqq y, |Y^m(\cdot, t) - Y^m(\cdot, t_1)| \geqq y) \leqq \frac{K^*}{y^3}(t_2 - t_1)^{\frac{3}{2}}.$$

For $t_1, \cdots, t_k \in [0, 1]$, denote by $\mu^m_{t_1,\cdots,t_k}$ the probability measure associated with the k-dimensional distribution function $F^m(y_1, \cdots, y_k; t_1, \cdots, t_k)$. If $0 \leqq t_1 \leqq t \leqq t_2$, by (2.2.15) we have

$$(2.2.16) \qquad \mu^m_{t_1,t,t_2}\{(\beta_1, \beta, \beta_2): |\beta - \beta_1| \geqq y, |\beta_2 - \beta| \geqq y\} \leqq \frac{K^*}{y^3}(t_2 - t_1)^{\frac{3}{2}}.$$

If $0 \leqq t < 1$, $h > 0$ and $t + h \in (0, 1)$, then

$$|X_T{}^m(\cdot, t+h) - X_T{}^m(\cdot, t)| \Rightarrow_{T\to\infty} |Y^m(\cdot, t+h) - Y^m(\cdot, t)|;$$

and it follows from (2.2.12) that, for $\varepsilon > 0$,

$$\mu^m_{t,t+h}\{(\beta_1, \beta_2): |\beta_1 - \beta_2| \geqq \varepsilon\} \leqq \frac{K'}{\varepsilon^3}(A_m{}^{-2}A^2_{[m(t+h)]} - A_m{}^{-2}A^2_{[mt]})^{\frac{3}{2}}.$$

Consequently,

$$(2.2.17) \qquad \mu^m_{t,t+h}\{(\beta_1, \beta_2): |\beta_1 - \beta_2| \geqq \varepsilon\} \to_{h\downarrow 0} 0.$$

By (2.2.16) and (2.2.17), there is a family of stochastic processes $\{Z^m(\cdot, t): 0 \leqq t \leqq 1\}$ in $(D, \mathscr{D})$ with the same finite-dimensional distributions as those of $\{Y^m(\cdot, t): 0 \leqq t \leqq 1\}$ (cf. [2] page 130, Theorem 15.7). Since $P_m{}^*(Z^m(\cdot, 1) \neq Z^m(\cdot, 1^-)) = 0$, it follows from (2.2.5) and (2.2.10) that $X_T{}^m \Rightarrow_{T\to\infty} Z^m$ in $(D, \mathscr{D})$ (cf. [2] page 133). Finally, from (2.2.9) and (2.2.15) we conclude that $Z^m \Rightarrow_{m\to\infty} W$ in $(D, \mathscr{D})$ (cf. [2] page 128, Theorem 15.6). Therefore, Theorem (2.2.2) is established. □

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