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# THE EXCESSIVE FUNCTIONS OF A CONTINUOUS TIME MARKOV CHAIN

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Boundary conditions in the form of equalities have been used by Feller, Dynkin, and others to characterize the range of the resolvent operator for certain continuous time Markov chains. Along similar lines Denzel, Kemeny, and Snell were able to establish a characterization and Riesz decomposition for the excessive functions of a more restricted class of Markov chains through the use of boundary conditions in the form of inequalities. The present paper sets out to clarify and build upon this work by reanalyzing these excessive functions in a more general setting. Here the boundary theory developed by Chung is brought to bear on the problem so that the results can be derived in canonical form for probabilistic interpretation.

**0. Introduction.** The Laplace transform of the transition matrix for a continuous time Markov chain can be considered as an operator on the class of bounded functions which are defined on the state space of the chain. In his pioneering work ([7], [8]), Feller characterized the range of this resolvent operator by certain equations called lateral conditions involving a "normal derivative" at the boundary. Subsequently, Dynkin ([6]) was able to accomplish this characterization with similar boundary conditions for a wider class of transition matrices. Since each excessive function for the Markov chain is the pointwise limit of a sequence chosen from the range of the resolvent, it would seem likely that the excessive functions should be characterized by boundary conditions which are related in some way to those of Feller or Dynkin. This was demonstrated in a paper by Denzel, Kemeny, and Snell ([5]) for a continuous time Markov chain which has one minimal "fast" exit boundary. These authors also took up the task of extending to these excessive functions the Riesz decomposition known for the superregular functions of the embedded jump chain. More recently, Pittenger [11] has shown that similar boundary conditions can also be produced for certain Markov processes.

The present paper seeks to extend more fully the results of Denzel, Kemeny, and Snell to the case where the passable part of the Martin exit boundary for the chain is completely atomic and finite. Moreover, by using the analytic and probabilistic results of Chung ([2], [3], [4]), and by working directly with time-dependent quantities rather than with their Laplace transforms, greater insight is afforded in the derivation of this material. For example, once boundary

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conditions are obtained in canonical form for the excessive functions of a chain, it is then a simple matter to characterize a non-recurrent boundary atom as sticky or non-sticky according to whether or not all excessive functions enjoy a certain right continuity at that atom. And Feller's normal derivative, when divorced from the Laplace transform, is a natural aid in the decomposition problem. But before these topics can be considered, certain preliminary facts must be established.

**1. Background and notation.** It is assumed that the reader has a knowledge of the results contained in [2], [3], and [4]. The following paragraphs will serve as a summary of some of the important points referred to later.

Let $(X_t, \mathscr{F}_t)$ be a continuous time Markov chain on a probability space $(\Omega, \mathscr{F}, \mathbf{P})$ with the standard, stochastic transition matrix $P(t) = (p_{ij}(t))$, where $t$ is positive and where $i$ and $j$ range over the denumerable state space $\mathbf{I}_\theta = \mathbf{I} \cup \{\theta\}$. Without loss of generality, we assume that the chain is Borel measurable and right separable with respect to the rationals. The matrix $P(t)$ satisfies

$$P(t) \geqq 0\,, \tag{1.1}$$

$$P(t)1 = 1\,, \tag{1.2}$$

$$P(s)P(t) = P(s+t)\,, \tag{1.3}$$

$$\lim_{t\to 0+} P(t) = I\,, \tag{1.4}$$

where $I$ is the identity matrix and 1 is the vector consisting of all ones. From these properties, it follows (see [4]) that $P(t)$ has a continuous derivative for all positive $t$ and also a right-hand derivative at $t = 0$, denoted by $Q = (q_{ij})$. It is assumed that on $\mathbf{I}$

$$0 < q1 < +\infty$$

and

$$Q1 = 0\,, \tag{1.5}$$

where $q$ is the diagonal matrix $(-\delta_{ij} q_{ij})$.

If $\tau_n$ is the time that the chain makes the $n$th change of state, then $\chi_n = X(\tau_n)$ defines the jump chain associated with $X_t$. This discrete time Markov chain has state space $\mathbf{I}_\theta$ and transition matrix $R = I + q^{-1}Q$. The chain $\chi_n$, with a strictly positive initial distribution $\gamma$, induces an exit boundary $\mathbf{B}$ and a metric on $\mathbf{I}$ and $\mathbf{B}$ so that almost surely $\chi_n$ converges to a point $\chi_\infty$ of $\mathbf{B}$. If $\mathbf{P}(\chi_\infty = b) > 0$ for some $b \in \mathbf{B}$, then $b$ is called an atomic boundary point. Letting $\tau = \lim_{n\to\infty} \tau_n$, the first infinity of the jump chain, there exists a subset $\mathbf{A}$ of $\mathbf{B}$ so that $\{\tau < \infty\}$ and $\{\chi_\infty \in \mathbf{A}\}$ differ by a null set. The set $\mathbf{A}$, the passable part of the exit boundary, is assumed throughout this paper to be nonvoid, to contain all of the atomic boundary points, and to be finite.

If $X_t$ is stopped at the first infinity $\tau$ and absorbed in some new state, then the resulting chain, the minimal chain, has a substochastic transition matrix $\Phi(t)$ which satisfies (1.1), (1.3), and (1.4). All $\Phi$-recurrent states are lumped together, and we assume that $\theta$ is the only $\Phi$-recurrent state if one exists. $\Phi(t)$ also has the initial derivative matrix $Q$, and, in fact, $\Phi(t)$ can be derived analytically from $Q$ (see [4]) as the increasing limit of the matrices $(\sigma_{ij}^{\langle n\rangle}(t))$ defined recursively by

$$\sigma_{ij}^{\langle 0\rangle}(t) = \delta_{ij}e^{-q_i t} \tag{1.6}$$

and

$$\sigma_{ij}^{\langle n+1\rangle}(t) = \sigma_{ij}^{\langle 0\rangle}(t) + \sum_{k\neq i} \int_0^t e^{-q_i(t-s)} q_{ik}\sigma_{kj}^{\langle n\rangle}(s)\, ds\,. \tag{1.7}$$

For the convenience of notation, we will let $\sigma_n(t)$ stand for the matrix $(\sigma_{ij}^{\langle n\rangle}(t))$, and $e^{-q}(t)$ will be the diagonal matrix with diagonal entries $e^{-q_i t}$. Then (1.6) becomes

$$\sigma_0(t) = e^{-q}(t)\,, \tag{1.8}$$

and (1.7) can be written as

$$\sigma_{n+1}(t) = \sigma_0(t) + e^{-q} * \langle qR, \sigma_n\rangle(t)\,. \tag{1.9}$$

Here and later, $*$ denotes convolution, and the notation $\langle\alpha, \beta\rangle$ or $\alpha\beta$ will stand for the inner product $\sum_{i\in \mathbf{I}_\theta} \alpha(i)\beta(i)$.

For each $a \in \mathbf{A}$, $\tau^a$ is defined to be $\tau$ when $\chi_\infty = a$ and infinity otherwise. Then for all nonnegative $t$, the vector function

$$L(t) = 1 - \langle\Phi(t), 1\rangle \tag{1.10}$$

can be written as

$$L(t) = \sum_{a\in\mathbf{A}} L^a(t)\,, \tag{1.11}$$

where $L_i(t) = \mathbf{P}_i(\tau \leqq t)$ and $L_i^a(t) = \mathbf{P}_i(\tau^a \leqq t)$. Each $L^a(t)$ satisfies

$$\langle\Phi(s), L^a(t)\rangle = L^a(s+t) - L^a(s) \tag{1.12}$$

and has a continuous nonnegative derivative $l^a(t)$ satisfying the exit law equation (see Section 4 of [2] and Section 13 of [3])

$$\Phi(s)e(t) = e(s+t)\,. \tag{1.13}$$

Further properties of this natural class $\{l^a(t)\}$ of exit solutions include:

$$l^a(t) = QL^a(t)\,, \qquad 0 \leqq t \leqq \infty\,; \tag{1.14}$$

$$\lim_{n\to\infty} L^a_{\chi_n}(t) = 1\,, \qquad 0 < t \leqq \infty\,, \tag{1.15}$$

almost surely on $\Delta^a = \{\tau^a < \infty\}$. At this point, some results for the minimal transition matrix $\Phi(t)$ are presented in order to set the scene for the investigation of functions which are excessive with respect to nonminimal chains.

**2. Excessive functions for the minimal chain.** Some of the theorems of this section are known, but most often in terms of the Laplace transform. To be complete, proofs have been supplied which use only the time-dependent quantities themselves.

(2.1) LEMMA. *On* $\mathbf{I} \times \mathbf{I}$,

$$\int_0^\infty \sigma_n(u)\,du = \sum_{k=0}^n R^k q^{-1}. \tag{2.2}$$

PROOF. For $n = 0$, by (1.6),

$$\int_0^\infty \sigma_{ij}^{\langle 0\rangle}(u)\,du = \delta_{ij} q_j^{-1}.$$

Assume (2.2) is true for $n = m$. Then by (1.7),

$$\begin{aligned}\int_0^\infty \sigma_{ij}^{\langle m+1\rangle}(u)\,du &= \delta_{ij} q_j^{-1} + \sum_{k\neq i} \int_0^\infty e^{-q_i t} q_{ik} \int_0^\infty \sigma_{kj}^{\langle m\rangle}(s)\,ds\,dt \\ &= \delta_{ij} q_j^{-1} + \sum_k r_{ik}\{\sum_{l=0}^m R^l(k, j)\} q_j^{-1} \\ &= \sum_{l=0}^{m+1} R^l(i, j) q_j^{-1},\end{aligned}$$

and induction completes the proof.

Following fairly standard notation, on $\mathbf{I} \times \mathbf{I}$ define the matrices

$$G = \int_0^\infty \Phi(u)\,du$$

and

$$N = \sum_{k=0}^\infty R^k.$$

Notice that $G$ has finite entries since $\theta$ is the only possible $\Phi$-recurrent state.

(2.3) COROLLARY. $G = Nq^{-1}$.

A sequence $C = \{C(i)\}$ of nonnegative finite numbers indexed by $\mathbf{I}_\theta$ is called a $\Phi$-excessive function if $\Phi(t)C \leqq C$ for all $t$. The same sequence is known as an $R$-superregular function if $RC \leqq C$.

(2.4) LEMMA. *$C$ is $\Phi$-excessive if and only if $C$ is $R$-superregular if and only if*

$$C \geqq 0 \tag{2.5}$$

*and*

$$QC \leqq 0. \tag{2.6}$$

PROOF. If $C$ is $\Phi$-excessive, then (2.5) is true and $t^{-1}\{\Phi(t)C - C\} \leqq 0$ for all positive $t$, which implies (2.6) by Fatou's Lemma. Now assume (2.5) and (2.6) are both true. Obviously, (2.6) is equivalent to

$$R^n C \leqq C \tag{2.7}$$

for $n = 0, 1, 2, \cdots$.
We prove inductively that, for all $t$,

$$\sigma_n(t)C \leqq C. \tag{2.8}$$

It will then follow by the Monotone Convergence Theorem that $\Phi(t)C \leqq C$. For $n = 0$, we have $\sigma_0(t)C = e^{-q}(t)C$, which is no greater than $C$. Assume (2.8) is true for $n = m$. Suppressing the time variable $t$, we have by (1.9)

$$\sigma_{m+1}C = \sigma_0 C + e^{-q} * \langle qR, \sigma_m C\rangle.$$

By (2.7) and (2.8),

$$\langle qR, \sigma_m C\rangle \leqq qC,$$

while

$$e^{-q} * qC = \{(I - e^{-q})q^{-1}\}qC = C - \sigma_0 C ,$$

and so (2.8) is true for $n = m + 1$.

The decomposition contained in the next theorem is suggested by the work of Feller ([8] Lemma 11.2) and Dynkin ([6] Lemma 2.2) on characterizing the range of the resolvent operator for the minimal transition matrix. For our purposes, with respect to the minimal chain, a finite, nonnegative $C$ is said to be $\Phi$-invariant if $\Phi(t)C = C$ for all $t$, whereas $C$ is a $\Phi$-potential if $C$ is $\Phi$-excessive and $\lim_{t\to\infty} \Phi(t)C = 0$. And with respect to the jump chain, $C$ is $R$-regular if $RC = C$, or $C$ is an $R$-potential if $C$ is $R$-superregular and $\lim_{n\to\infty} R^n C = 0$.

(2.9) THEOREM. *If $C$ is a $\Phi$-excessive function, then $C$ is continuous from the left at* $\mathbf{A}$ *in the sense that*

$$C(a) = \lim_{n\to\infty} C(\chi_n)$$

*exists and is a finite constant almost surely on $\Delta^a$, $a \in \mathbf{A}$. Moreover, on $\mathbf{I}_\theta$*

$$C \geqq \langle \Phi(t), C\rangle - \int_0^t \langle \Phi(u), QC\rangle \, du + \sum_{a\in\mathbf{A}} C(a)L^a(t) \tag{2.10}$$

*with equality if $\langle \gamma, C\rangle < +\infty$. A necessary and sufficient condition that equality hold in* (2.10) *for all $\Phi$-excessive functions is that every $R$-regular $\Phi$-potential be bounded.*

PROOF. Assume $C$ is $\Phi$-excessive and $\langle \gamma, C\rangle < +\infty$. $C$ is $R$-superregular by Lemma (2.4). Hence, by Theorem 10–41 and Lemma 10–42 of [10], we can decompose $C$ into the sum of an $R$-potential function and an $R$-regular function as follows:

$$C = \langle N, C - RC\rangle + \int_{\mathbf{B}_e} f(b)K(\cdot, b)\, d\mu(b) , \tag{2.11}$$

where $\mu$ is the harmonic boundary measure and $f$ is the fine boundary function of $C$. Now by Proposition 10–21 of [10],

$$K(i, b)\, d\mu(b) = d\mathbf{P}_i(\chi_\infty = b)$$

on $\mathbf{B}$. From this identity and the discussion on page 32 of [3], we have

$$\begin{aligned}\langle I - \Phi(t), \textstyle\int_{\mathbf{B}_e} f(b)K(\cdot, b)\, d\mu(b)\rangle(i) &= \textstyle\int_{\mathbf{B}_e} f(b)\mathbf{P}_i(\chi_\infty \in db;\, \tau \leqq t) \\ &= \textstyle\sum_{a\in\mathbf{A}} f(a)L_i^a(t) .\end{aligned}$$

Moreover, by Corollary (2.3), $\langle N, C - RC\rangle$ equals $-\langle G, QC\rangle$. Then equality in (2.10) follows directly from (2.11) upon calculating $C - \langle \Phi(t), C\rangle$, since by Theorem 10–43 and Proposition 10–45 of [10], we have almost surely

$$\lim_{n\to\infty} C(\chi_n) = f(\chi_\infty) ,$$

and, in particular, on $\Delta^a$

$$\lim_{n\to\infty} C(\chi_n) = f(a) ,$$

a finite constant which is our $C(a)$, $a \in \mathbf{A}$.

For the general $\Phi$-excessive $C$, $C(\chi_n)$ can be seen to converge almost surely to a finite limit by considering for each $i$ in $\mathbf{I}$ the nonnegative supermartingale formed by multiplying $C(\chi_n)$ with the indicator function of the set $\{\chi_0 = i\}$ (see, for example, Section 4–3 in [10]). Hence, for $a \in \mathbf{A}$ one can choose integer $m$ so that on a subset of positive probability of $\Delta^a$,

$$\lim_{n\to\infty} C \wedge m(\chi_n) < m$$

where $C \wedge m$, the minimum of $C$ and $m$, is bounded and $\Phi$-excessive. So, by what has previously been shown, $C \wedge m(\chi_n)$, and therefore $C(\chi_n)$, has a finite constant limit $C(a)$ almost surely on $\Delta^a$. The equality in (2.10) for $C \wedge m$ becomes an inequality for $C$, by letting $m$ tend to infinity, after several applications of the Monotone Convergence Theorem and Fatou's Lemma.

Assume equality holds in (2.10) for all $\Phi$-excessive $C$. If for some $R$-regular $C$ we know that $\langle \Phi(t), C\rangle$ converges to zero as $t$ approaches infinity, then from (2.10) we see that $C$ equals $\sum_{a\in\mathbf{A}} C(a)L^a(\infty)$, which is bounded since $\mathbf{A}$ is a finite set. Conversely, if $C$ is $\Phi$-excessive then it is easy to check by (2.10) and Lemma (2.4) that

$$U = C - \lim_{t\to\infty} \langle \Phi(t), C\rangle + \langle G, QC\rangle$$

is an $R$-regular $\Phi$-potential. Assuming that $U$ is bounded, by equality (2.10)

$$U = \langle \Phi(t), U\rangle + \textstyle\sum_{a\in\mathbf{A}} U(a)L^a(t)\,,$$

and therefore

$$C = \langle \Phi(t), C\rangle - \textstyle\int_0^t \langle \Phi(u), QC\rangle\, du + \sum_{a\in\mathbf{A}} U(a)L^a(t)\,,$$

which yields by the inequality (2.10) for $C$,

$$C \geqq \textstyle\sum_{a\in\mathbf{A}} U(a)L^a(t) \geqq \sum_{a\in\mathbf{A}} C(a)L^a(t)\,.$$

Taking the limit at each $a \in \mathbf{A}$, we get $U(a)$ to equal $C(a)$ and equality must hold in (2.10) for $C$.

The following corollary complements Lemma (2.4) and gives the counterpart to the decomposition in [5] of a $\Phi$-excessive function into the sum of an invariant function, a "space" potential, and a "boundary" potential.

(2.12) Corollary. *Assume that $C$ is $\Phi$-excessive and that equality holds in* (2.10). *Then*

(i) $C = \lim_{t\to\infty} \langle \Phi(t), C\rangle + \langle G, -QC\rangle + \sum_{a\in\mathbf{A}} C(a)L^a(\infty)$;
(ii) *$C$ is $\Phi$-invariant if and only if $C$ is $R$-regular and $C(a) = 0$ for all $a \in \mathbf{A}$;*
(iii) *$C$ is a $\Phi$-potential if and only if $C - \sum_{a\in\mathbf{A}} C(a)L^a(\infty)$ is an $R$-potential.*

Proof. The Monotone Convergence Theorem applied to the equality (2.10) establishes (i). Assume that $C$ is $\Phi$-excessive (or $R$-superregular by Lemma (2.4)). Then $\Phi(t)C = C$ is equivalent to

$$\textstyle\int_0^t \langle \Phi(u), QC\rangle\, du = \sum_{a\in\mathbf{A}} C(a)L^a(t) = 0\,. \tag{2.13}$$

Since $\Phi(t)$ is a standard transition matrix, (2.13) will be true for all $t$ when and only when $QC = C(a) = 0$, $a \in \mathbf{A}$. And lastly, by (i) $C$ is a $\Phi$-potential if and only if

$$C = \langle G, -QC \rangle + \sum_{a \in \mathbf{A}} C(a) L^a(\infty) ,$$

where $\langle G, -QC \rangle = \langle N, C - RC \rangle$ is an $R$-potential.

**3. Boundary conditions.** In the treatment of excessive functions for non-minimal chains, we employ Chung's "canonical decomposition" ([2], [3]). Under the assumption that the exit boundary atoms are distinguishable, the transition matrix $P(t)$ is written

$$P(t) = \Phi(t) + \sum_{a \in \mathbf{A}} l^a * \xi^a(t) , \tag{3.1}$$

where $\xi_j{}^a(t) = \mathbf{P}(X(\tau^a + t) = j \mid \tau^a < \infty)$ is the distribution of the post-$\tau^a$ chain, which has state space

$$\mathbf{I}^a = \{ j \in \mathbf{I}_\theta : \xi_j{}^a(t) > 0,\ t > 0 \} .$$

If $\rho_j{}^a(t)$ is the probability that the post-$\tau^a$ chain at time $t$ is at state $j$ and has not yet reached $\mathbf{A} - \{a\}$, and if $F^{ab}(t)$ is the probability that, starting at $a$, the chain first switches to boundary atom $b$ before time $t$, then on $I_\theta$

$$\xi_j{}^a(t) = \rho_j{}^a(t) + \sum_{b \neq a} \int_0^t \xi_j{}^b(t - s)\, dF^{ab}(s) . \tag{3.2}$$

For each $a \in \mathbf{A}$, there is a canonical $\Phi$-entrance law $\eta^a(t)$, satisfying the entrance law equation

$$e(s)\Phi(t) = e(s + t) ,$$

and there is a measure $E^a(t)$ so that

$$\rho^a(t) = \int_0^t \eta^a(t - s)\, dE^a(s) . \tag{3.3}$$

Properties and interrelationships of these and other canonical quantities which are of importance to the present paper are too numerous to list here, and so as they are needed references will be made to [2] and [3]. However, one remaining important formula should be noted, namely, the one representing $\xi(t)$ in terms of $\rho(t)$:

$$\xi^{a_0}(t) = \sum_{n=0}^{\infty} \sum_{a_1, a_2, \cdots, a_n} \int_0^t \rho^{a_n}(t - s) d(F^{a_0 a_1} * \cdots * F^{a_{n-1} a_n})(s) . \tag{3.4}$$

The definitions of excessive, invariant, and potential functions for a non-minimal chain are the same as those for the minimal chain, with $P(t)$ in place of $\Phi(t)$. The following result is trivial, but it is of such frequent (and often implicit) use that it should be stated at this point.

(3.5) LEMMA. *If $e(t)$ is a $P$-entrance law and $C$ is a $P$-excessive function, then $\langle e(t), C \rangle$ is non-increasing as $t$ increases.*

The next theorem introduces a boundary condition for excessive functions analogous to that of [5]. We note that Pittenger, in (5.5) of [11], has presented in a much more general setting a boundary condition which, after some modification, is of a similar form and, in fact, is an exact equality for the sticky atom case. With a different approach, we shall obtain a condition, based on the behavior of the chain between the "change of banners," which is expressed in terms of the canonical quantities $\eta^a(\cdot)$ and $F^{ab}(\cdot)$ of [3], and which enables us in Section 4 to decompose functions which are excessive with respect to nonminimal chains. The constant $d^{ab}$ is defined in Section 17 of [3] as the probability that, starting from $a$, the last exit from $a$ before switching occurs in finite time, and is a switch to $b$. The symbol $\delta^a$ is 0 if $a$ is a recurrent trap and 1 otherwise.

(3.6) THEOREM. *C is P-excessive if and only if C is Φ-excessive and C satisfies the boundary condition*

$$(3.7) \qquad \delta^a C(a) - \textstyle\sum_{b\neq a} d^{ab}C(b) - \lim_{t\to 0}\langle \eta^a(t), C - C(a)L^a(\infty)\rangle \geqq 0$$

*for all* $a \in \mathbf{A}$.

PROOF. Assume $C$ is $P$-excessive. Since $\xi^b(\cdot)$ is a $P$-entrance law, $\langle \xi^b(t), C\rangle$ is non-increasing in $t$ for each $b \in \mathbf{A}$. Furthermore, by (3.1) $C$ is $\Phi$-excessive, and also

$$C \geqq l^a * \langle \xi^a(\cdot), C\rangle(t) \geqq L^a(t)\langle \xi^a(t), C\rangle$$

for all $a \in \mathbf{A}$. Taking limits at $\mathbf{A}$ for $t > 0$, and using Fatou's Lemma for $t = 0$, we get for all $a \in \mathbf{A}$

$$(3.8) \qquad C(a) \geqq \langle \xi^a(t), C\rangle\,.$$

Fix $a \in \mathbf{A}$. From (3.2) and (3.8), we have that

$$C(a) - \langle \rho^a(t), C\rangle \geqq \textstyle\sum_{b\neq a}\langle \xi^b(\cdot), C\rangle * F^{ab}(t)\,,$$

and so, from Corollary 1 to Theorem 14.7 of [3],

$$(3.9) \qquad \{\delta^a C(a) - \langle \eta^a(\cdot), C - C(a)L^a(\infty)\rangle\} * E^a(t) \geqq \textstyle\sum_{b\neq a}\langle \xi^b(\cdot), C\rangle * F^{ab}(t)\,.$$

Subtracting

$$\textstyle\sum_{b\neq a} C(b)[F^{ab}(\infty) - \sigma^{ab}(\cdot)] * E^a(t) = \sum_{b\neq a} F^{ab}(t)C(b)$$

(see Theorem 14.7 of [3]) from both sides of (3.9) yields

$$(3.10) \qquad \{\delta^a C(a) - \textstyle\sum_{b\neq a}[F^{ab}(\infty) - \sigma^{ab}(\cdot)]C(b) - \langle \eta^a(\cdot), C - C(a)L^a(\infty)\rangle\} * E^a(t) \geqq \sum_{b\neq a}\{\langle \xi^b(\cdot), C\rangle - C(b)\} * F^{ab}(t)\,.$$

For $b \in \mathbf{A}$, the function $\langle \xi^b(t), C\rangle$ increases, as $t$ tends to 0, to a limit which is no larger than $C(b)$ by (3.8). Also, $C - C(a)L^a(\infty)$ is $\Phi$-excessive, and so $\lim_{t\to 0}\langle \eta^a(t), C - C(a)L^a(\infty)\rangle$ also exists. Then dividing both sides of the inequality (3.10) by $E^a(t)$ and letting $t$ decrease to 0, we obtain

$$(3.11) \qquad \delta^a C(a) - \textstyle\sum_{b\neq a} d^{ab}C(b) - \lim_{t\to 0}\langle \eta^a(t), C - C(a)L^a(\infty)\rangle \geqq \sum_{b\neq a} d^{ab}\{\lim_{t\to 0}\langle \xi^b(t), C\rangle - C(b)\}\,,$$

where the last sum must be 0, as the following reasoning makes clear. If $b$ is nonsticky, then $d^{ab} = 0$ by Corollary 2 to Theorem 17.1 of [3]. And if $b$ is sticky, then

$$C(b) \geqq \lim_{t\to 0} \langle \xi^b(t), C\rangle \geqq \lim_{t\to 0} \langle \xi^b(t), L^b(\infty)C(b)\rangle = L^{bb}(\infty)C(b) = C(b)$$

by Theorem 17.2 of [3]. So (3.7) follows from (3.11).

Conversely, assume $\Phi$-excessive $C$ satisfies (3.7) for each $a \in A$. Since $C - \sum_{b\in A} C(b)L^b(\infty)$ is $\Phi$-excessive, we have

$$\lim_{t\to 0} \langle \eta^a(t), C - \textstyle\sum_{b\in A} C(b)L^b(\infty)\rangle = \sup_{t>0} \langle \eta^a(t), C - \textstyle\sum_{b\in A} C(b)L^b(\infty)\rangle .$$

And since $E^a(\cdot)$ is absolutely continuous for positive $t$,

$$\begin{aligned} 0 &\leqq E^a(t)\{\delta^a C(a) - \textstyle\sum_{b\neq a} d^{ab}C(b) - \lim_{t\to 0} \langle \eta^a(t), C - C(a)L^a(\infty)\rangle\} \\ &= E^a(t)\{\delta^a C(a) - \textstyle\sum_{b\neq a} F^{ab}(\infty)C(b) \\ &\qquad - \sup_{t>0} \langle \eta^a(t), C - \textstyle\sum_{b\in A} C(b)L^b(\infty)\rangle\} \\ &\leqq E^a(t)\{\delta^a C(a) - \textstyle\sum_{b\neq a} F^{ab}(\infty)C(b)\} \qquad (3.12) \\ &\qquad - \langle \eta^a(\cdot), C - \textstyle\sum_{b\in A} C(b)L^b(\infty)\rangle * E^a(t) \\ &= C(a)\{\delta^a + \sigma^{aa}(\cdot)\} * E^a(t) - \textstyle\sum_{b\neq a} C(b)\{F^{ab}(\infty) - \sigma^{ab}(\cdot)\} * E^a(t) \\ &\qquad - \langle \eta^a(\cdot), C\rangle * E^a(t) \\ &= C(a) - \textstyle\sum_{b\neq a} F^{ab}(t)C(b) - \langle \rho^a(t), C\rangle \end{aligned}$$

for each $a \in A$. Hence the telescoping series

$$\sum_{n=0}^{\infty} \sum_{a_1,\cdots,a_n} \int_0^t \{C(a_n) - \textstyle\sum_b F^{a_n b}(t-s)C(b)\} d(F^{aa_1} * \cdots * F^{a_{n-1}a_n})(s) ,$$

which has value $C(a)$, is no smaller than

$$\sum_{n=0}^{\infty} \sum_{a_1,\cdots,a_n} \int_0^t \langle \rho^{a_n}(t-s), C\rangle d(F^{aa_1} * \cdots * F^{a_{n-1}a_n})(s) ,$$

which equals $\langle \xi^a(t), C\rangle$, by (3.4) and the Tonelli Theorem, for each $a \in A$. So by (2.10),

$$C \geqq \langle \Phi(t), C\rangle + \textstyle\sum_{a\in A} C(a)L^a(t) \geqq \langle \Phi(t), C\rangle + \sum_{a\in A} l^a * \langle \xi^a(\cdot), C\rangle(t)$$

or

$$C \geqq \langle P(t), C\rangle$$

by (3.1).

(3.13) COROLLARY. *If $C$ is $\Phi$-excessive, then $C$ is $P$-excessive if and only if for all $a \in A$, $C(a) \geqq E^a\{C(X_t)\}$ for $t \geqq 0$.*

There are other equivalent expressions for the boundary condition (3.7) in Theorem (3.6) which are of some use and interest. One which we notice in the second part of the proof of that theorem is

$$\begin{aligned} (3.14) \qquad &\delta^a C(a) - \textstyle\sum_{b\neq a} F^{ab}(\infty)C(b) \\ &\qquad - \lim_{t\to 0} \langle \eta^a(t), C - \textstyle\sum_{b\in A} C(b)L^b(\infty)\rangle \geqq 0 . \end{aligned}$$

Another form resembles the type of equation which Dynkin ([6] Theorem 1.13) used to determine the range of the resolvent operator. The fundamental difference, as noted in [5], is that for excessive functions the boundary condition is an inequality. From (14.28), (17.9), and Theorem 14.5 of [3], and by (1.11),

$$\begin{aligned}\delta^a &= \delta^a(\textstyle\sum_{b \in \mathbf{A}} F^{ab}(\infty) + \lim_{t\to\infty} \langle \rho^a(t), 1\rangle) \\ &= \textstyle\sum_{b \in \mathbf{A}} F^{ab}(\infty) + \lim_{t\to 0} \langle \eta^a(t), 1 - L(\infty)\rangle \\ &= \textstyle\sum_{b \in \mathbf{A}} d^{ab} + \lim_{t\to 0} \langle \eta^a(t), 1 - L^a(\infty)\rangle\ .\end{aligned}$$

So (3.7) is equivalent to

$$\text{(3.15)} \qquad \textstyle\sum_{b \in \mathbf{A}} d^{ab}\{C(b) - C(a)\} + \lim_{t\to 0} \langle \eta^a(t), C - C(a)\rangle \leqq 0\ .$$

A final form, involving the normal derivative of Feller's ([8] Theorem 14.1) lateral conditions will be developed in the next section.

Some partial probabilistic results could now be derived from Theorem (3.6). However, a sharpening of this theorem will yield the more satisfying Corollary (3.21). The weakness of Theorem (3.6) is in not explicitly describing the behavior of $P$-excessive functions at the recurrent boundary atoms. Since these recurrent atoms also play a critical role in the next section, the following material is now developed.

To say a boundary atom $a$ in $\mathbf{A}$ is a recurrent non-trap for $P$ is equivalent to saying that $a$ is a recurrent state for the discrete parameter Markov chain $\{z_n;\ n \geqq 0\}$, discussed in Section 19 of [3], which has state space $\mathbf{A}$ and transition matrix $F = (F^{ab}(\infty))$. The notions of communication and recurrence class for boundary atoms will be defined, as usual, in terms of this chain (see, e.g., Sections I.3, I.4 of [4]). A recurrent boundary class $\mathbf{A}_1$ for $P$ then is either a singleton, containing a recurrent trap, or contains two or more elements which enjoy these properties: for any $a_1 \in \mathbf{A}_1$,

$$\text{(3.16)} \qquad \mathbf{A}_1 = \{b \in \mathbf{A}\colon F^n(a_1, b) > 0 \ \text{ for some } \ n \geqq 0\}\ ;$$

$$\text{(3.17)} \qquad F|_{\mathbf{A}_1 \times \mathbf{A}_1} \ \text{ is stochastic.}$$

Now from (3.3), (3.4), and (3.16), it is easy to see that the class of states

$$\mathbf{I}^{\mathbf{A}_1} = \textstyle\bigcup_{b \in \mathbf{A}_1} \{i\colon \eta_i^b(\cdot) \not\equiv 0\}$$

is equal to $\mathbf{I}^a$ for any $a \in \mathbf{A}_1$. For any $\Phi$-excessive $C$, the symbol $C(\mathbf{A}_1)$ will be used when and only when $C$ is constant on $\mathbf{A}_1$, in which case $C(\mathbf{A}_1)$ is defined to be this constant.

(3.18) THEOREM. *Let $\mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_m$ be any collection of recurrence classes of boundary atoms for $P$, and let $\mathbf{A}_0 = \mathbf{A} - \bigcup_{k=1}^m \mathbf{A}_k$. Then the class of $P$-excessive functions consists of all $\Phi$-excessive $C$ which satisfy*:

$$C = C(\mathbf{A}_k) \qquad on \qquad \mathbf{I}^{\mathbf{A}_k}\ , \qquad\qquad 1 \leqq k \leqq m\ ;$$

$$\delta^a C(a) - \textstyle\sum_{b\neq a} d^{ab} C(b) - \lim_{t\to 0} \langle \eta^a(t), C - C(a)L^a(\infty)\rangle \geqq 0\ ,$$

*for all $a \in \mathbf{A}_0$.*

PROOF. By Theorem 14.8 of [3] and (3.17) above, $\sigma^{ab}(0) = 0$ for $a \in \mathrm{A}_k$ and $b \notin \mathrm{A}_k$. So by (3.14), all we need to do is fix a $\Phi$-excessive $C$ and, for example, prove that

$$(3.19) \qquad \delta^a C(a) - \sum_{b \in \mathrm{A}_1} F^{ab}(\infty) C(b) - \lim_{t \to 0} \langle \eta^a(t), C - \sum_{b \in \mathrm{A}_1} C(b) L^b(\infty) \rangle \geqq 0$$

for all $a \in \mathrm{A}_1$ if and only if

$$(3.20) \qquad C = C(\mathrm{A}_1) \quad \text{on} \quad \mathrm{I}^{\mathrm{A}_1}.$$

If $\mathrm{A}_1$ is a singleton containing only the recurrent trap $a$, then (3.19) reduces to

$$\langle \eta^a(\cdot), C - C(a) L^a(\infty) \rangle = 0$$

since this function of $t$ is nonnegative and non-decreasing as $t$ decreases. But $L^a(\infty) = 1$ on $\mathrm{I}^a$ since $a$ is a recurrent trap, and so (3.19) and (3.20) are equivalent in this case.

Assume now that $\mathrm{A}_1$ contains more than one element, and assume (3.19) is true. Take $C(\underline{a})$ as the minimum value among the values $C(a)$ for $a \in \mathrm{A}_1$. By (3.17) and (3.19),

$$0 \leqq C(\underline{a}) - \sum_{b \in \mathrm{A}_1} F^{\underline{a}b}(\infty) C(b) \leqq C(\underline{a}) \{1 - \sum_{b \in \mathrm{A}_1} F^{\underline{a}b}(\infty)\} = 0,$$

and so necessarily $C(b) = C(\underline{a})$ if $F(\underline{a}, b) > 0$. It follows by induction that $C(b) = C(\underline{a})$ if $F^n(\underline{a}, b) > 0$ for some positive integer $n$. Hence, by (3.16), $C$ is constant on $\mathrm{A}_1$, and $C(\mathrm{A}_1)$ is well defined. By (3.17) and (3.19) again,

$$-\lim_{t \to 0} \langle \eta^a(t), C - \sum_{b \in \mathrm{A}_1} C(b) L^b(\infty) \rangle \geqq 0$$

for $a \in \mathrm{A}_1$. This inequality, as in the recurrent trap case, implies that

$$C = \sum_{b \in \mathrm{A}_1} C(b) L^b(\infty) = C(\mathrm{A}_1) L(\infty) = C(\mathrm{A}_1)$$

on

$$\mathrm{I}^{\mathrm{A}_1} = \bigcup_{b \in \mathrm{A}_1} \{i : \eta_i^b(\cdot) \not\equiv 0\}$$

since, by Theorem 12.2 of [3], $L(\infty) = 1$ on $\mathrm{I}^{\mathrm{A}_1}$. Conversely, if (3.20) is assumed to be true, then (3.19) follows since $\langle \eta^a(\cdot), C - C(\mathrm{A}_1) \rangle = 0$ for all $a \in \mathrm{A}_1$.

(3.21) COROLLARY. *If $a$ is a recurrent boundary atom, then for each P-excessive $C$, the boundary condition* (3.7) *is an equality and*

$$C(a) = \mathrm{E}^a\{C(X_t)\}, \qquad t > 0.$$

*If $a$ is a sticky boundary atom, then for each P-excessive $C$,*

$$C(a) = \lim_{t \to 0} \mathrm{E}^a\{C(X_t)\}.$$

*Conversely, if $a$ is nonrecurrent, then there exists a bounded R-regular, $\Phi$-potential $C$ satisfying*

$$C(a) > \mathrm{E}^a\{C(X_t)\}, \qquad t > 0.$$

*Furthermore, if $a$ is also nonsticky, then*

$$C(a) > \lim_{t \to 0} \mathrm{E}^a\{C(X_t)\}.$$

PROOF. The first statement follows from Theorem (3.18) since $C = C(a)$ on $\{i: \xi_i^a(\cdot) \not\equiv 0\}$ by (3.3), (3.4), and (3.17), while the proof of the second statement is contained in the proof of Theorem (3.6). From (3.14), (3.12), (3.2), and Corollary (3.13), we have for any $P$-excessive $C$

$$\begin{aligned} E^a(t)\{\delta^a C(a) - \textstyle\sum_{b\in A} F^{ab}(\infty)C(b) - \lim_{t\to 0}\langle \eta^a(t), C - \sum_{b\in A} C(b)L^b(\infty)\rangle\} \\ \leqq C(a) - \textstyle\sum_{b\in A} F^{ab}(t)C(b) - \langle \rho^a(t), C\rangle \\ \leqq C(a) - \textstyle\sum_{b\in A}\langle \xi^b(\cdot), C\rangle * F^{ab}(t) - \langle \rho^a(t), C\rangle \\ = C(a) - \langle \xi^a(t), C\rangle \end{aligned}$$

for all positive $t$. Obviously, the remainder of the corollary follows upon demonstrating in those two cases the existence of a bounded $R$-regular $\Phi$-potential $C$ which is $P$-excessive, and for which the boundary condition (3.14) is a strict inequality at $a$.

Let $A_1$ be the set of all $P$-recurrent atoms of $A$, and assume $A_0 = A - A_1$ is nonempty. From the theory of discrete time Markov chains, we have that $\lim_{n\to\infty} T^n = 0$, where $T$ is the substochastic matrix gotten by restricting $F$ to $A_0 \times A_0$. Then for fixed $a_0 \in A_0$, it is possible to choose a nonnegative integer $n$ so that

$$\textstyle\sum_{b\in A_0} T^{n+1}(a, b) \leqq \sum_{b\in A_0} T^n(a, b) \tag{3.22}$$

for $a \in A_0$, with strict inequality if $a = a_0$. Define $C$ to be $\sum_{a\in A_0} C(a)L^a(\infty)$ where $C(a)$ equals $\sum_{b\in A_0} T^n(a, b)$ for $a \in A_0$. Then the boundary conditions (3.14) for $C$ are just

$$\delta^a C(a) - \textstyle\sum_{b\in A} F^{ab}(\infty)C(b) \geqq 0 \tag{3.23}$$

for all $a \in A$. Now the inequality (3.23) is an equality if $a \in A_1$ by (3.17). And (3.22) implies (3.23) if $a \in A_0$, with strict inequality when $a = a_0$.

**4. A representation theorem.** As has already been mentioned, the boundary conditions for excessive functions turn out to be the boundary conditions used to characterize the range of the resolvent operator with equalities replaced by inequalities. In [5] and in the preceding section, the comparison was made with the Dynkin [6] boundary conditions. Here, in Lemma (4.2), the analogy will be made to the canonical boundary conditions for the range of the resolvent in [1], which involve the normal derivative introduced by Feller ([8] Theorem 11.1). In order to do this, the normal derivative must first be defined for excessive functions.

(4.1) LEMMA. *If $\eta(\cdot)$ is a $\Phi$-entrance law, and if $C$ is a $\Phi$-excessive function for which* (2.10) *is an equality, then* $-\infty \leqq [\eta, QC] \leqq 0$ *where*

$$[\eta, QC] = -\langle \eta(t), C\rangle + \textstyle\int_0^t \langle \eta(u), QC\rangle\, du + \sum_{b\in A}\langle \eta(t), L^b(\infty)\rangle C(b)$$

*is constant for* $t > 0$.

PROOF. Let $\bar{C} = C - \sum_{b \in A} C(b)L^b(\infty)$. From (1.12), (1.14), and the equality (2.10), we get

$$\bar{C} = \langle \Phi(t), \bar{C} \rangle - \int_0^t \langle \Phi(u), Q\bar{C} \rangle \, du .$$

Consider the nonpositive quantity

$$D(t) = -\langle \eta(t), \bar{C} \rangle + \int_0^t \langle \eta(u), Q\bar{C} \rangle \, du .$$

For $s \geqq 0$ and $t > 0$,

$$D(t+s) = -\langle \eta(t), \Phi(s)\bar{C} \rangle + \int_0^t \langle \eta(u), Q\bar{C} \rangle \, du + \int_t^{t+s} \langle \eta(u), Q\bar{C} \rangle \, du ,$$

where, by the Tonelli Theorem, the last integral equals

$$\int_0^s \langle \eta(t+u), Q\bar{C} \rangle \, du = \langle \eta(t), \int_0^s \langle \Phi(u), Q\bar{C} \rangle \, du \rangle .$$

So

$$\begin{aligned} D(t+s) &= -\langle \eta(t), \Phi(s)\bar{C} - \int_0^s \langle \Phi(u), Q\bar{C} \rangle \, du \rangle + \int_0^t \langle \eta(u), Q\bar{C} \rangle \, du = D(t) \\ &= -\langle \eta(t), C \rangle + \textstyle\sum_{b \in A} \langle \eta(t), L^b(\infty) \rangle C(b) \\ &\qquad + \int_0^t \langle \eta(u), QC \rangle \, du = [\eta, QC] . \end{aligned}$$

(4.2) LEMMA. *A $\Phi$-excessive $C$ for which* (2.10) *is an equality is $P$-excessive if and only if*

$$\delta^a C(a) - \textstyle\sum_{b \in A} F^{ab}(\infty)C(b) + [\eta^a, QC] \geqq 0 \tag{4.3}$$

*for all $a \in A$.*

PROOF. If $C$ is $P$-excessive, then by two applications of Fatou's Lemma, the Tonelli Theorem, and by (2.10),

$$\begin{aligned} \int_0^t \langle \eta^a(u), -QC \rangle \, du &\leqq \liminf_{s \to 0} \int_0^t \langle \eta^a(u+s), -QC \rangle \, du \\ &\leqq \liminf_{s \to 0} \langle \eta^a(s), \int_0^t \langle \Phi(u), -QC \rangle \, du \rangle \\ &\leqq \lim_{s \to 0} \langle \eta^a(s), C - C(a)L^a(\infty) \rangle , \end{aligned}$$

which is finite by (3.7). Also, since this last limit is an increasing limit, $\langle \eta^a(t), C - C(a)L^a(\infty) \rangle$ is finite for all $t > 0$, as is $\langle \eta^a(t), C(a)L^a(\infty) \rangle$, or $C(a)\sigma^{aa}(t)$, and hence so is their sum $\langle \eta^a(t), C \rangle$. Then by Lemma (4.1), $[\eta^a, QC]$ is finite, so

$$[\eta^a, QC] = -\lim_{t \to 0} \langle \eta^a(t), C - \textstyle\sum_{b \in A} C(b)L^b(\infty) \rangle , \tag{4.4}$$

and (4.3) follows from (3.14). Conversely, if (4.3) holds for all $a \in A$, then $[\eta^a, QC]$ necessarily is finite for all $a \in A$, and (3.14) follows from (4.4).

(4.5) THEOREM. *Let $A_0$ be the set of all nonrecurrent atoms for $P$. If $C$ is a $P$-excessive function for which* (2.10) *is an equality, then for $t > 0$,*

$$C = \langle P(t), C \rangle - \int_0^t \langle P(u), QC \rangle \, du + \textstyle\sum_{a \in A_0} \int_0^t C(a, t-u) l^a(u) \, du$$

*where, for $a \in A_0$,*

$$C(a, t) = \textstyle\sum_{n=0}^{\infty} \sum_{a_1, \cdots, a_n \in A_0} B(a_n) \int_0^t E^{a_n}(t-s) d(F^{aa_1} * \cdots * F^{a_{n-1}a_n})(s) ,$$

*and $B(a)$ is the nonnegative quantity $\delta^a C(a) - \sum_b F^{ab}(\infty)C(b) + [\eta^a, QC]$.*

PROOF. From Lemma (4.1), we know that, for $0 \leqq s < t$,

$$[\eta^a, QC] = -\langle \eta^a(t-s), C\rangle + \int_0^{t-s} \langle \eta^a(u), QC\rangle\, du + \sum_{b\in A} \sigma^{ab}(t-s)C(b)\,.$$

Then since $E^a(t)$ is absolutely continuous for positive $t$, as in (3.12),

$$(4.6)\qquad \begin{aligned} B(a)E^a(t) &= \int_0^t \{\delta^a C(a) - \sum_{b\in A} F^{ab}(\infty)C(b) + [\eta^a, QC]\}\, dE^a(s) \\ &= C(a) - \sum_{b\in A} F^{ab}(t)C(b) - \langle \rho^a(t), C\rangle + \int_0^t \langle \rho^a(u), QC\rangle\, du \end{aligned}$$

is nonnegative and equals zero if $a \notin A_0$ by Corollary (3.21). Then a simple computation, of the type used in the last part of the proof of Theorem (3.6), using (3.4) and (3.17) and the equality (4.6), yields

$$C(a, t) = C(a) - \langle \xi^a(t), C\rangle + \int_0^t \langle \xi^a(u), QC\rangle\, du\,,$$

a quantity which is nonnegative and equals 0 unless $a \in A_0$. The proof is now complete since (3.1) and (2.10) allow us to write

$$\begin{aligned} C = \langle P(t), C\rangle &- \int_0^t \langle P(u), QC\rangle\, du \\ &+ \sum_{a\in A_0} \int_0^t \{C(a) - \langle \xi^a(t-u), C\rangle + \int_0^{t-u} \langle \xi^a(s), QC\rangle\, ds\} l^a(u)\, du\,. \end{aligned}$$

The next corollary is the multiple boundary atom analogue to the representation of Section 10 in [5], with the important exception that here, $P$ is not assumed to be transient. This result is the extension of the representation in Corollary (2.12) to functions which are excessive with respect to a nonminimal chain.

(4.7) COROLLARY. *Any P-excessive function C for which* (2.10) *is an equality can be written as the sum of a P-invariant function, a P-potential function, and a $\Phi$-potential function. In the notation of Theorem* (4.5),

$$(4.8)\qquad C = \lim_{t\to\infty} \langle P(t), C\rangle - \langle \int_0^\infty P(u)\, du, QC\rangle + \sum_{a\in A_0} C(a, \infty)L^a(\infty)\,,$$

*where on* $A_0$, $I_0$ *is the identity matrix and*

$$C(\cdot, \infty) = \{I_0 - F(\infty)|_{A_0\times A_0}\}^{-1}B\,.$$

PROOF. The decomposition (4.8) follows from Theorem (4.5) by the Monotone Convergence Theorem and the fact that $E^a(\cdot)$ is a probability measure when $a$ is not a recurrent trap.

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