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CONVERGENCE TO TOTAL OCCUPANCY IN AN INFINITE PARTICLE SYSTEM WITH INTERACTIONS[1]

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Let $p(x, y)$ be the transition function for an irreducible, positive recurrent, reversible Markov chain on the countable set S. Let η_t be the infinite particle system on S with the simple exclusion interaction and one-particle motion determined by p. The principal result is that there are no nontrivial invariant measures for η_t which concentrate on infinite configurations of particles on S. Furthermore, it is proved that if the system begins with an arbitrary infinite configuration, then it converges in probability to the configuration in which all sites are occupied.

1. Introduction. Let $p(x, y)$ be the transition function for a Markov chain on the countable set S. In [5], Spitzer introduced the infinite particle system on S with the simple exclusion interaction as the strong Markov process with state space $X = \{0, 1\}^S$ whose infinitesimal generator Ω is obtained in the following way. Define Ωf for functions f in $C(X)$ which depend on only finitely many coordinates by

$$\Omega f(\eta) = \sum_{\eta(x)=1,\eta(y)=0} p(x, y)[f(\eta_{xy}) - f(\eta)], \tag{1.1}$$

where η_{xy} denotes the configuration which satisfies

$$\begin{aligned} \eta_{xy}(u) &= \eta(u) && \text{if} \quad u \neq x, y \\ &= \eta(x) && \text{if} \quad u = y \\ &= \eta(y) && \text{if} \quad u = x . \end{aligned}$$

It was proved in [2] that this operator has a closure which is the generator of a strong Markov process on X, provided that p satisfies

$$\sup_y \sum_x p(x, y) < \infty . \tag{1.2}$$

This assumption will be made throughout the entire paper.

This process describes the behavior of particles on S whose basic motion is that of independent continuous time Markov chains with transition probabilities given by $p(x, y)$ and exponential waiting times with constant parameter one. Superimposed on this motion is the simple exclusion interaction, which prohibits transitions to occupied sites. A particle which attempts a transition to an occupied site is required to remain where it is until it next attempts a transition.

The ergodic theory of the process η_t was studied by Spitzer and the present

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author in [3], [4], and [6] under the assumption that $p(x, y)$ is irreducible and symmetric. The results obtained there are of two types. First, the class of invariant measures of η_t is described in terms of the functions on S which are harmonic for p. Then necessary and sufficient conditions on the initial distribution of the process are given in order to have convergence to any of the extreme invariant measures.

The symmetry of p played a crucial role in these considerations. One of the key places at which this assumption was used was in the reduction of the study of the infinite particle system to the study of a corresponding problem involving only finitely many interacting particles. Part of the motivation for the present paper is to study ergodic properties of the process η_t in a context in which the result which permits this reduction (e.g., Theorem 1.1 of [3]) fails to hold. A further point of interest is that the behavior of the process in the present context is quite different from what it is in the symmetric case.

Our concern here is with the case that $p(x, y)$ is positive recurrent. If S is infinite and p is irreducible, then p cannot be both symmetric and positive recurrent, so there is no overlap with the previously studied case. In order to state the main result, let $\mathscr{I}$ be the set of all invariant (probability) measures of η_t, and let $\mathscr{I}_e$ be the set of extreme points of $\mathscr{I}$. Let μ_0 and μ_∞ be the point masses on the configurations in which, respectively, all sites are vacant or all sites are occupied. Then $\mu_0 \in \mathscr{I}_e$. If p is positive recurrent, and has (finite) stationary distribution $\{\pi(x), x \in S\}$, then it is called reversible provided that $\pi(x)p(x, y) = \pi(y)p(y, x)$ for all $x, y \in S$. Examples of reversible, positive recurrent Markov chains are provided by positive recurrent birth and death chains on the nonnegative integers.

(1.3) THEOREM. *Assume that p is positive recurrent, irreducible, reversible and satisfies* (1.2). *Then*

(a) *For each positive integer*, n, *there is a unique* $\mu_n \in \mathscr{I}$ *which satisfies* $\mu_n\{\eta \mid \sum_x \eta(x) = n\} = 1$.

(b) $\mathscr{I}_e = \{\mu_n \mid 0 \leqq n \leqq \infty\}$.

(c) *If* $\eta \in X$ *and* $\sum_x \eta(x) = \infty$, *then for all* $x \in S$,

$$\lim_{t\to\infty} P^\eta[\eta_t(x) = 1] = 1 .$$

If the initial configuration of η_t is finite (say of size n), then the process is simply an irreducible Markov chain on the set of all subsets of S of size n, so (a) says merely that this chain is positive recurrent. The ordinary convergence theorem for positive recurrent Markov chains then says that the process converges in distribution to μ_n whenever the initial configuration is of size n. Therefore (c) describes the ergodic behavior of the process in the only nontrivial case.

Section 2 is devoted to the proofs of several monotonicity results which permit the reduction of parts (b) and (c) of Theorem 1.3 to problems involving only finitely many particles. These are independent of the assumption of reversibility. Theorem 1.3 is proved in Section 3 by using the additional observation that if p is reversible, it is possible to exhibit explicitly a collection of measures in $\mathscr{I}$ which concentrate on $\{\eta \in X \mid \sum_x \eta(x) < \infty\}$.

We conjecture that Theorem 1.3 is true without the assumption of reversibility. The proof in that generality will be more difficult, because it appears to be impossible in the nonreversible case to produce an explicit collection of invariant measures for the process. In Section 4, however, Theorem 1.3 will be proved by a different technique for a class of nonreversible cases. In Section 5, some comments are made concerning the general nonreversible case. A conjecture concerning Markov chains which arises in this connection is also stated there.

2. The general case. Let $S(t)$ be the semigroup of contractions on $C(X)$ whose generator is Ω. Let

$$\mathscr{M} = \{f \in C(X) \mid \zeta, \eta \in X, \zeta \leqq \eta \Rightarrow f(\zeta) \leqq f(\eta)\},$$

where the inequality relating ζ and η is interpreted coordinatewise. The following monotonicity result is similar to ones obtained in [1] for a different model of an infinite particle system.

(2.1) THEOREM. *If $f \in \mathscr{M}$ and $t \geqq 0$, then $S(t)f \in \mathscr{M}$.*

PROOF. Since $\mathscr{M}$ is closed and $S(t)f = \lim_{n\to\infty} (I - (t/n)\Omega)^{-n} f$ for each $f \in C(X)$, it suffices to prove that if $f \in \mathscr{D}(\Omega)$, $f - \lambda\Omega f = g$, $\lambda \geqq 0$, and $g \in \mathscr{M}$, then $f \in \mathscr{M}$ also. Take such an f, g, and λ. Since f is continuous, it suffices to prove that $\zeta \leqq \eta \Rightarrow f(\zeta) \leqq f(\eta)$ for finite configurations ζ and η. For fixed $n \geqq 1$, let

$$\mathscr{A} = \{\eta \in X \mid \textstyle\sum_x \eta(x) = n\},$$

$\mathscr{B}$ be the collection of all bounded functions on $\mathscr{A}$ with the supremum norm, and

$$\mathscr{C} = \{h \in \mathscr{B} \mid h(\eta) \geqq f(\zeta) \text{ whenever } \eta \geqq \zeta\}.$$

$\mathscr{C}$ is nonempty, since it contains all sufficiently large constant functions. Define a transformation T on $\mathscr{B}$ by

$$Th(\eta) = (1 + \lambda n)^{-1}\{g(\eta) + \lambda \textstyle\sum_{\eta(x)=1,\eta(y)=0} p(x, y)h(\eta_{xy}) + \lambda h(\eta) \sum_{\eta(x)=1,\eta(y)=1} p(x, y)\}.$$

Then $||Th_1 - Th_2|| \leqq \lambda n(1 + \lambda n)^{-1}||h_1 - h_2||$, so T is a strict contraction. Since $f - \lambda\Omega f = g$, the unique fixed point of T is given by the restriction of f to $\mathscr{A}$. Therefore, it suffices to prove that T leaves $\mathscr{C}$ invariant. In order to do this, consider an $h \in \mathscr{C}$. If $\eta \in \mathscr{A}$ and $\eta \geqq \zeta$, then $g(\eta) \geqq g(\zeta)$, $h(\eta) \geqq f(\zeta)$, and $h(\eta_{xy}) \geqq f(\zeta_{xy})$ for all x and $y \in S$. It follows from these inequalities that $Th(\eta) \geqq f(\zeta)$, and therefore that $Th \in \mathscr{C}$, which completes the proof.

REMARK. The proof of Theorem 2.1 could also be accomplished by constructing a Markov process (ζ_t, η_t) on $\{(\zeta, \eta) \in X \times X \mid \zeta \leqq \eta\}$ in such a way that both marginals ζ_t and η_t are Markovian with generator Ω. The desired result would follow immediately from this construction since $S(t)f(\eta) = E^\eta[f(\eta_t)]$. This approach is similar to that used in [1], and is used also in the case of finite configurations in the proofs of Lemmas 4.2 and 4.6.

(2.2) COROLLARY. *If* $\zeta, \eta \in X$, $\zeta \leqq \eta$, *and* $x_1, \cdots, x_n \in S$, *then*

(a) $P^\zeta[\eta_t(x_i) = 1$ *for* $1 \leqq i \leqq n] \leqq P^\eta[\eta_t(x_i) = 1$ *for* $1 \leqq i \leqq n]$ *for all* $t \geqq 0$, *and hence*

(b) $\liminf_{t\to\infty} P^\zeta[\eta_t(x_i) = 1$ *for* $1 \leqq i \leqq n] \leqq \liminf_{t\to\infty} P^\eta[\eta_t(x_i) = 1$ *for* $1 \leqq i \leqq n]$.

PROOF. This follows from an application of Theorem 2.1 to the function $f(\eta) = \prod_{i=1}^n \eta(x_i)$, which is in $\mathscr{M}$.

Assume now that p is irreducible. The process η_t restricted to $\{\eta \in X \mid \sum_x \eta(x) = n\}$ is then an irreducible Markov chain (see, for example, the proof of Lemma 2.1 of [4]), which will be referred to as the n-particle process. Whenever it is positive recurrent, let μ_n be its (unique) stationary distribution.

(2.3) COROLLARY. *If p is irreducible and the n-particle process is positive recurrent for each (finite) n, then*

$$\mu_n\{\eta \in X \mid \eta(x) = 1\} \leqq \mu_{n+1}\{\eta \in X \mid \eta(x) = 1\}$$

for each $x \in S$.

PROOF. Let $\zeta, \gamma \in X$ satisfy $\zeta \leqq \gamma$, $\sum_x \zeta(x) = n$, and $\sum_x \gamma(x) = n + 1$. Then

$$\mu_n\{\eta \in X \mid \eta(x) = 1\} = \lim_{t\to\infty} P^\zeta[\eta_t(x) = 1],$$

and

$$\mu_{n+1}\{\eta \in X \mid \eta(x) = 1\} = \lim_{t\to\infty} P^\gamma[\eta_t(x) = 1].$$

Therefore the result follows from part (b) of Corollary 2.2.

(2.4) THEOREM. *Assume that p is irreducible and the n-particle process is positive recurrent for each (finite) n. Assume in addition that*

$$\lim_{n\to\infty} \mu_n\{\eta \in X \mid \eta(x) = 1\} = 1 \tag{2.5}$$

for each $x \in S$. *Then*

(a) *if* $\mu \in \mathscr{I}$ *and* $\mu\{\eta \in X \mid \sum_x \eta(x) = \infty\} = 1$, *then* $\mu = \mu_\infty$, *and*

(b) $\lim_{t\to\infty} P^\eta[\eta_t(x) = 1] = 1$ *for all* $x \in S$ *and* $\eta \in X$ *such that* $\sum_y \eta(y) = \infty$.

PROOF. Part (a) follows from part (b) and the bounded convergence theorem. To prove (b), let $\eta \in X$ with $\sum_x \eta(x) = \infty$. Fix $n \geqq 1$ and choose $\zeta \in X$ such that $\zeta \leqq \eta$ and $\sum_x \zeta(x) = n$. From part (b) of Corollary (2.2), it follows that

$$\liminf_{t\to\infty} P^\eta[\eta_t(x) = 1] \geqq \lim_{t\to\infty} P^\zeta[\eta_t(x) = 1] = \mu_n\{\gamma \in X \mid \gamma(x) = 1\}.$$

Since n is arbitrary, (b) now follows from (2.5).

3. The reversible case. Throughout this section, we will assume that p is irreducible, positive recurrent, and reversible, so that $\pi(x)p(x, y) = \pi(y)p(y, x)$ for all $x, y \in S$, where π is the stationary distribution of p. The reason that the reversible case is easier to deal with than is the general case, is that it is possible to compute the n-particle stationary distributions explicitly.

(3.1) THEOREM. *For $\rho > 0$, let μ_ρ be the product measure on X with marginals given by*

$$\mu_\rho\{\eta \in X \mid \eta(x) = 1\} = \frac{\rho\pi(x)}{1 + \rho\pi(x)}$$

for $x \in S$. Then

(a) $\mu_\rho \in \mathscr{I}$, *and*
(b) $\mu_\rho\{\eta \in X \mid \sum_x \eta(x) < \infty\} = 1$.

PROOF. A probability measure μ on X is in $\mathscr{I}$ if and only if $\int \Omega f \, d\mu = 0$ for all $f \in \mathscr{D}(\Omega)$. Since Ω is the closure of its restriction to functions depending on only finitely many coordinates, it suffices to verify this only for such functions f. So, suppose F is a finite subset of S, and f depends only on the coordinates in F. Then

$$\begin{aligned}(3.2) \qquad \Omega f(\eta) = {} & \textstyle\sum_{x \text{ or } y \in F} \eta(x)[1 - \eta(y)]p(x, y)f(\eta_{xy}) \\ & - \textstyle\sum_{x \text{ or } y \in F} \eta(x)[1 - \eta(y)]p(x, y)f(\eta),\end{aligned}$$

since the contributions to the sum in (1.1) which correspond to other pairs x, y vanish, and both sums in (3.2) converge by assumption (1.2). Let $\mu = \mu_\rho$ for some $\rho > 0$, and μ_{xy} be the probability measure on X obtained from μ by interchanging the x and y coordinates. Then

$$\begin{aligned}\int [\textstyle\sum_{x \text{ or } y \in F} & \eta(x)[1 - \eta(y)]p(x, y)f(\eta_{xy})] \, d\mu \\ & = \textstyle\sum_{x \text{ or } y \in F} p(x, y)[\int \eta(x)[1 - \eta(y)]f(\eta_{xy}) \, d\mu] \\ & = \textstyle\sum_{x \text{ or } y \in F} p(x, y)[\int \eta(y)[1 - \eta(x)]f(\eta) \, d\mu_{xy}] \\ & = \textstyle\sum_{x \text{ or } y \in F} p(y, x)[\int \eta(x)[1 - \eta(y)]f(\eta) \, d\mu_{xy}] .\end{aligned}$$

For fixed $x, y \in S$, let g be the function which depends only on coordinates in $F \backslash \{x, y\}$ such that

$$f(\eta) = g(\eta) \qquad \text{whenever} \quad \eta(x) = 1 \quad \text{and} \quad \eta(y) = 0 .$$

Then, since $\eta(x)$, $\eta(y)$, and $g(\eta)$ are independent relative to μ_{xy},

$$\begin{aligned}\int \eta(x)[1 - \eta(y)]f(\eta) \, d\mu_{xy} & = \int \eta(x)[1 - \eta(y)]g(\eta) \, d\mu_{xy} \\ & = \frac{\rho\pi(y)}{[1 + \rho\pi(y)][1 + \rho\pi(x)]} \int g(\eta) \, d\mu .\end{aligned}$$

Similarly,

$$\int \eta(x)[1 - \eta(y)]f(\eta) \, d\mu = \frac{\rho\pi(x)}{[1 + \rho\pi(x)][1 + \rho\pi(y)]} \int g(\eta) \, d\mu .$$

Therefore, since p is reversible, $\int \Omega f \, d\mu = 0$. To prove part (b), it suffices to note that

$$\int \sum_x \eta(x)\, d\mu_\rho = \sum_x \mu_\rho\{\eta \in X \mid \eta(x) = 1\} = \sum_x \frac{\rho\pi(x)}{1 + \rho\pi(x)},$$

which is finite since $\sum_x \pi(x) = 1$.

(3.3) COROLLARY. *The n-particle system is positive recurrent, and μ_n is given by*

$$\mu_n(A) = \mu_\rho(A, \textstyle\sum_x \eta(x) = n)/\mu_\rho(\sum_x \eta(x) = n). \tag{3.4}$$

PROOF. This follows immediately from the observations that

$$\mu_\rho\{\eta \in X \mid \textstyle\sum_x \eta(x) = n\} > 0 \qquad \text{for each} \quad n,$$

and that

$$\{\eta \in X \mid \textstyle\sum_x \eta(x) = n\}$$

is invariant for the process η_t.

(3.5) COROLLARY. *For all $x \in S$,*

$$\lim_{n\to\infty} \mu_n\{\eta \in X \mid \eta(x) = 1\} = 1.$$

PROOF. By Corollary 2.3, this limit exists. By (3.4) and part (b) of Theorem 3.1, for each $\rho > 0$

$$\begin{aligned}\frac{\rho\pi(x)}{1 + \rho\pi(x)} &= \mu_\rho\{\eta \in X \mid \eta(x) = 1\}\\ &= \textstyle\sum_{n=0}^{\infty} \mu_n\{\eta(x) = 1\}\mu_\rho\{\sum_x \eta(x) = n\}\\ &\leqq \lim_{n\to\infty} \mu_n\{\eta(x) = 1\}.\end{aligned}$$

Since $\lim_{\rho\to\infty} \rho\pi(x)/(1 + \rho\pi(x)) = 1$, the required result follows.

Theorem 1.3 is now an immediate consequence of Theorem 2.4 and Corollaries 3.3 and 3.5.

4. A nonreversible example. In this section, we take $S = \{0, 1, \cdots\}$; $p(0, x) = p_x$ where $\sum_x p_x = 1$, $\sum_x x p_x < \infty$, and $p_x > 0$ for infinitely many x; $p(x, x-1) = 1$ for $x \geqq 1$. This makes p positive recurrent, irreducible, and nonreversible. Our aim is to show that the conclusions of Theorem 1.3 hold for this class of examples.

(4.1) LEMMA. *If $\mu \in \mathscr{I}$, then*

$$\mu\{\eta \in X \mid \eta(x) = 1 \textit{ for infinitely many } x \textit{ and } \eta(x) = 0 \textit{ for infinitely many } x\} = 0.$$

PROOF. Since $\mu \in \mathscr{I}$, $\int \Omega f\, d\mu = 0$ for all $f \in \mathscr{D}(\Omega)$. For any $z \in S$, $f(\eta) = \sum_{x=0}^{z} \eta(x)$ depends on only finitely many coordinates, and hence is in $\mathscr{D}(\Omega)$. For this f,

$$\Omega f(\eta) = \eta(z+1)[1 - \eta(z)] - \eta(0) \textstyle\sum_{y>z} p_y[1 - \eta(y)], \qquad \text{so}$$

$$\begin{aligned}\mu\{\eta \mid \eta(z) = 0, \eta(z+1) = 1\} &= \textstyle\sum_{y=z+1}^{\infty} p_y \mu\{\eta \mid \eta(0) = 1, \eta(y) = 0\}\\ &\leqq \textstyle\sum_{y=z+1}^{\infty} p_y.\end{aligned}$$

Since $\sum_x xp_x < \infty$, it follows that

$$\sum_z \mu\{\eta \mid \eta(z) = 0, \eta(z+1) = 1\} < \infty ,$$

and the required conclusion follows immediately from the Borel–Cantelli lemma.

(4.2) LEMMA. *For each $n > 0$, there is a unique $\mu_n \in \mathscr{I}$ such that*

$$\mu_n\{\eta \mid \sum_x \eta(x) = n\} = 1 .$$

PROOF. The n-particle process is irreducible, so it suffices to prove that it is positive recurrent. The proof is by induction on n. For $n = 1$, it is true because $\sum_x xp_x < \infty$. In order to prove the induction step, assume that it is true for $n-1$ and false for n, where $n \geqq 2$. Then if η and ζ are configurations of size n,

$$\lim_{t\to\infty} P^\eta[\eta_t = \zeta] = 0 . \tag{4.3}$$

Let $(X_1(t), \cdots, X_n(t))$ be the Markov chain on $M = \{\mathbf{x} \in S^n \mid x_i \neq x_j \text{ for } i \neq j\}$ which moves according to the following rules:

(a) The chain has exponential waiting times with parameter $1/n$.

(b) At each transition, a coordinate is chosen at random—each with probability $1/n$—and if the ith one is chosen, then a point $u \in S$ is chosen according to the probabilities $p(x_i, \cdot)$.

(c) If $u \neq x_j$ for all $j \neq i$, the ith coordinate changes to u; if $u = x_j$ for some $j < i$, the vector $\mathbf{x}$ remains unchanged; if $u = x_j$ for some $j > i$, the ith and jth coordinates are interchanged. Then if $1 \leqq k \leqq n$, the motion of the set $\{X_1(t), \cdots, X_k(t)\}$ has the Markov property, and has the same transition probabilities as the k-particle process. By the induction assumption, the distribution of $(X_1(t), \cdots, X_{n-1}(t))$ remains relatively compact as $t \to \infty$, while by (4.3),

$$\lim_{t\to\infty} P^{\mathbf{x}}[X_1(t) = y_1, \cdots, X_n(t) = y_n] = 0$$

for all $\mathbf{x}, \mathbf{y} \in M$. Therefore

$$\lim_{t\to\infty} P^{\mathbf{x}}[X_n(t) = y] = 0 \tag{4.4}$$

for all $y \in S$. Now define a function g on M by

$$g(\mathbf{x}) = \sum_y p(x_n, y)(y - x_n) + \sum_{i=1}^{n-1} [p(x_i, x_n) - p(x_n, x_i)](x_i - x_n) .$$

This is just the generator of the chain $\mathbf{X}(t)$ applied to the function $f(\mathbf{x}) = x_n$, so

$$E^{\mathbf{x}}[X_n(t)] = x_n + \int_0^t E^{\mathbf{x}} g(X_1(s), \cdots, X_n(s))\, ds . \tag{4.5}$$

Note that g is bounded and satisfies

$$\lim_{x_n\to\infty} g(x_1, \cdots, x_n) = -1$$

for fixed $x_1, \cdots, x_{n-1}$. Therefore

$$\lim_{s\to\infty} E^{\mathbf{x}} g(X_1(s), \cdots, X_n(s)) = -1 ,$$

which contradicts (4.5), since $E^{\mathbf{x}}[X_n(t)] \geqq 0$.

(4.6) LEMMA. *If* $\mu \in \mathscr{I}$ *and* $\mu\{\eta \mid \eta(x) = 0$ *for at most finitely many* $x\} = 1$, *then* $\mu = \mu_\infty$.

PROOF. This proof is similar to that of Lemma 4.2—the main difference is that we concentrate on the vacant sites rather than on the occupied sites. If η_t begins in $\{\eta \in X \mid \eta(x) = 0$ for exactly n points $x \in S\}$, it remains in it for all $t > 0$. (In general, this is a consequence of (1.2).) Therefore it suffices to prove that if $\eta, \zeta \in X$ and $\sum_x [1 - \eta(x)] = \sum_x [1 - \zeta(x)] = n$, then

$$\lim_{t\to\infty} P^\eta[\eta_t = \zeta] = 0 . \tag{4.7}$$

In order to do this, consider the Markov chain $(Z_1(t), \cdots, Z_n(t))$ on $M = \{\mathbf{x} \in S^n \mid x_i \neq x_j \text{ for } i \neq j\}$ which has infinitesimal parameters given by

$$P^{\mathbf{z}}[Z_i(t) = z_i \quad \text{for} \quad i \neq j,\ Z_j(t) = u] = p(u, z_i)t + o(t)$$

if $1 \leqq j \leqq n$ and $u \neq z_i$ for all i,

$$P^{\mathbf{z}}[Z_i(t) = z_i \quad \text{for} \quad i \neq j, k,\ Z_j(t) = z_k,\ Z_k(t) = z_j] = p(z_k, z_j)t + o(t)$$

if $1 \leqq j < k \leqq n$, and $P^{\mathbf{z}}[\mathbf{Z}(t) = \mathbf{w}] = o(t)$ for all other choices of $\mathbf{w} \neq \mathbf{z}$. For every $1 \leqq k \leqq n$, the motion of the set $\{Z_1(t), \cdots, Z_k(t)\}$ has the Markov property, and has the same transition probabilities as the process obtained from η_t by following the set of empty sites, if the initial configuration η has exactly k empty sites. The Markov chain $Z_1(t)$ is transient, since $\prod_{x=1}^{\infty} (1 + p_x)^{-1} > 0$, so $\lim_{t\to\infty} P^z(Z_1(t) = w) = 0$ for all $z, w \in S$. Therefore,

$$P^{\mathbf{z}}[\{Z_1(t), \cdots, Z_n(t)\} = \{w_1, \cdots, w_n\}] \leqq \sum_{i=1}^{n} P^{\mathbf{z}}(Z_1(t) = w_i)$$

tends to zero as $t \to \infty$. It follows then from the identification of the chain $\mathbf{Z}(t)$ in terms of the set of empty sites for the process η_t that (4.7) holds.

In order to complete the proof of Theorem 1.3 in the present case, it suffices by Theorem 2.4 to prove the following result.

(4.8) THEOREM. *For all* $x \in S$,

$$\lim_{n\to\infty} \mu_n\{\eta \mid \eta(x) = 1\} = 1 .$$

PROOF. By Corollary 2.2, the functions $\mu_n\{\eta \mid \eta(x_i) = 1 \text{ for } 1 \leqq i \leqq k\}$ are monotonically increasing in n for $x_1, \cdots, x_k \in S$. This, together with the compactness of X, implies that there is a probability measure ν on X such that $\mu_n \to \nu$ weakly. Since $\mu_n \in \mathscr{I}$ for each n, $\nu \in \mathscr{I}$ also. If $f \in \mathscr{M}$, $\int f\, d\mu_n$ increases to $\int f\, d\nu$ by Theorem 2.1. Therefore, since the indicator function of $\{\eta \mid \sum_{x=0}^{z} \eta(x) \geqq k\}$ is in $\mathscr{M}$,

$$\nu\{\eta \mid \textstyle\sum_{x=0}^{z} \eta(x) \geqq k\} \geqq \mu_k\{\eta \mid \sum_{x=0}^{z} \eta(x) \geqq k\} .$$

Letting z tend to ∞, we see that $\nu\{\eta \mid \sum_x \eta(x) \geqq k\} = 1$ for each k, and therefore $\nu\{\eta \mid \sum_x \eta(x) = \infty\} = 1$. By Lemmas 4.1 and 4.6, $\nu = \mu_\infty$. But $\mu_\infty\{\eta \mid \eta(x) = 1\} = 1$ for any $x \in S$, so $\lim_{n\to\infty} \mu_n\{\eta \mid \eta(x) = 1\} = 1$.

5. Comments and conjectures. As mentioned in the introduction, we conjecture that Theorem 1.3 is true without the assumption of reversibility. Even the problem of proving that the n-particle process is positive recurrent is open in general. The idea used in Lemma 4.2 generalizes significantly by replacing the function $f(\mathbf{x}) = x_n$ by $f(\mathbf{x}) = E^{x_n}(\tau)$, where τ is the hitting time of a fixed point in S for the one-particle process. However, it does not appear to suffice for the general case. The proof that $\lim_{n\to\infty} \mu_n\{\eta \in X \mid \eta(x) = 1\} = 1$ will probably be harder, and the idea used in Lemma 4.1 seems not to generalize very much at all.

It is interesting to note what part (c) of Theorem 1.3 would say for special initial configurations η. Suppose for example that η has only one vacant site. Then the conclusion would be that the Markov chain obtained by following the empty site is either null recurrent or transient, but not positive recurrent. Therefore a special case of Theorem 1.3 in the general nonreversible situation is the following result which we conjecture to be true:

Suppose $q(x, y)$ satisfies

(i) $q(x, y) \geqq 0$ for $x, y \in S$,
(ii) $\sup_x \sum_y q(x, y) < \infty$, and
(iii) $\sup_y \sum_x q(x, y) < \infty$.

Let $X(t)$ and $Y(t)$ be the Markov chains on S with infinitesimal parameters given by $P^u(X(t) = v) = q(u, v)t + o(t)$, and $P^u(Y(t) = v) = q(v, u)t + o(t)$ for $u \neq v$. Then if S is infinite and the chains are irreducible, not both of the chains can be positive recurrent.

If S is finite, then of course both of the chains are positive recurrent. In this situation, we make a more quantitative conjecture which, if true, could serve as a means of proving the above:

Assume in addition to (i), (ii), (iii) that either

(iv) $\sum_y q(x, y)$ is independent of x, or
(v) $\sum_x q(x, y)$ is independent of y.

If S is finite and the two chains are irreducible with stationary distributions $\pi_1(x)$ and $\pi_2(x)$ respectively, then

$$\sum_x \pi_1(x)\pi_2(x) \leqq \frac{1}{n} \tag{5.1}$$

with equality holding if and only if both (iv) and (v) hold. If both (iv) and (v) hold, then $\pi_1(x) = \pi_2(x) = 1/n$, so (5.1) is immediate. In the reversible case. $\pi_1(x)\pi_2(x)$ is constant, so (5.1) is a consequence of the Schwartz inequality. Furthermore, direct computations lead to a verification of (5.1) if $n = 2$ or $n = 3$.

ADDED IN PROOF. Harry Kesten has provided the following example of a positive recurrent chain on the nonnegative integers for which the two-particle process is not positive recurrent. The transition probabilities are given by $p(x, x+1) = p(x, 0) = p_x$ and $p(x, x) = 1 - 2p_x$ where p_x is monotone decreasing, $0 < p_x \leqq \frac{1}{2}$, $\sum (2^x p_x)^{-1} < \infty$, and $\sum (r^x p_x)^{-1} = \infty$ for all $r < 2$.

REFERENCES

[1] HARRIS, T. (1974). Contact interactions on a lattice. *Ann. Probability* **2** 968–988.

[2] LIGGETT, T. (1972). Existence theorems for infinite particle systems, *Trans. Amer. Math. Soc.* **165** 471–481.

[3] LIGGETT, T. (1973). A characterization of the invariant measures for an infinite particle system with interactions. *Trans. Amer. Math. Soc.* **179** 433–453.

[4] LIGGETT, T. (1974). A characterization of the invariant measures for an infinite particle system with interactions II. *Trans. Amer. Math. Soc.* To appear.

[5] SPITZER, F. (1970). Interaction of Markov processes. *Adv. Math.* **5** 246–290.

[6] SPITZER, F. (1974). Recurrent random walk of an infinite particle system. *Trans. Amer. Math. Soc.* To appear.

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