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RECURRENT RANDOM WALKS ON COUNTABLE ABELIAN GROUPS OF RANK 2

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Let r_n be the probability that a recurrent random walk on a countable Abelian group fails to return to the origin in the first n steps. For two-dimensional walk, Kesten and Spitzer have shown that r_n is slowly varying. I.e. $\lim_{n\to\infty} r_{2n}/r_n = 1$. We strengthen this result and show that for any countable Abelian group of rank 2, r_n is super slowly varying in the sense that $\lim_{n\to\infty} r_{[nr_n]}/r_n = 1$. We use the superslow variation of r_n to obtain the limit law for the number of returns to the origin for all recurrent random walks on these groups.

1. Introduction. Let G be a countable Abelian group and $\{X_n, n \geqq 1\}$ a sequence of independent, identically distributed G-valued random variables. The sequence $\{S_n, n \geqq 1\}$ of partial sums of the $\{X_n\}$ ($S_1 = X_1$, $S_{n+1} = X_{n+1} + S_n$) is said to be a random walk on G. We let $P_n(x, y) = P(S_n = y - x)$,

$$f_n = P(S_1 \neq 0, S_2 \neq 0, \cdots, S_{n-1} \neq 0, S_n = 0) \qquad \text{and} \qquad r_n = 1 - \sum_{k=1}^n f_k .$$

A random walk on G is said to be recurrent if $\sum_{n=1}^\infty f_n = 1$ (equivalently: $r_n \to 0$ as $n \to \infty$). The walk is said to be *aperiodic* (with respect to G) if $\{x \mid P_1(0, x) \neq 0\}$ generates G. Dudley [2] has shown that there exist aperiodic recurrent random walks on a countable Abelian group G if and only if the rank of G is less than or equal to 2.

Recurrent random walks which are aperiodic with respect to groups isomorphic to Z^2 exhibit a remarkable homogeneity in their behavior. Kesten and Spitzer [7] have shown that in this case r_n is slowly varying, i.e. $\lim_{n\to\infty} r_{2n}/r_n = 1$. More recently Jain and Pruitt [6] have shown that the strong law of large numbers holds for the range of all such walks.

These results both follow from the condition

$$(*) \qquad \sup_{x<G} P_n(0, x) = O\left(\frac{1}{n}\right)$$

which holds for all such walks. We show in Theorem A of Section 2 that this homogeneity of behavior extends to all recurrent walks which are aperiodic with respect to Abelian groups of rank 2 by showing that (*) holds in this generality.

We prove (Theorem B of Section 3) that for recurrent walks on a countable Abelian group which satisfy (*), r_n is *super slowly varying* in the sense that $\lim_{n\to\infty} r_{[nr_n]}/r_n = 1$. We remark that there are slowly varying functions which

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are not super slowly varying, e.g. $e^{-(\log n)^{\frac{1}{2}}}$. Thus Theorem B strengthens the aforementioned result of Kesten and Spitzer.

We use Theorem B to prove Theorem C of Section 4 stating that for recurrent walks on a countable Abelian group which satisfy (*), $\lim_{n\to\infty} P(r_n \mathscr{R}_n < t) = 1 - e^{-t}$ $(t > 0)$, $\mathscr{R}_n$ equalling the number of times the walk visits the origin 0 during the first n steps.

Theorem C extends the results of Erdös and Taylor ([3] pages 141–143) for simple walk on Z^2 and Spitzer ([9] page 231) for aperiodic two-dimensional walk with mean zero and finite second moment. We remark that the same result may be obtained using only the slow variation of r_n, by employing the occupation time theorem of Darling and Kac [1] (although this possibility seems to have been overlooked in the literature).

In view of Theorem A, Theorems B and C apply to all recurrent walks which are aperiodic with respect to Abelian groups of rank 2. We end this paper with examples of recurrent random walks on groups of rank 0 and 1 for which (*) holds and to which these theorems apply as well. We believe these examples are of some interest in connection with the open questions concerning the range of recurrent random walks, for in view of the proof of Jain and Pruitt's strong law of large numbers for recurrent walks on Z^2 the strong law of large numbers would apply in these instances as well.

2. Condition (*) for recurrent walks on groups of rank 2. We say that a random walk on a countable Abelian group G has *genuine rank two* if $\{x \in G \mid P_1(0, x) \neq 0\}$ generates a subgroup of G of rank two.

THEOREM A. *If $\{S_n\}$ is a recurrent random walk of genuine rank two then*

$$(*) \qquad \sup_{x\in G} P_n(0, x) = O\left(\frac{1}{n}\right).$$

REMARK. Suppose that G has a subgroup H such that $G/H \cong Z^2$ (it is easily shown that this is equivalent to $G \cong Z^2 \oplus T$ where T is a countable Abelian torsion group). For these groups the (*) condition for G follows trivially from the (*) condition for Z^2 by projecting the walk on G onto a walk on Z^2. Not every countable Abelian group of rank two is however of this type (e.g. Q^2, Q being the additive group of rationals) and for the latter groups the simple projection argument breaks down. For such groups, we provide the following

PROOF. Let G_1 be the group generated by $\{x \in G \mid P_1(0, x) \neq 0\}$, and let μ be the measure on G_1 defined by

$$\mu(x) = P_1(0, x).$$

Let ν be the measure on G_1 defined by

$$\nu(x) = \sum_{y\in G_1} \mu(y)u(y - x).$$

It is easily shown that the recurrence of the walk implies that the support of ν generates a subgroup G_1' of G_1 of rank two.

Let G_1 be endowed with the discrete topology. Let Γ be the dual group of G_1 and $P_\Gamma(\gamma)$ the Haar measure on Γ. Recall that if m is a probability measure on G_1 the characteristic function of m is the function $\hat{m}$ defined on Γ by $\hat{m}(\gamma) = \sum_{x \in G_1} (\gamma, x)m(x)$, $\forall \gamma \in \Gamma$. We may assume ([8] page 22) that the Haar measure on Γ has been normalized so that the Fourier inversion theorem $m(x) = \int_\Gamma \overline{(\gamma, x)}\hat{m}(\gamma)\, dP_\Gamma(\gamma)$, $\forall x \in G_1$, holds.

Now

$$\hat{\nu}(\gamma) \equiv |\hat{\mu}(\gamma)|^2 \qquad \text{and}$$

$$\begin{aligned}\sup_{x \in G} P_{2n}(0, x) &= \sup_{x \in G_1} \int_\Gamma \overline{(\gamma, x)}(\hat{\mu}(\gamma))^{2n}\, dP_\Gamma(\gamma) \\ &\leqq \int_\Gamma |\hat{\mu}(\gamma)|^{2n}\, dP_\Gamma(\gamma) = \int_\Gamma (\hat{\nu}(\gamma))^n\, dP_\Gamma(\gamma)\,.\end{aligned}$$

Also, $\sup_{x \in G} P_{2n+1}(0, x) \leqq \sup_{x \in G} P_{2n}(0, x)$, so that to prove (*) it suffices to show that $\int_\Gamma (\hat{\nu}(\gamma))^n\, dP_\Gamma(\gamma) = O(1/n)$.

As, G_1' has rank two, we may find $x_1, x_2 \in G_1$, such that $\nu(x_1) > 0$, $\nu(x_2) > 0$, and the subgroup H of G_1 generated by $\{x_1, x_2\}$ is isomorphic to Z^2. $\hat{\nu}(\gamma)$ is real and nonnegative so that

$$\begin{aligned}0 \leqq \hat{\nu}(\gamma) &= \textstyle\sum_{x \in G_1} \operatorname{Re}(\gamma, x)\nu(x) \\ &\leqq \nu(x_1)\operatorname{Re}(\gamma, x_1) + \nu(x_2)\operatorname{Re}(\gamma, x_2) + (1 - \nu(x_1) - \nu(x_2))\,.\end{aligned}$$

We write $\Psi(\gamma)$ in place of the last expression and conclude that

$$0 \leqq \int_\Gamma (\hat{\nu}(\gamma))^n\, dP_\Gamma(\gamma) \leqq \int_\Gamma (\Psi(\gamma))^n\, dP_\Gamma(\gamma)\,.$$

We now let $\Gamma_1 = \{\gamma \in \Gamma \mid \gamma \equiv 1 \text{ on } H\}$ = the annihilator of H. It follows from the Pontryagin duality theorem ([8] page 35) that Γ/Γ_1 is (isomorphic to) the character group of H and it is thus isomorphic to T^2, the torus. Writing $dP_{\Gamma/\Gamma_1}(\gamma + \Gamma_1)$ for the Haar measure on the quotient group Γ/Γ_1 and $dP_{\Gamma_1}(\gamma_1)$ for the Haar measure on Γ_1, we may assume ([8] page 54) that the Haar measures have been adjusted so that

$$\int_\Gamma (\Psi(\gamma))^n\, dP_\Gamma(\gamma) = \int_{\Gamma/\Gamma_1} dP_{\Gamma/\Gamma_1}(\gamma + \Gamma_1) \int_\Gamma (\Psi(\gamma + \gamma_1))^n\, dP_{\Gamma_1}(\gamma_1)\,. \tag{2.1}$$

Observe now that $(\gamma + \gamma_1, x) = (\gamma, x)\cdot(\gamma_1, x)$, and, for $x \in H$, $(\gamma_1, x) = 1$ so that $\Psi(\gamma + \gamma_1) = \Psi(\gamma)$ for all $\gamma_1 \in \Gamma_1$. Thus $\Psi(\gamma + \gamma_1)$ is constant on any given coset $\gamma + \Gamma_1$ and we denote this constant value as $\Psi(\gamma + \Gamma_1)$. It follows then from (2.1) that

$$\int_\Gamma (\Psi(\gamma))^n\, dP_\Gamma(\gamma) = c\int_{\Gamma/\Gamma_1} (\Psi(\gamma + \Gamma_1))^n\, dP_{\Gamma/\Gamma_1}(\gamma + \Gamma_1)$$

where $c = \int_{\Gamma_1} dP_{\Gamma_1}(\gamma_1) < \infty$. The last integral, however, is in effect an integral over T^2, and the argument employed by Spitzer ([9] page 73) to show that $\sup_{x \in Z^2} P_n(0, x) = O(1/n)$ for genuinely two dimensional recurrent walk applies directly to this integral, and shows that it is $O(1/n)$.

3. Super slow variation of r_n. We will need as lemmas the slow variation of r_n and an inequality employed in Spitzer and Kesten's proof of slow variation. We state these below as Lemmas 1 and 2; the proofs may be found in ([7] pages

307–309) under the assumption that the walk is "genuinely two dimensional." The reader may observe, however, that the only way in which this hypothesis enters the argument is through (*).

LEMMA 1. *If a recurrent random walk* $\{S_n\}$ *satisfies* (*) *then*

$$\exists c_1 \ni \text{if} \quad m_1 + m_2 + m_3 = m \quad (m_1, m_2, m_3 > 0)\,, \qquad \text{then} \quad f_m \leqq c_1 m_1^{-1} r_{m_2} r_{m_3}\,.$$

LEMMA 2. *If* $\{S_n\}$ *has property* (*) *then* r_n *is slowly varying.*

THEOREM B. *If* $\{S_n\}$ *is a recurrent random walk on a discrete Abelian group* G *which satisfies* (*) *then* r_n *is super slowly varying.*

PROOF. Given m, we apply Lemma 1 taking $m_2 = m_3 = [m/4]$, $m_1 = m - m_2 - m_3$ to obtain

$$f_m \leqq c_1 m_1^{-1} r^2_{[m/4]} \leqq c_2 m^{-1} r^2_{[m/4]}\,.$$

Since r_m is slowly varying $r^2_{[m/4]} \sim r_m{}^2$ so that $f_m \leqq c_3 m^{-1} r_m{}^2$. In particular, for $n \leqq m \leqq 2n$ we have $f_m \leqq c_3 n^{-1} \cdot r_n{}^2$, thus

$$r_n - r_{2n} = \textstyle\sum_{n+1}^{2n} f_m \leqq c_3 r_n{}^2\,. \tag{3.1}$$

Now write $t_k = 1/r_{2^k}$. By Lemma 2, once again, we have $t_{k+1}/t_k \to 1$ as $k \to \infty$, and (since our walk is assumed recurrent) $t_k \uparrow \infty$. Finally, in view of (3.1) we have $1/t_k - 1/t_{k+1} \leqq c_3/t_k{}^2$, which is equivalent to $t_{k+1} - t_k \leqq c_3 t_{k+1}/t_k$, from which it follows that $t_{k+1} - t_k \leqq c_4$. In particular we have

$$t_k - t_{k-[\log_2 t_k]-1} \leqq c_4([\log_2 t_k] + 1)$$

so that

$$\begin{aligned} 1 \geqq \frac{t_{k-[\log_2 t_k]-1}}{t_k} &= \frac{t_k - (t_k - t_{k-[\log_2 t_k]-1})}{t_k} \\ &\geqq \frac{t_{k-c_4([\log_2 t_k]+1)}}{t_k} \geqq 1 - c_4\frac{[\log_2 t_k] + 1}{t_k} \to 1 \end{aligned} \tag{3.2}$$

(since $\log_2 t_k = o(t_k)$).

Reintroducing the r_n we find that (3.2) yields

$$\frac{r_{2^k}}{r_{2^k - [\log_2 1/r_{2^k}] - 1}} \to 1\,.$$

However,

$$2^{-[\log_2 1/r_{2^k}]-1} \leqq 2^{-\log_2 1/r_{2^k}} = r_{2^k}\,,$$

therefore

$$2^{k-[\log_2 1/r_{2^k}]-1} \leqq 2^k r_{2^k}\,,$$

and in view of the monotonicity of $\{r_n\}$, it follows that

$$\frac{r_{2^k}}{r_{[2^k r_{2^k}]}} \to 1\,. \tag{3.3}$$

We have now shown that $r_n/r_{[nr_n]} \to 1$ if n is restricted to the subsequence of indices of the form $n = 2^k$. In order to extend this result to the full sequence we observe that

$$\exists c_5 \ni j \leqq n \rightarrow j r_j \leqq c_5 n r_n . \qquad \text{([6] Lemma 2.1)}$$

If $2^k \leqq n < 2^{k+1}$ we have

$$\frac{r_{2^{k+1}}}{r_{[2^k r_{2^k}/c_5]}} \leqq \frac{r_n}{r_{[n r_n]}} \leqq \frac{r_{2^k}}{r_{[2^{k+1} r_{2^{k+1}} c_5]}}$$

and both extremes of this last inequality go to 1 in view of (3.3).

4. Returns to the origin. We appropriate the usual terminology for random walks on Z^d and refer to $\mathscr{R}_n$, the number of indices $k \leqq n$ for which $S_k = 0$, as the number of returns of the walk $\{S_n\}$ to the origin. In this section we use the superslow variation of r_n to obtain the convergence in distribution of the number of returns to the origin for all recurrent random walks satisfying (*).

THEOREM C. *If $\{S_n\}$ is a recurrent random walk on a discrete Abelian group G which satisfies (*) then for any $t > 0$,*

$$\lim_{n\to\infty} P(r_n \mathscr{R}_n < t) = 1 - e^{-t} .$$

REMARK. Note that

$$E(\mathscr{R}_n) = P(S_1 = 0) + P(S_2 = 0) + \cdots + P(S_n = 0) .$$

But the sum is asymptotic to $1/r_n$ ([5] page 281) when r_n is slowly varying. Consequently the conclusion of the theorem may be formulated as

$$\lim_{n\to\infty} P(\mathscr{R}_n / E(\mathscr{R}_n) < t) = 1 - e^{-t} .$$

PROOF. Write $W_k =$ the waiting time for the kth return to the origin, and let $T_1 = W_1$, $T_k = W_k - W_{k-1}$, $k > 1$. The T_k's are independent, identically distributed random variables. We have

$$\begin{aligned} P(r_n \mathscr{R}_n \geqq t) &\geqq P(W_{[t/r_n]+1} \leqq n) \\ &\geqq \textstyle\prod_{k=1}^{[t/r_n]+1} P(T_k \leqq n/[t/r_n] + 1) \\ &= P(T_1 \leqq n/[t/r_n] + 1)^{[t/r_n]+1} . \end{aligned}$$

But, in view of the superslow variation of r_n

$$\begin{aligned} P(T_1 \leqq n/[t/r_n] + 1) &= 1 - r_{[n/[t/r_n]+1]} \\ &= 1 - r_n + o(r_n) . \end{aligned}$$

Therefore,

$$P(r_n \mathscr{R}_n \geqq t) \geqq (1 - r_n + o(r_n))^{t/r_n + O(1)} \rightarrow e^{-t} .$$

On the other hand,

$$\begin{aligned} P(r_n \mathscr{R}_n \geqq t) &\leqq P(W_{[t/r_n]} \leqq n) \\ &\leqq \textstyle\prod_{k=1}^{[t/r_n]} P(T_k \leqq n) \\ &= (1 - r_n)^{[t/r_n]} \rightarrow e^{-t} . \end{aligned}$$

5. Examples. We conclude with examples of recurrent random walks on Z (rank 1) and $Z_2 \oplus Z_2 \oplus \cdots$ (rank 0) for which (*) holds (Z_2 denotes here the additive group mod 2).

EXAMPLE 1. We consider the random walk on the line with transition function $P(0, 0) = 1 - 2/\pi$, $P(0, x) = 2/\pi(4x^2 - 1)^{-1}$ for $\neq 0$. This walk is recurrent and has the characteristic function $\phi(\theta) = 1 - |\sin \theta/2|$ ([9] page 89).

Observe that

$$\sup_{x \in Z^1} P^n(0, x) = 1/2\pi \int_{-\pi}^{\pi} (1 - |\sin \theta/2|)^n \, d\theta$$
$$= \frac{2}{\pi} \int_0^{\pi/2} (1 - \sin \theta)^n \, d\theta \leqq \frac{2}{\pi} \int_0^{\pi/2} \left(1 - \frac{2\theta}{\pi}\right)^n d\theta = \frac{1}{n+1} .$$

EXAMPLE 2. We consider the walk on $G = Z_2 \oplus Z_2 \oplus \cdots$ for which $P(0, x) = 0$ unless $x = x_k = (\delta_1{}^k, \cdots, \delta_j{}^k, \cdots)$, $1 \leqq k < \infty$, and $P(0, x_k) = 2^{-k}$ ($\delta_j{}^k$ denotes here the Kronecker delta symbol). The walk is known to be recurrent ([9] page 95). Let $\phi(\gamma) = \sum_{x \in G} P(0, x)(\gamma, x) = \sum_{k=1}^{\infty} 2^{-k}(\gamma, x_k)^n$ $\gamma \in \Gamma$, be the characteristic function of the walk, Γ being the dual of G. Then $P_n(0, x) = \int_\Gamma [\phi(\gamma)]^n \, dP_\Gamma(\gamma)$ so that

$$\sup_{x \in G} P_n(0, x) \leqq \int_\Gamma |\phi(\gamma)|^n \, dP_\Gamma(\gamma) . \tag{5.1}$$

It is shown in [4] that the measure space Γ endowed with the Haar measure P_Γ can be identified with the unit interval [0, 1] endowed with Lebesgue measure. In this identification (γ, x_k) becomes the kth Rademacher function $\phi_k(t)$, $0 \leqq t \leqq 1$, where $\phi_1 = 1$ on $[0, \frac{1}{2})$, -1 on $[\frac{1}{2}, 1]$, etc. We thus replace γ by t and write $\phi(t) = \sum_{k=1}^{\infty} 2^{-k}\phi_k(t)$, $0 \leqq t \leqq 1$. The estimate (5.1) becomes

$$\sup_{x \in G} P_n(0, x) \leqq \int_0^1 |\phi(t)|^n \, dt . \tag{5.2}$$

Since $\phi(t) = -\phi(1 - t)$, we have $\int_0^1 |\phi(t)|^n \, dt = 2 \int_0^{\frac{1}{2}} |\phi(t)|^n \, dt$. It is convenient to write $\Psi(t) = 1 - \phi(t) = \sum_{k=1}^{\infty} 2^{-k}(1 - \phi_k(t))$ and to study $\Psi(t)$ on each of the intervals $E_j = [2^{-(j+1)}, 2^{-j})$, $1 \leqq j < \infty$. To begin with $\Psi(t) \leqq 1$ for all $t \in [0, \frac{1}{2})$ while for $t \in E_j$, we have $1 - r_k(t) = 0$ for $k = 1, 2, \cdots, j$, $1 - r_{j+1}(t) = 2$ so that

$$\Psi(t) = 2/2^{j+1} + \sum_{k=j+2}^{\infty} 1/2^k(1 - r_k(t)) \geqq 1/2^j \geqq t .$$

But, $[0, \frac{1}{2}) = \bigcup_{j=1}^{\infty} E_j$, so that $1 \geqq \Psi(t) \geqq t$ for all $t \in [0, \frac{1}{2})$, therefore

$$|\phi(t)|^n = |1 - \Psi(t)|^n \leqq (1 - t)^n \qquad \text{for } t \in [0, \tfrac{1}{2})$$

and

$$\sup_{x \in G} P_n(0, x) \leqq 2 \int_0^{\frac{1}{2}} (1 - t)^n \, dt = O(1/n) .$$

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