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A NEW BOUND FOR STANDARD p-FUNCTIONS

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Let $p(t)$, $t \in (0, \infty)$ be the standard p-function of a regenerative phenomenon as defined in Kingman's theory. Let $p(1) = M$ and $\min\{p(t), 0 \leqq t \leqq 1\} = m$. Griffeath (1973) has derived a new upper bound for M for given m by using the Kingman inequalities of order $\leqq 3$. Here Griffeath's result is generalized by using the Kingman equalities of order $\leqq n$. Further taking limits as $n \to \infty$ a new upper bound is obtained which is uniformly strictly superior to the present known upper bound. Thus a part of the uncharted region in the $M - m$ diagram becomes charted by being shown inaccessible. This gives also an improved upper bound for the constant ν_0.

1. Introduction. Let $p(t)$, $t \in (0, \infty)$ be the p-function of a standard regenerative phenomenon Z as defined in J. F. C. Kingman's theory [8]. This means that there is a stochastic process $Z = \{Z(t); t > 0\}$ which assumes values in the two-point set $\{0, 1\}$ such that for every increasing sequence $0 = t_0 < t_1 < t_2 \cdots < t_n$,

$$P_r\{Z(t_1) = Z(t_2) = \cdots = Z(t_n) = 1\} = \prod_{r=1}^{n} p(t_r - t_{r-1}) . \tag{1}$$

The p-function is said to be standard if

$$\lim_{t \to 0} p(t) = 1 . \tag{2}$$

For standard p-functions we put $p(0) = 1$.

In the following we consider only standard p-functions. $\mathscr{P}$ denotes the class of all standard p-functions. Suppose we know that for some given s, $p(s) = M$. What is the maximum possible fluctuation in the value of $p(t)$ for a fixed $t \in (0, s)$? As $\mathscr{P}$ is closed under constant dilations and contractions of the time scale, for notational convenience, without loss of generality we take $s = 1$. Thus the problem is to determine for fixed $t \in (0, 1)$

$$\begin{aligned} \bar{\pi}_M(t) &= \sup_{p \in \mathscr{P}} \{p(t) \mid p(1) = M\} , \\ \underline{\pi}_M(t) &= \inf_{p \in \mathscr{P}} \{p(t) \mid p(1) = M\} . \end{aligned} \tag{3}$$

This is termed by Griffeath [7] the maximal oscillation problem for regenerative phenomena.

Until now interest has centered mostly on the following part of the problem. Let $m(p) = \min\{p(t), 0 \leqq t \leqq 1\}$. An ordered pair (M, m) is termed accessible if there exists $\hat{p} \in \mathscr{P}$ for which $m(\hat{p}) = m$, and $\hat{p}(1) = M$, where obviously $1 \geqq M \geqq m \geqq 0$. Further as shown by Davidson [5], if a pair (M, m) is accessible,

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then so is every pair (M', m) for which $m < M' < M$. Hence there exists a boundary which separates the accessible and inaccessible pairs in what Davidson [5] refers to as the $M - m$ diagram. We refer here to the $M - m$ diagram given by Griffeath ([7] Fig. 1, page 408). Its shaded portion shows the region which at present is uncharted; for pairs (M, m) falling within this region it is not known whether they are accessible or not. A connected problem is that of determining the value of a constant ν_0 defined by

$$(4) \qquad I_M = \inf\{\underline{\pi}_M(t), 0 \leqq t \leqq 1\}, \qquad \nu_0 = \inf\{M \mid I(M) > 0\}$$

where $\underline{\pi}_M(t)$ is as defined in (3).

The latest contribution to this topic is of Griffeath who gives a summary of previously known results and proves ([7] Proposition 1) a new upper bound for M for given m. This new bound does not however make any contribution to the $M - m$ diagram being uniformly higher than the upper bound obtained already by combining a result of Bloomfield [2] and Davidson [6] with that of Cornish [4] (cf. Equations (9) and (15) in [7]). This upper bound is expressible as

$$(5) \qquad M \leqq \phi_1(m)$$

where $\phi_1(m) = 1 - e^{-1}$ if $m \leqq e^{-1}$, $\phi_1(m) = 1 + m \log m$ if $m \geqq e^{-1}$. Hence ν_0 in (4) satisfies

$$(6) \qquad \nu_0 \leqq 1 - e^{-1}.$$

As against (5) the new bound derived by Griffeath [7] is expressible as

$$(7) \qquad \text{if} \quad m \leqq \tfrac{1}{3}, \quad M \leqq \tfrac{2}{3}; \qquad \text{if} \quad m \geqq \tfrac{1}{3}, \quad M \leqq \tfrac{1}{4}(3m^2 - 2m + 3)$$

yielding $\nu_0 \leqq \frac{2}{3}$. But Griffeath's new bound is of interest because it holds for a wider class $\mathscr{K}_3$ of functions p. For each integer n, $\mathscr{K}_n$ denotes the class of continuous real valued functions p defined on $[0, \infty)$ such that $p(0) = 1$ and for every non-decreasing sequence $\{t_k; 1 \leqq k \leqq n\}$ p satisfies the nth order Kingman equalities viz.

$$(8\,\text{i}) \qquad f(t_n) \geqq 0,$$

$$(8\,\text{ii}) \qquad g(t_n) \geqq 0,$$

where for each s, $s = 1, 2, \cdots, n$ the functions $f(t_s) = f(t_1, t_2, \cdots, t_s)$ and $g(t_s) = g(t_1, t_2, \cdots, t_s)$ are defined recursively by

$$(9\,\text{i}) \qquad f(t_s) = p(t_s) - \textstyle\sum_{k=1}^{s-1} f(t_k)p(t_s - t_k),$$

$$(9\,\text{ii}) \qquad g(t_s) = 1 - \textstyle\sum_{k=1}^{s} f(t_s).$$

NOTE 1.1. Obviously the value of $f(t_s)$ depends not on t_s alone but also on $t_1, t_2, \cdots, t_{s-1}$. Thus $f(t_s)$ is a convenient notational abbreviation for $f(t_1, t_2, \cdots, t_s)$. Similarly for $g(t_s)$.

For $p \in \mathscr{P}$, the inequalities (8) hold for all n. Also the nth order Kingman inequalities include all lower order ones as special cases as the increments in the sequence $\{t_k\}$ can be assigned the value 0. Hence

$$(10)\qquad \mathscr{K}_1 \supset \mathscr{K}_2 \supset \mathscr{K}_3 \cdots \supset \mathscr{P}, \qquad \lim_{n\to\infty} \mathscr{K}_n = \mathscr{P}.$$

In deriving (7) use is made only of the Kingman inequalities of order $\leqq 3$. Hence (7) holds for $p \in \mathscr{K}_3$. An earlier result of Davidson [5] (also derived independently for Markov processes by Blackwell and Freedman [1]) is expressible as: for $p \in \mathscr{K}_2$,

$$(11)\qquad \text{if } m \leqq \tfrac{1}{2}, \quad M \leqq \tfrac{3}{4}; \qquad \text{if } m \geqq \tfrac{1}{2}, \quad M \leqq 1 + m^2 - m.$$

Generalizing the argument of Griffeath ([7] Proposition 1, page 410), we derive in Section 2 an upper bound for M for given m, which holds for $p \in \mathscr{K}_n$ for each $n \geqq 2$. (11) and (7) thus become particular cases of our result for $n = 2$ and 3. Further taking limits as $n \to \infty$, we derive an upper bound, $M \leqq \phi_2(m)$ for $p \in \mathscr{P}$, which is uniformly strictly superior for $0 \leqq m < 1$ to the existing bound $\phi_1(m)$ in (5). A portion of the uncharted region in the $M - m$ diagram ([7] Fig. 1 page 408), thus becomes charted as inaccessible. Our result also yields $\nu_0 \leqq .590$ as compared to the bound of approximately .632 in (6).

2. Main result.

We state the main result as a proposition.

PROPOSITION 2.1. *Let* $p(1) = M$, $\inf\{p(t), 0 \leqq t \leqq 1\} = m$. *Then,*
(A) *for* $p \in \mathscr{K}_n$, *where* $n \geqq 2$,

$$(12\text{i})\qquad \text{if} \quad m \leqq \rho, \quad M \leqq 1 + \rho^2 - \left(1 - \frac{1-\rho}{n-1}\right)^{n-1},$$

$$(12\text{ii})\qquad \text{if} \quad m \geqq \rho, \quad M \leqq 1 + m^2 - \left(1 - \frac{1-m}{n-1}\right)^{n-1},$$

where $2\rho = (1 - (1-\rho)/(n-1))^{n-1}$;
(B) *for* $p \in \mathscr{P}$,

$$(13\text{i})\qquad \text{if} \quad m \leqq K, \quad M \leqq (1-K)^2,$$

$$(13\text{ii})\qquad \text{if} \quad m \geqq K, \quad M \leqq 1 + m^2 - \exp(m-1),$$

where $\exp(K-1) = 2K$. *Hence*

$$(13\text{iii})\qquad \nu_0 \leqq (1-K)^2.$$

PROOF. Select arbitrarily any $t' \in (0, 1)$; let $p(t') = \alpha$. Then select $t_1 \in (0, 1)$ such that either

$$(14)\qquad \begin{aligned} p(t_1) &= \alpha, \qquad p(1-t_1) \leqq \alpha, \qquad \text{or} \\ p(t_1) &\geqq \alpha, \qquad p(1-t_1) = \alpha. \end{aligned}$$

For any given t', t_1 can always be chosen so that (14) is satisfied. For let w be the first $t \in (0, 1)$ for which $p(w) = \alpha$. If $p(1-w) \leqq \alpha$, put $t_1 = w$; if $p(1-w) > \alpha$ put $t_1 - 1 - w$. Having selected t_1 we form a non-decreasing sequence $t_1 \leqq t_2 \leqq t_3 \cdots \leqq t^n = 1$ such that

$$(15)\qquad p(t_n - t_{r+1}) \geqq p(t_n - t_r) > 0, \qquad r = 2, 3, \cdots, (n-1).$$

After choosing $t_1, t_2, t_3, \cdots, t_{n-1}$ satisfying (15) can always be found because of (2) and the continuity of p on $[0, \infty)$. Next using (8 ii), (9 ii) and (9 i) after putting $s = n$ and that $p(t_n) = p(1) = M$, we obtain

$$0 \leqq g(t_n) = 1 - f(t_n) - \sum_{k=1}^{n-1} f(t_k) = 1 - M - \sum_{k=1}^{n-1} f(t_k)[1 - p(t_n - t_k)] .$$

Taking out the term for $k = n - 1$ from the summation, we get

$$(16) \qquad 1 - M \geqq f(t_{n-1})[1 - p(t_n - t_{n-1})] + \sum_{k=1}^{n-2} f(t_k)[1 - p(t_n - t_k)] .$$

In the right-hand side of (16) substitute for $f(t_{n-1})$ by (9 i) putting $s = n - 1$. This gives

$$(17) \quad 1 - M \geqq p(t_{n-1})[1 - p(t_n - t_{n-1})] + \sum_{k=1}^{n-2} f(t_k)\{1 - p(t_n - t_k) - p(t_{n-1} - t_k)[1 - p(t_n - t_{n-1})]\} .$$

We next use the 2nd order Kingman inequalities (8 ii), (9 ii) and (9 i) for the sequence $\{v_1, v_2\}$ where $v_1 = t_n - t_{n-1}$, $v_2 = t_n = 1$, so that

$$0 \leqq g(v_2) = 1 - f(v_1) - f(v_2) = 1 - p(t_n - t_{n-1}) - M + p(t_n - t_{n-1})p(t_{n-1})$$

which gives, since $p(t_n - t_{n-1}) > 0$ by (15),

$$(18) \qquad p(t_{n-1}) \geqq 1 - \frac{1 - M}{p(t_n - t_{n-1})} .$$

Similarly taking another sequence $\{v_1', v_2'\}$ with $v_1' = t_{n-1} - t_k$, $v_2' = t_n - t_k$ and applying (8 i) and (9 i) for $s = 2$ we get $p(v_2') \geqq p(v_1')p(v_2' - v_1')$ so that

$$(19) \qquad p(t_{n-1} - t_k) \leqq \frac{p(t_n - t_k)}{p(t_n - t_{n-1})} .$$

Substituting in (17) by (18) and (19) and simplifying,

$$(20) \qquad \frac{1 - M}{p(t_n - t_{n-1})} \geqq \sum_{k=1}^{n-2} f(t_k)\left[1 - \frac{p(t_n - t_k)}{p(t_n - t_{n-1})}\right] + [1 - p(t_n - t_{n-1})] .$$

Suppose that $n \geqq 4$. We extend the result (20) by induction.

INDUCTIVE HYPOTHESIS. For some integer j, $1 \leqq j \leqq n - 3$, the following inequality holds viz.

$$(21) \qquad \frac{1 - M}{p(t_n - t_{n-j})} \geqq \sum_{k=1}^{n-j-1} f(t_k)\left[1 - \frac{p(t_n - t_k)}{p(t_n - t_{n-j})}\right] + \sum_{k=1}^{j}\left[1 - \frac{p(t_n - t_{n-k})}{p(t_n - t_{n-k+1})}\right].$$

Note that in the right-hand side of (21) for $k = 1$, $p(t_n - t_{n-k+1}) = p(0) = 1$.

The following inductive argument runs closely parallel to that from (16) to (21). From the first summation in (21) take out the term for $k = n - j - 1$. Then substitute for $f(t_{n-j-1})$ by (9 i) putting $s = n - j - 1$. This gives

$$\tag{22} \begin{aligned} \frac{1-M}{p(t_n - t_{n-j})} \geqq p(t_{n-j-1})\left[1 - \frac{p(t_n - t_{n-j-1})}{p(t_n - t_{n-j})}\right] \\ + \textstyle\sum_{k=1}^{n-j-2} f(t_k)\left\{\left[1 - \frac{p(t_n - t_k)}{p(t_n - t_{n-j})}\right]\right. \\ \left. - p(t_{n-j-1} - t_k)\left[1 - \frac{p(t_n - t_{n-j-1})}{p(t_n - t_{n-j})}\right]\right\} \\ + \textstyle\sum_{k=1}^{j}\left[1 - \frac{p(t_n - t_{n-k})}{p(t_n - t_{n-k+1})}\right]. \end{aligned}$$

In place of (18) and (19) we use

$$\tag{23} p(t_{n-j-1}) \geqq 1 - \frac{1-M}{p(t_n - t_{n-j-1})},$$

$$\tag{24} p(t_{n-j-1} - t_k) \leqq \frac{p(t_n - t_k)}{p(t_n - t_{n-j-1})}$$

(23) is derived by applying the 2nd order Kingman inequalities to $\{v_1, v_2\}$ where $v_1 = t_n - t_{n-j-1}$, $v_2 = t_n = 1$ so that

$$\begin{aligned} 0 \leqq g(v_2) &= 1 - f(v_1) - f(v_2) \\ &= 1 - p(t_n - t_{n-j-1}) - M + p(t_{n-j-1})p(t_n - t_{n-j-1}) . \end{aligned}$$

Similarly (24) is derived by applying (8 i) and (9 i) to $\{v_1', v_2'\}$ where $v_1' = t_{n-j-1} - t_k$, $v_2' = t_n - t_k$, so that $0 \leqq f(v_2') = p(v_2') - p(v_1')p(v_2' - v_1')$. We now substitute in the right-hand side of (22) by (23) and (24). Substitution by (24) is possible because by (15)

$$\tag{25} 1 - \frac{p(t_n - t_{n-j-1})}{p(t_n - t_{n-j})} \geqq 0 \qquad \text{for } j = 1, 2, \cdots, (n-3) .$$

On substituting in (22) by (23) and (24) and rearranging the terms we obtain that (21) holds also on changing j into $j + 1$. By (20), (21) holds for $j = 1$. Hence it holds for $j = 1, 2, \cdots, (n - 2)$. Putting $j = n - 2$ in (21) we obtain, using that $f(t_1) = p(t_1)$ by (9 i),

$$\tag{26} \begin{aligned} \frac{1-M}{p(t_n - t_2)} \geqq p(t_1)\left[1 - \frac{p(t_n - t_1)}{p(t_n - t_2)}\right] \\ + \textstyle\sum_{k=1}^{n-2}\left[1 - \frac{p(t_n - t_{n-k})}{p(t_n - t_{n-k+1})}\right]. \end{aligned}$$

(26) has been proved by induction for $n \geqq 4$. For $n = 3$, (26) is identical with (20). Thus (26) holds for $n \geqq 3$. As $t_n = 1$,

(27) the first term in the right-hand side of (26) $\geqq \alpha\left[1 - \frac{\alpha}{p(t_n - t_2)}\right]$

by (14). Put

$$\beta_r = p(t_n - t_{n-r}) \qquad r = 1, 2, \cdots, n-2 . \tag{28}$$

Substituting in (26) by (27) and (28) we obtain after a slight rearrangement

$$1 - M \geqq \alpha\beta_{n-2} - \alpha^2 + \beta_{n-2}\left\{(1-\beta_1) + \sum_{r=1}^{n-3}\left[1 - \frac{\beta_{r+1}}{\beta_r}\right]\right\} \tag{29}$$

where in the last summation we have put $r = k - 1$. Denote the expression in the right-hand side of (29) by $F = F(\beta_1, \beta_2, \cdots, \beta_{n-2}, \alpha)$. Since (29) holds for all possible $\beta_1, \beta_2, \cdots, \alpha$, subject to (15) we maximize F for fixed α. Putting $\partial F/\partial\beta_k = 0$ for $k = 1, 2, \cdots, (n-3)$, $\beta_1 = \beta_2/\beta_1 = \cdots = \beta_{n-2}/\beta_{n-3}$. Hence

$$\beta_k = \beta_1^{\,k} \qquad \text{for} \quad k = 1, 2, \cdots, (n-2) . \tag{30}$$

Note that (30) is consistent with (15). Next putting $\partial F/\partial\beta_{n-2} = 0$, we obtain using (30)

$$\begin{aligned} 0 &= \alpha + 1 - \beta_1 + \sum_{r=1}^{n-4}\left(1 - \frac{\beta_{r+1}}{\beta_r}\right) + \left[1 - \frac{2\beta_{n-2}}{\beta_{n-3}}\right] \\ &= \alpha + (n-1)(1-\beta_1) - 1 , \end{aligned}$$

so that

$$1 - \beta_1 = \frac{1}{n-1}(1-\alpha) . \tag{31}$$

Substituting in (29) by (30) and (31) we obtain

$$\begin{aligned} 1 - M &\geqq -\alpha^2 + \beta_1^{\,n-2}[\alpha + (n-2)(1-\beta_1)] \\ &= -\alpha^2 + \left[1 - \frac{1}{n-1}(1-\alpha)\right]^{n-1} . \end{aligned} \tag{32}$$

Denote the right-hand side of (32) by $g(\alpha)$. $g(\alpha)$ is maximized when $g'(\alpha) = 0$ which gives

$$2\alpha = \left[1 - \frac{1}{n-1}(1-\alpha)\right]^{n-2} . \tag{33}$$

It is easily seen that the equation $2X = [1 - (n-1)^{-1}(1-x)]^{n-2}$ has a unique root ρ say in (0, 1). Let $m = \min\{p(t), 0 \leqq t \leqq 1\}$. Since $\alpha = p(t')$ for some $t' \in (0, 1)$, $\alpha \geqq m$. Hence α can assume the value ρ satisfying (33) only if $m \leqq \rho$. Then putting $\alpha = \rho$ in the right-hand side of (32) yields (12i). Suppose $m > \rho$. By (32), $g'(\alpha) = -2\alpha + [1 - (n-1)^{-1}(1-\alpha)]^{n-2}$ and it is easily verified that for $\alpha > \rho$, $g'(\alpha) < 0$. Hence for $m > \rho$, $g(\alpha)$ is maximized for $\alpha = m$. Putting $\alpha = m$ in the right-hand side of (32) we obtain (12ii).

The above argument holds for $n \geqq 3$. If $n = 2$, note that by the 2nd order Kingman inequalities (26) holds for $n = 2$ also if the 2nd term in its right-hand side is equated to zero. Then substituting in (26) by (27) and maximizing the right-hand side for α we obtain (12i) and (12ii) for $n = 2$.

This completes the proof of part (A) of the proposition. Part (B) follows immediately by taking limits as $n \to \infty$ because of (10).

REMARK. (13 iii) gives $K \doteqdot .232$ where $\doteqdot$ denotes approximate equality. Hence $\nu_0 \leqq (1-K)^2 \doteqdot .590$. Combining (13 ii) with (6), we obtain $.368 \doteqdot e^{-1} \leqq \nu_0 \leqq (1-K)^2 \doteqdot .590$. According to a remark of Williams [9], it is important to know whether $\nu_0 \leqq \frac{1}{2}$ as this has an important application to Markov semigroups (cf. remark in [7] page 411). This question remains open.

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