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ITERATED LOGARITHM RESULTS FOR WEIGHTED AVERAGES OF MARTINGALE DIFFERENCE SEQUENCES[1]

BY R. J. TOMKINS

University of Regina

Let $(X_n, \mathscr{F}_n, n \geqq 1)$ be a martingale difference sequence with $E(X_n^2 \mid \mathscr{F}_{n-1}) = 1$ a.s. This paper presents iterated logarithm results involving $\limsup_{n\to\infty} \sum_{m=1}^n f(m/n)X_m/(2n \log\log n)^{\frac{1}{2}}$, where f is a continuous function on $[0, 1]$. For example, it is shown that the above limit superior equals the L_2-norm of f if the X_n's are uniformly bounded and f is a power series with radius in excess of one. These results generalize (and correct the proof of) a previous theorem due to the author.

A generalization of the strong law of large numbers is also established.

1. Introduction. Let $X_1, X_2, \cdots$ be a sequence of random variables (rv) and let f be a continuous function on $[0, 1]$. This article investigates the limiting behavior of

$$Z_n(f) \equiv (2n \log\log n)^{-\frac{1}{2}} \sum_{m=1}^n f(m/n)X_m \qquad \text{as } n \to \infty .$$

This problem was considered, in the case where $X_1, X_2, \cdots$ are independent, by the author (Tomkins (1971) and (1974)) and Wichura (1973). This paper focuses on the case in which $\{X_n\}$ is a martingale difference sequence.

Following a discussion of notation and the presentation of several useful lemmas in Section 2, upper and lower bounds for $\limsup_{n\to\infty} Z_n(f)$ are derived in Section 3. Section 4 considers some ramifications of Lemma 2 (Section 2), concerning the strong law of large numbers.

2. Some preliminary lemmas. This paper considers only continuous functions on $[0, 1]$. Let $f^* = \max_{0\leqq x\leqq 1} |f(x)|$, the sup-norm of f, and $||f||_2 \equiv (\int_0^1 f^2(t)\,dt)^{\frac{1}{2}}$, the L_2-norm of f. If f is a function of bounded variation (BV), its total variation will be denoted by V_f.

$(S_n, \mathscr{F}_n, n \geqq 1)$ is a stochastic process with sigma-fields $\mathscr{F}_1 \subset \mathscr{F}_2 \subset \cdots$ and S_n being $\mathscr{F}_n$-measurable. $\mathscr{F}_0$ is the trivial sigma-field. Write $a_n \sim b_n$ when $a_n/b_n \to 1$.

The first lemma is a generalization of Theorem 1(i) of Tomkins (1972) and of a theorem of Csáki (1968).

LEMMA 1. *Let $(S_n, \mathscr{F}_n, n \geqq 1)$ be a submartingale. Let $\{\alpha_n\}$, $\{B_n\}$ and $\{c_n\}$ be positive real sequences satisfying*

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(i) $B_n \uparrow \infty$
(ii) $B_{n+1} \sim B_n$
(iii) $c_n^2 \log\log B_n^2 \to 0$ *and*
(iv) *for some* $C > 0$ *and* $N > 0$, $E\exp\{tS_n/(\alpha_n B_n)\} \leqq C\exp\{(t^2/2)(1 + tc_n)\}$ *for all* $n \geqq N$ *and all* t *in* $[0, c_n^{-1}]$.

Then

$$\limsup_{n\to\infty} S_n/(2B_n^2 \log\log B_n^2)^{\frac{1}{2}} \leqq \limsup_{n\to\infty} \alpha_n \quad \textit{almost surely (a.s.)}\,.$$

PROOF. This lemma may be proved by following the proof of Theorem 1(i) of Tomkins (1972) except that, instead of using Lévy's inequality, one should employ Doob's inequality (Doob (1953) page 314), noting that $(e^{\lambda S_n}, \mathscr{F}_n, n \geqq 1)$ is a submartingale for each $\lambda > 0$.

LEMMA 2. *Let* $Y_1, Y_2, \cdots$ *be any* rv. *Suppose*

$$\limsup_{n\to\infty} |Y_1 + \cdots + Y_n|/b_n \leqq 1 \quad \text{a.s.} \tag{1}$$

for some positive sequence $\{b_n\}$. *Let* $\{a_{nm}\}$, $m, n \geqq 1$, *be a double sequence of reals such that either*

(i) $\sum_{m=1}^{\infty} |a_{nm}|E|Y_m| < \infty$ *for all* n, *or*,
(ii) $\sum_{m=1}^{\infty} |a_{nm} - a_{n,m+1}|E|\sum_{j=1}^{m} Y_j| < \infty$ *for all* n. *Assume, moreover, that*
(iii) $a_{nm} \sum_{j=1}^{m-1} Y_j \to 0$ a.s. $m \to \infty$ *for each* $n \geqq 1$ *and*
(iv) *for some number* L *and all* $m \geqq 1$, $\lim_{n\to\infty} a_{nm} = L$. *Then, for any sequence* $\{\beta_n\}$ *with* $\liminf_{n\to\infty} \beta_n > 0$,

$$\limsup_{n\to\infty} \beta_n^{-1}|\textstyle\sum_{m=1}^{\infty} a_{nm} Y_m| \leqq \limsup_{n\to\infty} b_n\beta_n^{-1} \sum_{m=1}^{\infty} |a_{nm} - a_{n,m+1}| \quad \text{a.s.}$$

PROOF. Define $T_m \equiv \sum_{j=1}^{m} Y_j$ and $d_{nm} = a_{nm} - a_{n,m+1}$. Using (iii) and Abel's partial summation formula (Apostol (1960) page 365), $\sum_{m=1}^{\infty} d_{nm} T_m = \sum_{m=1}^{\infty} a_{nm} Y_m$ a.s.; these two series are well defined since one or the other converges a.s. by (i) or (ii).

Let A be the event on which (1) holds. Let $\varepsilon > 0$. For each $\omega \in A$, a number $N = N(\omega)$ exists such that $|T_n| < (1 + \varepsilon)b_n$ for $n \geqq N$. Then

$$\begin{aligned}
&\limsup_{n\to\infty} \beta_n^{-1}|\textstyle\sum_{m=1}^{\infty} a_{nm} Y_m(\omega)| \\
&\quad \leqq \limsup_{n\to\infty} \{\beta_n^{-1} \textstyle\sum_{m=1}^{N-1} |d_{nm}| \max_{j<N} |T_j(\omega)| + (1+\varepsilon)b_n\beta_n^{-1} \sum_{m=N}^{\infty} |d_{nm}|\} \\
&\quad \leqq (1+\varepsilon) \limsup_{n\to\infty} b_n\beta_n^{-1} \textstyle\sum_{m=1}^{\infty} |d_{nm}|\,, \qquad \text{by (iv).} \quad \square
\end{aligned}$$

The next lemma will be the crucial tool in the proof of Theorem 1.

LEMMA 3. *Let* $Y_1, Y_2, \cdots$ *be any* rv *satisfying*

$$\limsup_{n\to\infty} (2n\log\log n)^{-\frac{1}{2}}|Y_1 + \cdots + Y_n| \leqq 1 \quad \text{a.s.} \tag{2}$$

Then, for any function g *of* BV,

$$\limsup_{n\to\infty} (2n\log\log n)^{-\frac{1}{2}}\, |\textstyle\sum_{m=1}^{n} g(m/n)Y_m| \leqq V_g + |g(1)| \quad \text{a.s.}$$

PROOF. Letting $b_n = \beta_n = (2n\log\log n)^{\frac{1}{2}}$ and $a_{nm} = g(m/n)$ or 0, accordingly as $m \leqq n$ or $m > n$, in Lemma 2, we have

$$\limsup_{n\to\infty} (2n \log\log n)^{-\frac{1}{2}} |\sum_{m=1}^{n} g(m/n) Y_m|$$
$$\leqq \limsup_{n\to\infty} (\sum_{m=0}^{n-1} |g(m/n) - g(m/n+1)| - |g(0) - g(1/n)|) + |g(1)|$$
$$\leqq V_g + |g(1)| \quad \text{a.s.} \qquad \square$$

REMARK. The quantity $V_f + |f(1)|$ is a norm on the space of all functions of BV, but is not the usual norm for that space, namely $||f||_{\mathrm{BV}} \equiv V_f + |f(0+)|$ (cf. Dunford and Schwartz (1958) page 241). It is not hard to show that convergence in the norm $V_f + |f(1)|$ and in the norm $||f||_{\mathrm{BV}}$ are equivalent; such convergence has been called strong convergence by Morse (1937). It is easily shown (see proof of Theorem 1) that $V_f + |f(1)|$ is no smaller than the sup-norm on [0, 1] and, hence, is at least as large as the L_2-norm.

Clearly, then, Lemma 3 is not as sharp as others which hold for independent rv (cf. Tomkins (1971) and Theorem 1 herein), but it is valid for a wider range of rv. Heyde (1973) and Révész (1972) give examples of dependent rv satisfying (2).

Our final lemma presents two special cases of Lemmas 4.1 and 4.2 of Stout's (1967) generalization of Kolmogorov's exponential bounds.

LEMMA 4. *Let* $(S_n \equiv \sum_{m=1}^{n} X_m, \mathscr{F}_n, n \geqq 1)$ *be a martingale and let* $\mathscr{G} \subset \mathscr{F}_1$ *be a sigma-field. Suppose that, for some* $c > 0$, $\max_{1 \leqq m \leqq n} |X_m|/s_n \leqq c$ a.s., *where* $s_n^2 \equiv ES_n^2$. *Assume, moreover, that* $E(X_m^2 | \mathscr{F}_{m-1}) = EX_m^2$ a.s. *Let* $\varepsilon > 0$.

(a) *If* $\varepsilon c \leqq 1$ *then* $P(S_n > \varepsilon s_n | \mathscr{G}) \leqq \exp\{-(\varepsilon^2/2)(1 - \varepsilon c/2)\}$ a.s.;

(b) *for any* $\gamma > 0$ *there exist numbers* $\varepsilon_0 > 0$ *and* $\eta_0 > 0$ *such that* $P(S_n > \varepsilon s_n | \mathscr{G}) > \exp\{-(1+\gamma)\varepsilon^2/2\}$ a.s. *if* $\varepsilon_0 > \varepsilon$ *and* $\varepsilon c < \eta_0$.

3. The main results. This section contains two theorems which generalize the work of Tomkins (1971).

THEOREM 1. *Let* $(X_n, \mathscr{F}_n, n \geqq 1)$ *be a martingale difference sequence. Suppose a number* $N > 0$ *and a positive sequence* $c_n = o((\log\log n)^{-\frac{1}{2}})$ *exist such that*

$$E(\exp(tn^{-\frac{1}{2}} X_m) | \mathscr{F}_{m-1}) \leqq \exp\{(t^2/(2n))(1 + |t| c_n)\} \quad \text{a.s.} \tag{3}$$

provided $n > N$, $m \leqq n$ *and* $|t| \leqq c_n^{-1}$.

Let f be a function on [0, 1] *such that*

(i) *a sequence* $\{p_m\}$ *of polynomials exists such that* $p_m(1) \to f(1)$ *and* $V_{p_m - f} \to 0$ *as* $m \to \infty$. *Then*

$$\limsup_{n\to\infty} \sum_{m=1}^{n} f(m/n) X_m / (2n \log\log n)^{\frac{1}{2}} \leqq ||f||_2 \quad \text{a.s.} \tag{4}$$

In particular, (4) *holds if* $f(x) = \sum_{j=0}^{\infty} a_j x^j$ *is a power series with either*

(ii) *radius of convergence greater than* 1, *or*

(iii) $\limsup_{n\to\infty} |a_n|^{1/n} = 1$, $\sum_{j=0}^{\infty} |a_j| < \infty$.

REMARK. If f satisfies (i) then f is absolutely continuous (AC) by Theorem

2.1 of Edwards and Wayment (1971). Since it will be shown that (ii) and (iii) each imply (i), Theorem 1 is applicable only to AC functions.

PROOF OF THEOREM 1. For brevity's sake, define $r_n^2 = 2n \log\log n$, $t_n^2 = 2\log\log n$, and $Z_n(g) \equiv r_n^{-1} \sum_{m=1}^n g(m/n)X_m$ for any function g on $[0, 1]$. If $||f||_2 = 0$ the result is trivial, so there is no harm in assuming $||f||_2 = 1$.

Suppose, first of all, that f is a polynomial, say, $f(x) = \sum_{j=0}^p a_j x^j$. Let $\varepsilon > 0$. Define $A \equiv \sum_{j=0}^p |a_j|$ and $\delta = \varepsilon/A$. Choose $c > 1$ so close to 1 that

$$\sum_{j=0}^p |a_j|(c^j - 1)c^{-j} < \varepsilon,\ c^{2p+1} < 2,\ c < 1 + \varepsilon,\ \delta > 2(c^{2p+1} - 1)^{\frac{1}{2}}\,. \tag{5}$$

For each $k \geqq 1$, let $n_k = [c^k] + 1$, where $[x]$ is the integral part of x. Thus $n_k \sim c^k$, and, for all large k (say, $k \geqq K_0$), $n_{k-1} < n_k$. Assume $k \geqq K_0$ hereafter.

Following the proof of Theorem 2 of Tomkins (1972), one can deduce from (3) and Lemma 1 that, for each $0 \leqq j$,

$$\limsup_{n\to\infty} |\textstyle\sum_{m=1}^n m^j X_m|/(n^j r_n) \leqq (2j + 1)^{-\frac{1}{2}} \leqq 1 \quad \text{a.s.} \tag{6}$$

Now

$$\begin{aligned} R_{k1} &\equiv \max_{n_{k-1}<n\leqq n_k} r_{n_{k-1}}^{-1} |\textstyle\sum_{m=1}^{n_{k-1}} \{f(m/n) - f(m/n_{k-1})\}X_m| \\ &= \max_{n_{k-1}<n\leqq n_k} r_{n_{k-1}}^{-1} |\textstyle\sum_{j=0}^p a_j(n^{-j} - n_{k-1}^{-j}) \sum_{m=1}^{n_{k-1}} m^j X_m| \\ &\leqq \textstyle\sum_{j=0}^p |a_j|(n_k^j - n_{k-1}^j)n_k^{-j}| \sum_{m=1}^{n_{k-1}} m^j X_m|/(n_{k-1}^j r_{n_{k-1}}) \end{aligned} \tag{7}$$

so that, in view of (5) and (6),

$$\limsup_{k\to\infty} R_{k1} < \varepsilon \quad \text{a.s.} \tag{8}$$

Now for any $0 \leqq j \leqq p$, let $W_k = \sum_{n_{k-1}<m\leqq n_k} m^j X_m$ and $w_k^2 \equiv \sum_{n_{k-1}<m\leqq n_k} m^{2j}$. Then $n_{k-1}^{-(2j+1)} w_k^2 \to (c^{2j+1} - 1)/(2j + 1) \leqq (c^{2p+1} - 1) < (\delta/2)^2$ by (5). Moreover, it follows easily from (3) that $E\exp\{tW_k/w_k\} \leqq \exp\{t^2/2)(1 + |t|c_k^*)\}$ provided $|t|c_k^* \leqq 1$, where $c_k^* \equiv 2c^{p+\frac{1}{2}}(2p + 1)^{\frac{1}{2}}c_{n_k}$. But $(\exp(t_{n_{k-1}} \sum_{n_{k-1}<m\leqq n} m^j X_m), \mathscr{F}_n, n_{k-1} < n \leqq n_k)$ is a submartingale. Again by Doob's inequality, we have, for k satisfying $t_{n_k} c_k^* \leqq 1$,

$$\begin{aligned} P_k^* &\equiv P[\max_{n_{k-1}<n\leqq n_k} \textstyle\sum_{n_{k-1}<m\leqq n} m^j X_m \geqq \delta n_{k-1}^j r_{n_{k-1}}] \\ &= P[\max_{n_{k-1}<n\leqq n_k} t_{n_{k-1}} \textstyle\sum_{n_{k-1}<m\leqq n} m^j X_m/w_k \geqq \delta n_{k-1}^{j+\frac{1}{2}} t_{n_{k-1}}^2 w_k^{-1}] \\ &\leqq \exp\{-\delta n_{k-1}^{j+\frac{1}{2}} t_{n_{k-1}}^2 w_k^{-1} + (t_{n_{k-1}}^2/2)(1 + t_{n_{k-1}} c_k^*)\} \\ &\leqq \exp\{-t_{n_{k-1}}^2\} \leqq \{(\log c)(k - 1)\}^{-2}\,, \end{aligned}$$

so $\sum P_k^* < \infty$.

By a similar argument $\sum_{k=1}^\infty P_k^{**} < \infty$ where P_k^{**} is defined like P_k^* except with $-X_m$ in place of X_m. For each $0 \leqq j \leqq p$, then, the Borel–Cantelli lemma shows that

$$\limsup_{k\to\infty} \max_{n_{k-1}<n\leqq n_k} |\textstyle\sum_{n_{k-1}<m\leqq n} m^j X_m|/(n_{k-1}^j r_{n_{k-1}}) \leqq \delta \quad \text{a.s.} \tag{9}$$

Therefore, letting $R_{k2} \equiv \max_{n_{k-1}<n\leqq n_k} r_{n_{k-1}}^{-1} |\sum_{n_{k-1}<m\leqq n} f(m/n)X_m|$,

$$\begin{aligned} &\limsup_{k\to\infty} R_{k2} \\ &\quad\leqq \limsup_{k\to\infty} \textstyle\sum_{j=0}^p |a_j| \max_{n_{k-1}<n\leqq n_k} |\sum_{n_{k-1}<m\leqq n} m^j X_m|/(n_{k-1}^j r_{n_{k-1}}) \\ &\quad\leqq \delta A = \varepsilon\,. \end{aligned} \tag{10}$$

By (8) and (10),

(11) $\limsup_{k\to\infty} \max_{n_{k-1}<n\leq n_k} r_{n_{k-1}}^{-1} |\sum_{m=1}^{n} f(m/n)X_m - \sum_{m=1}^{n_{k-1}} f(m/n_{k-1})X_m| < 2\varepsilon$.

Noting that $\sum_{m=1}^{n} f^2(m/n) \sim n\|f\|_2^2 = n$, we have,

$$\begin{aligned} P[Z_{n_{k-1}}(f) &\geqq (1+\varepsilon)] \\ &\leqq \exp\{-(1+\varepsilon)t_{n_{k-1}}^2 + (t_{n_{k-1}}^2 \sum_{m=1}^{n_{k-1}} f^2(m/n_{k-1})/(2n_{k-1}))(1 + t_{n_{k-1}} c_k')\} \\ &\leqq \exp\{-(t_{n_{k-1}}^2/2)(2 + 2\varepsilon - (1+\varepsilon/2)^2\} \leqq (\log n_{k-1})^{-(1+\varepsilon/2)} \\ &= O((k-1)^{-(1+\varepsilon/2)}) , \end{aligned}$$

where $c_k' \equiv f^* c_{n_{k-1}}$ and k is so large that $t_{n_{k-1}} c_k' < \varepsilon/2$ and $\sum_{m=1}^{n_{k-1}} f^2(m/n_{k-1}) \leqq (1+\varepsilon/2)n_{k-1}$. Again by the Borel–Cantelli lemma, $\limsup_{k\to\infty} Z_{n_{k-1}}(f) \leqq 1 + \varepsilon$ a.s. This inequality, together with (11), establishes (4) if f is a polynomial.

Now suppose (i) holds. Using a well-known result (cf. Apostol (1960) page 164), for each $x \in [0, 1]$, $|p_m(x) - f(x)| \leqq V_{f-p_m} + |f(1) - p_m(1)| \to 0$ as $m \to \infty$. Hence $p_m \to f$ uniformly, so $\|p_m\|_2 \to \|f\|_2$ as $m \to \infty$.

Letting $C = 1$, $\alpha_n = 1$ and $B_n = n^{\frac{1}{2}}$, and using (3), Lemma 1 shows that (2) holds (with X_k in place of Y_k). For each fixed $m \geqq 1$ we can now apply Lemma 3 to get

$$\begin{aligned} \limsup_{n\to\infty} Z_n(f) &\leqq \limsup_{n\to\infty} Z_n(f - p_m) + \limsup_{n\to\infty} Z_n(p_m) \\ &\leqq V_{f-p_m} + |f(1) - p_m(1)| + \|p_m\|_2 . \end{aligned}$$

Letting $m \uparrow \infty$, (4) obtains.

Now let $f(x) = \sum_{j=0}^{\infty} a_j x^j$ and, for each $m \geqq 1$, define $p_m(x) = \sum_{j=0}^{m} a_j x^j$. Suppose that either (ii) or (iii) holds; note that $p_m(1) \to f(1)$ and $\sum_{j=0}^{\infty} |a_j| < \infty$ in either case. Choose any partition $0 = x_0 < x_1 < \cdots < x_n = 1$. Then

$$\begin{aligned} \sum_{k=1}^{n} &|(f - p_m)(x_k) - (f - p_m)(x_{k-1})| \\ &= \sum_{k=1}^{n} |\sum_{j=m+1}^{\infty} a_j(x_k^j - x_{k-1}^j)| \\ &\leqq \sum_{j=m+1}^{\infty} \sum_{k=1}^{n} |a_j|(x_k^j - x_{k-1}^j) = \sum_{j=m+1}^{\infty} |a_j| . \end{aligned}$$

Hence $V_{f-p_m} \leqq \sum_{j=m+1}^{\infty} |a_j| \to 0$ as $m \to \infty$; i.e. (i) holds. □

REMARKS 1. In the special case where $X_1, X_2, \cdots$ are independent with mean zero and variance one, Theorem 1 improves the theorem of Tomkins (1971). Note, however, that Wichura (1973 page 279) has shown that equality holds in (4) for a wide range of independent rv, provided only that f is of BV.

REMARK 2. N. S. Chen has observed that, while (7) holds even when $p = \infty$, Fatou's lemma was inappropriately applied by Tomkins (1971) to establish (8). The proof just completed corrects the error by using a different approach.

The next theorem, complementary to Theorem 1, is analogous to the result of Tomkins (1971) and generalizes Gaposhkin's (1965) result.

THEOREM 2. *Let* $(X_n, \mathscr{F}_n, n \geqq 1)$ *be a martingale difference sequence with* $EX_1 = 0$, $E(X_n^2 | \mathscr{F}_{n-1}) = 1$ a.s. *and* $\max_{m\leqq n} |X_m| \leqq M_n$ a.s. *where* $\{M_n\}$ *is a* *positive sequence such that* $(\log\log n)M_n^2/n \to 0$. *Then, for any continuous function* f *on* $[0, 1]$,

(12) $$\limsup_{n\to\infty} (2n \log\log n)^{-\frac{1}{2}} \sum_{m=1}^{n} f(m/n)X_m \geqq \|f\|_2 \quad \text{a.s.}$$

REMARK 1. It is not hard to show that (3) holds under the hypotheses of Theorem 2 (see pages 254–255 of Loève (1963)), so that equality holds in (12) if any of (i), (ii) or (iii) of Theorem 1 holds.

PROOF OF THEOREM 2. As in the proof of Theorem 1, we may assume $\|f\|_2 = 1$. Let $0 < \varepsilon < \frac{1}{2}$. For $0 \leqq x \leqq 1$, define $I(x) \equiv \int_0^x f^2(t)\,dt$. Choose a positive integer p such that $I(p^{-1}) < \varepsilon^3$. Define $\nu \equiv I(p^{-1})$.

For each $k \geqq 1$, let $n_k = p^k$, $U_k = \sum_{m=1}^{n_{k-1}} f(m/n_k)X_m$, $u_k^2 = EU_k^2$, $V_k = \sum_{n_{k-1}<m\leqq n_k} f(m/n_k)X_m$ and $v_k^2 = EV_k^2$. Let $t_n^2 = 2\log\log n$.

Since f is continuous, $n_k^{-1}u_k^2 \to \nu$. But $\sum_{m=1}^{n} f^2(m/n) \sim n$, so $v_k^2 \sim n_k(1-\nu)$. Choose $K > 0$ so large that, for all $k \geqq K$, $(1-2\varepsilon)n_k^{\frac{1}{2}} < (1-\varepsilon)v_k/(1-\nu)^{\frac{1}{2}}$.

For each $k \geqq K$, $(f(m/n_k)X_m, \mathscr{F}_m, n_{k-1} < m \leqq n_k)$ is a martingale difference sequence; note that $|f(m/n_k)X_m|/v_k \leqq f^*M_{n_k}v_k^{-1}$ a.s. Hence, using Lemma 4(b), with $\mathscr{G} = \mathscr{F}_{n_{k-1}}$ and $\gamma = (1-\varepsilon)^{-1} - 1$,

$$\begin{aligned} P(V_k > (1-2\varepsilon)(2n_k \log\log n_k)^{\frac{1}{2}} \mid \mathscr{F}_{n_{k-1}}) &\geqq P(V_k > (1-\varepsilon)(1-\nu)^{-\frac{1}{2}} v_k t_{n_k} \mid \mathscr{F}_{n_{k-1}}) \\ &> \exp\{-(1+\gamma)(1-\varepsilon)^2 t_{n_k}^2 (1-\nu)^{-1}/2\} \\ &> (2k\log p)^{-1} \quad \text{a.s.} \qquad \text{for all large } k\,. \end{aligned}$$

Hence $\sum_{k=K}^{\infty} P(V_k > (1-2\varepsilon)(2n_k\log\log n_k)^{\frac{1}{2}} \mid \mathscr{F}_{n_{k-1}}) = \infty$ a.s. so that, by Levy's generalization of the Borel Zero–One Law (see Loève (1963) page 398),

(13) $$\limsup_{k\to\infty} V_k/(2n_k \log\log n_k)^{\frac{1}{2}} > 1 - 2\varepsilon \quad \text{a.s.}$$

Assume $\nu > 0$. Then applying Lemma 4(a) with $\mathscr{G} = \mathscr{F}_0$

$$\begin{aligned} P_k' \equiv P[|U_k| > 2\varepsilon(2n_k\log\log n_k)^{\frac{1}{2}}] &\leqq P[|U_k| > \varepsilon u_k t_{n_k} \nu^{-\frac{1}{2}}] \\ &\leqq 2\exp\{-(\varepsilon^2 t_{n_k}^2 \nu^{-1}/2)(1 - \varepsilon t_{n_k} f^* M_{n_k} \nu^{-\frac{1}{2}} u_k^{-1}/2)\} \\ &\leqq 2(2k\log p)^{-\varepsilon^3/\nu} \end{aligned}$$

for all k so large that $\varepsilon t_{n_k} f^* M_{n_k} \leqq (1-\varepsilon)\nu^{\frac{1}{2}} u_k$ and $2(n_k\nu)^{\frac{1}{2}} > u_k$. Since $\nu < \varepsilon^3$, $\sum_{k=1}^{\infty} P_k' < \infty$ and, by the Borel-Cantelli lemma,

(14) $$\limsup_{k\to\infty} |U_k|/(2n_k\log\log n_k)^{\frac{1}{2}} \leqq 2\varepsilon \quad \text{a.s.}$$

If $\nu = 0$ then $f(x) \equiv 0$ on $[0, p^{-1}]$. Hence $U_k = 0$ and (14) holds trivially.

(13) and (14) together imply (12). □

The following corollary shows that Gaposhkin's (1965) result remains true in a martingale setting. Since the function $f(x) = (1-x)^\alpha$ is continuous on $[0, 1]$, satisfies (iii) of Theorem 1, and $\|f\|_2 = (2\alpha+1)^{-\frac{1}{2}}$, the corollary follows immediately from Theorems 1 and 2.

COROLLARY 1. *Let* $(X_n, \mathscr{F}_n, n \geqq 1)$ *be a martingale difference sequence with*

$EX_1 = 0$, $E(X_n^2 | \mathscr{F}_{n-1}) = 1$ a.s. *and* $|X_n| \leqq M$ a.s. *for some* $M > 0$ *and all* $n \geqq 1$. *Then, for every* $\alpha > 0$,

$$\limsup_{n\to\infty} (2n \log\log n)^{-\frac{1}{2}} \sum_{m=1}^{n} (1 - m/n)^{\alpha} X_m = (2\alpha + 1)^{-\frac{1}{2}} \quad \text{a.s.}$$

REMARK. It is also possible to establish Theorem 2 by using Lemmas 1 and 3 of Stout (1970).

4. A strong law from Lemma 2. It is worth observing that Lemma 2 has some consequences connected with the strong law of large numbers (SLLN). We will present one such consequence and an example.

THEOREM 3. *Let* $Y_1, Y_2, \cdots$ *be any integrable* rv *satisfying the* SLLN (i.e. $(Y_1 + \cdots + Y_n)/n \to 0$ a.s.). *Let* $\{a_{nm}\}$ *be a double sequence of reals satisfying, for each* $n \geqq 1$,

(i) $\sum_{m=1}^{\infty} |a_{nm} - a_{n,m+1}| \leqq Kn^{-1}$ *for some* $K > 0$, *and,*
(ii) $\sum_{m=1}^{\infty} |a_{nm}| E|Y_m| < \infty$ *or* $\sum_{m=1}^{\infty} |a_{nm} - a_{n,m+1}| E|\sum_{j=1}^{m} Y_j| < \infty$, *and*
(iii) $\limsup_{m\to\infty} m a_{nm} < \infty$.

Assume, moreover, that

(iv) *for some* L *and all* $m \geqq 1$, $\lim_{n\to\infty} a_{nm} = L$. *Then*

$$\lim_{n\to\infty} \sum_{m=1}^{\infty} a_{nm} Y_m = 0 \quad \text{a.s.}$$

PROOF. Let $\varepsilon > 0$. Define $b_n = n\varepsilon/K$ and $\beta_n = 1$. Clearly (1) holds. Since (iii) and the SLLN hold, it is easy to show that (iii) of Lemma 2 holds. In view of (ii) and (iv) we can use Lemma 2 and (i) to get

$$\limsup_{n\to\infty} |\sum_{m=1}^{\infty} a_{nm} Y_m| \leqq \limsup_{n\to\infty} \varepsilon n K^{-1} \sum_{m=1}^{\infty} |a_{nm} - a_{n,m+1}| \leqq \varepsilon \quad \text{a.s.} \quad \square$$

In the special case where $Y_1, Y_2, \cdots$ are independent, Theorem 3 is reminiscent of results of Chow (1965), Pruitt (1966) and Stout (1968), but does not seem to be a consequence of the wide-ranging theorems of these three authors. For example, let $Y_1, Y_2, \cdots$ be independent rv with $EY_n = 0$, $EY_n^2 = 1$ and let $a_{nm} = (n+m)^{-1}$ for all $n, m \geqq 1$. Note that Kolmogorov's SLLN applies. (i), (iii) and (iv) of Theorem 3 are clear. But (ii) also holds since $E|\sum_{j=1}^{m} Y_j| = O(m^{\frac{1}{2}})$ and $\sum_{m=1}^{\infty} |a_{nm} - a_{n,m+1}| m^{\frac{1}{2}} = \sum_{m=1}^{\infty} m^{\frac{1}{2}} (m+n)^{-1} (m+n+1)^{-1} < \sum_{m=1}^{\infty} m^{-\frac{3}{2}} < \infty$.
Therefore, Theorem 3 shows that $\sum_{m=1}^{\infty} Y_m/(n+m) \to 0$ a.s.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF REGINA
REGINA, SASKATCHEWAN, CANADA