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ON BUILDING RANDOM VARIABLES OF A GIVEN DISTRIBUTION

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Given $(X_t)_{t \geqq 1}$, independent random variables on some measurable space $(I, \mathscr{B})$ with the same distribution m, and a positive function f of $L^1(m)$ with $\|f\|_1 = 1$, this paper studies how to build a stopping time T with respect to the σ-fields $\mathscr{F}_t$ generated $X_1, X_2, \cdots, X_t$, such that the distribution of X_T in $(I, \mathscr{B})$ is exactly $f\,dm$.

1. Introduction. The construction of random variables with a given distribution by using independent rv uniformly distributed in [0, 1], called "random numbers" was the subject of a lecture [2] by J. Von Neumann, in 1951. This problem is called Problem B by Von Neumann, the making of random numbers by arithmetical, physical or other techniques is called Problem A, which shall not concern us here. Problem B corresponds to Title 5.1 in the classified bibliography made in [1].

It can be formulated in the following way: let $(X_1, X_2, \cdots)$ be independent rv from a probability space $(\Omega, \mathscr{A}, P)$ valued in a measurable space $(I, \mathscr{B})$, with the same distribution m. Call $\mathscr{F}_t$ the sub σ-field of $\mathscr{A}$ generated by $X_1, X_2, \cdots, X_t$. Consider also another probability space $(E, \mathscr{E}, \mu)$. Problem B is to find on Ω:

(a) A stopping time T with respect to $(\mathscr{F}_t)_{t \geqq 1}$, with $P(T < \infty) = 1$,

(b) An integer n and rv $N_1, N_2, \cdots, N_n$, $\mathscr{F}_T$ measurable and valued in $\{1, 2, \cdots, T\}$,

(c) An rv K, $\mathscr{F}_T$ measurable and valued in $\{1, 2, \cdots\}$ and for each k such that $P[K = k] > 0$ a measurable map g_k from $(I^n, \mathscr{B}^{\otimes n})$ to $(E, \mathscr{E})$, such that the distribution of $g_K(X_{N_1}, \cdots, X_{N_n})$ is μ.

For instance, if $(I, \mathscr{B}, m)$ is [0, 1] with Lebesgue measure, $(E, \mathscr{E}, \mu)$ is $[0, \infty)$ with Borel sets and $\mu(dx) = e^{-x}\,dx$, Von Neumann [1] suggests, taking $S_0 = 0$, $S_{k+1} = \inf\{k' > S_k; X_{S_k+1} > X_{S_k+2} > \cdots > X_{k'-1}$ and $X_{k'-1} \leqq X_{k'}\}$, $K = \inf\{k > 0;$ $S_k - S_{k-1}$ is even$\}$, $T = S_K$, $n = 1$, $N_1 = 1 + S_{K-1}$, $g_k(x) = k - 1 + x$.

We want to study here a rather restricted form of Problem B since we take from now $(E, \mathscr{E}) = (I, \mathscr{B})$, $n = K = 1$, $g_1 =$ identity and $N_1 = T$. In other terms, given a probability distribution μ on $(I, \mathscr{B})$, we want to find a stopping time T with respect to $(\mathscr{F}_t)_{t \geqq 1}$, with $P[T < \infty] = 1$, such that the distribution of X_T is μ. It is fairly obvious that a necessary condition for the existence of such a T is that μ is absolutely continuous with respect to m.

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Let us introduce some notation:

$$\mathscr{L} = \{f \in L^1(m); f \geqq 0 \text{ and } \|f\|_1 = 1\}.$$

If f is in $\mathscr{L}$, $S(f)$ is the set of stopping times T with respect to $(\mathscr{F}_t)_{t\geqq 1}$ such that the distribution of X_T is $f\,dm$. If k is a positive integer and f is in $\mathscr{L}$, $S_k(f)$ is the set of T in $S(f)$ such that $T \equiv 0 \bmod k$, and $\{T = t\}$ is measurable with respect to the sub σ-field generated by $X_{t-k+1}, X_{t-k+2}, \cdots, X_t$ and restricted to $T \geqq t$.

In order to understand the meaning of $S_k(f)$ we give the following example of a T in $S_2(f)$, due to Von Neumann [1]: $(I, \mathscr{B}, m)$ is again [0, 1] with Lebesgue measure and f in $\mathscr{L}$ is bounded by a number b. Then:

$$T = \inf\{t > 0;\ t \text{ is even and } bX_{t-1} \leqq f(X_t)\}.$$

We come back to general $(I, \mathscr{B}, m)$. The set $S(f)$ can be quite big; but there are two ways to choose interesting T in $S(f)$: small $E(T)$ or simplicity of building T (or both$\cdots$). We prove in Theorem 1 that T in $S(f)$ implies $\|f\|_\infty \leqq E(T)$ and T in $S_k(f)$ implies $k\|f\|_\infty \leqq E(T)$ ($\|f\|_\infty$ is $L^\infty(m)$ norm, we denote $\|f\|_\infty = \infty$ if f is not bounded). We prove in Theorem 2 that if m has no atoms, there exists T in $S(f)$ such that $E(T) = \|f\|_\infty$. In Section 4, we find an algorithm to construct a T in $S_2(f)$; if m has no atoms $E(T) = 2\|f\|_\infty$ (this was true also in the Von Neumann example when $b = \|f\|_\infty$).

In Section 5, we find an algorithm to construct a T in $S_1(f)$, but not for all f in $\mathscr{L}$. When $(I, \mathscr{B}, m)$ is [0, 1] with Lebesgue measure, this algorithm is a graphical one and could really give a practical method.

2. Inequalities.

THEOREM 1. *If f is in $\mathscr{L}$ and T is in $S(f)$, denote $\nu_t(B) = P(X_t \in B; T = t)$ and $f_t = d\nu_t/dm$. Then*

(i) $\|f_t\|_\infty \leqq P[T \geqq t]$

(ii) $\|f\|_\infty \leqq E(T)$ *and, if* $T \in S_k(f)$, $k\|f\|_\infty \leqq E(T)$.

PROOF. If there exists B in $\mathscr{B}$ such that $m(B) > 0$ and $f_t > P(T \geqq t)$ on B, then:

$$\int_B f_t\,dm = P[T = t \text{ and } X_t \in B] > P[T \geqq t]m(B).$$

But $\{T = t \text{ and } X_t \in B\} \subset \{T \geqq t \text{ and } X_t \in B\}$. Now T is a stopping time, $\{T \geqq t\} \in \mathscr{F}_{t-1}$ and

$$P[T \geqq t \text{ and } X_t \in B] = P[T \geqq t]m(B),$$

using the fact that $(X_t)_{t=1}^\infty$ are independent. Hence we get (i), from which (ii) is an obvious corollary.

REMARK 2.1. Values of f_1 are necessarily 0 or 1.

REMARK 2.2. It is worth mentioning here another simple inequality (that we do not need in the sequel). Suppose for a moment that all X_t have the same distribution m in $(I, \mathscr{B})$ but are not necessarily independent. Consider also a random variable N, integer valued, possibly dependent on $(X_t)_{t=1}^{\infty}$, but not necessarily a stopping time. The distribution μ of X_N is again absolutely continuous with respect to m. Then, if $f = d\mu/dm$, we have the following inequality, for each $\alpha > 0$:

$$E(N^\alpha) \geqq \frac{1}{1+\alpha} \int_I f^{\alpha+1}\, dm \tag{2.1}$$

with the limit cases $||N||_\infty \geqq ||f||_\infty$ when $\alpha \to \infty$ and $1 + E(\log N) \geqq \int_I f \log f\, dm$ when $\alpha \to 0$ (using $E(\log N) = \lim_{\alpha\to 0} E((N^\alpha - 1)/\alpha)$). To prove (2.1) introduce the set $B(y) = \{x \in I;\, f(x) \geqq y\}$. Hence:

$$\mu(B(y)) = \sum_{t=1}^{\infty} P[N = t;\, X_t \in B(y)] = \sum_{t \leqq y} P[X_t \in B(y)] + P[N > y]\,.$$

We can write

$$\alpha y^{\alpha-1}\mu(B(y)) \leqq \frac{\alpha}{\alpha+1}\,(\alpha+1)y^\alpha m(B(y)) + \alpha y^{\alpha-1}P[N > y]$$

and an integration on $[0, +\infty)$ with respect to dy of the last inequality gives (2.1).

3. A minimal stopping time. We prove in this section that if $f \in \mathscr{L}$, there exists T in $S(f)$ such that, with the notation of Theorem 1, $||f_t||_\infty = P[T \geqq t]$ and $||f||_\infty = E(T)$ when m has no atoms. To do this we need the following lemma, the proof of which is deferred to the end of the section.

LEMMA. *Let $(Y, \mathscr{Y}, Q)$ a measured space, where Q is a positive measure without atoms such that $Q(Y) \leqq 1$. Then there exists a map $\alpha \mapsto A(\alpha)$ from $[0, +\infty)$ to $\mathscr{Y}$ such that*

(i) $Q(A(\alpha)) = \inf(\alpha, Q(Y))$

(ii) *for any measurable map g from $(I, \mathscr{B})$ to $[0, +\infty)$ (with Borel sets), the subset of $Y \times I$,*

$$\bigcup_{x \in I} A(g(x)) \times \{x\}\,,$$

is $\mathscr{Y} \otimes \mathscr{B}$ measurable.

THEOREM 2. *If f is in $\mathscr{L}$ and m has no atoms, then there exists T in $S(f)$ such that $||f_t||_\infty = P[T \geqq t]$ for all t and, when f is bounded, $||f||_\infty = E(T)$.*

PROOF. For $t > 0$, denote by π_t the map from Ω to I^t defined by $(X_1, X_2, \cdots, X_t)$, $\mathscr{B}^t$ the product σ-field on I^t, and m^t the measure carried from P by π_t. For $t = 0$, I° is a set with one element, π_0 is the unique map from Ω to I°, $\mathscr{B}^\circ$ the unique σ-field on I°, m° the mass one in I°. We shall define T in a recursive way, with $\{T = 0\} = \emptyset$. Let us keep $t \geqq 0$ fixed.

Induction hypothesis. Suppose that the sets $\{T = i\}_{i=0}^{t}$ are defined on Ω and are such that, if ν_i is the measure on I carried from $\mathbf{1}_{\{T=i\}}P$ by X_i and if f_i denotes $d\nu_i/dm$, then for $i = 1, 2, \cdots, t$:

$$f_i(x) = \inf\{f(x) - \sum_{j=1}^{i-1} f_j(x),\, 1 - \sum_{j=1}^{i-1} P[T = j]\} \tag{3.1}$$

(an empty sum is zero).

Observe that if $t = 0$, this hypothesis is reduced to "$\{T = 0\}$ is defined." We choose $\{T = t + 1\}$ in Ω in such a way that the induction hypothesis is also true when we change t in $t + 1$. To do this, define $g_t(x) = f(x) - \sum_{j=1}^{t} f_i(x)$. We apply the lemma to $g = g_t$, $Y = \pi_t[\Omega\setminus\bigcup_{i=0}^{t}(T = i)] \subset I^t$; $\mathscr{Y}$ and Q are the restriction of $\mathscr{B}^t$ and m^t to Y. The sets $A(\alpha) \in \mathscr{Y}$ constructed in the lemma are now denoted $A_t(\alpha)$. Observe that $A_0(\alpha) = \emptyset$ if $\alpha < 1$ and $A_0(\alpha) = I^0$ if $\alpha \geqq 1$. We can now define the event:

$$\{T = t + 1\} = \pi_{t+1}^{-1}\{\bigcup_{x \in I} A_t(g_t(x)) \times \{x\}\} .$$

In other terms:

$$\{T = 1\} = \{\omega; f(X_1(\omega)) \geqq 1\}$$

$$\{T = 2\} = \{\omega; f(X_1) < 1 \text{ and } X_1 \in A_1(f(X_2) - f_1(X_2))\}$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\{T = t + 1\} = \{\omega; \omega \notin \bigcup_{i=1}^{t}(T = i) \text{ and } (X_1, \cdots, X_t) \in A_t(g_t(X_{t+1}))\} .$$

Hence, for B in $\mathscr{B}$, we have:

$$\nu_{t+1}(B) = \int_B m^t(A_t(g_t(x)))m(dx) , \quad \text{and}$$

$$f_{t+1}(x) = \frac{d\nu_{t+1}}{dm} = m^t(A_t(g_t(x))) = \inf\{g_t(x), 1 - \textstyle\sum_{i=1}^{t} P[T = i]\}$$

and the induction hypothesis is extended.

Since $\{T = t\}$ is now defined for all $t > 0$, it remains to prove that $\sum_{t=1}^{\infty} P[T = t] = 1$, that $||f_{t+1}||_\infty = P[T > t]$ for $t \geqq 0$ and that $||f||_\infty = E(T)$. Keeping $g_t = f - \sum_{i=1}^{t} f_i$, equality (3.1) implies:

$$g_{t+1} = [g_t - \textstyle\int_I g_t\, dm]^+ \tag{3.2}$$

(where $C^+ = \max(C, 0)$) since $\int_I g_t\, dm = 1 - \sum_{i=1}^{t} P[T = i]$.

Let $\lambda(x)$ be the limit of the decreasing sequence $(g_t(x))_{t=0}^{\infty}$.

From monotone convergence, we deduce from (3.2) that:

$$0 \leqq \lambda(x) = [\lambda(x) - \textstyle\int_I \lambda\, dm]^+ .$$

Hence: $\lambda = 0$ and $P[T < \infty] = 1$, and $T \in S(f)$.

If $||f_{t+1}||_\infty = P[T > t]$ is false for some $t \geqq 0$, then $g_t(x) < P[T > t]$ for all x which is impossible since $\int_I g_t\, dm = P[T > t)$. Denote $I_t = \{x; g_t(x) \geqq P[T > t]\}$. If $P[T > t] \neq 0$, then for x in $I_t\setminus I_{t-1}$, $f_t(x) = g_{t-1}(x)$ and $g_t(x) = f_t(x) - g_{t-1}(x) = 0 \geqq P[T > t]$. Hence $I_t \subset I_{t-1}$ when $t > 0$ and $P[T > t] \neq 0$. We get

$$||f||_\infty = \textstyle\sum_{t=1}^{\infty} ||f_t||_\infty = \sum_{t=0}^{\infty} P[T > t] = E(T) .$$

We proceed now to the proof of the lemma: Since Q has no atoms, for all integer $n > 0$, it is easy (by induction on n) to choose a family $B(k/2^n)$ with k integer and $0 \leqq k \leqq Q(Y)2^n$ such that

(i) $B(k/2^n) \in \mathscr{Y}$;

(ii) $B(k/2^n) \subset B(k'/2^n)$ if $k < k'$;

(iii) $Q(B(k/2^n)) = k/2^n$.

Set $A(\alpha) = \bigcup_{k/2^n<\alpha} B(k/2^n)$ if $\alpha < Q(Y)$ and $A(\alpha) = Y$ if $\alpha \geqq Q(Y)$. Clearly

(3.3) $$A(\alpha) = \bigcup_{\alpha'<\alpha} A(\alpha') .$$

Introduce now a sequence g_n of simple functions on I (simple = countable range at most), $\mathscr{B}$-measurable, such that $g_n(x)$ is nonnegative, the sequence $(g_n(x))_{n=1}^\infty$ is monotone and $\lim_n g_n(x) = g(x)$. Since g_n is simple, $\bigcup_{x\in I} A(g_n(x)) \times \{x\}$ is $\mathscr{B} \otimes \mathscr{Y}$ measurable; hence, using (3.3), this is also true for $\bigcup_{x\in I} A(g(x)) \times \{x\}$.

REMARK. If m has atoms (but not an atom of mass 1, of course) minor modifications to the statement of the lemma and proof of Theorem 2 would allow us to show that $S(f)$ is not empty; but the lower bound $||f||_\infty$ is not necessarily reached by $E(T)$.

4. An element of $S_2(f)$. We begin this section by some considerations relative to the set $S_k(f)$, $k \geqq 1$.

Recall that m^k is the product measure on $(I^k, \mathscr{B}^k)$. If $x = (x_1, x_2, \cdots, x_k)$ belongs to I^k we denote $p(x) = x_k$. If A is in $\mathscr{B}^k$, the measure on I carried from $\mathbf{1}_A m^k$ by p is absolutely continuous with respect to m. Let us denote its derivative f_A, that is to say, if $B \in \mathscr{B}$:

$$\int_B f_A \, dm = m^k(A \cap p^{-1}(B)) .$$

It is easily seen that to choose T in $S_k(f)$ is equivalent to define a sequence $(A_t)_{t=1}^\infty$ with A_t in $\mathscr{B}^k$ such that:

(4.1) $$\sum_{t=1}^\infty m^k(A_t) = \infty$$

(4.2) $$f(x) = \sum_{t=1}^\infty f_{A_t}(x) \prod_{i=1}^{t-1} (1 - m^k(A_i)) \qquad m\text{-almost surely.}$$

If such a sequence is given, the stopping time

$$T = \inf \{kt; (X_{(k-1)t+1}, X_{(k-1)t+2}, \cdots, X_{kt}) \in A_t\}$$

is finite (condition (4.1), with Borel–Cantelli) and is in $S_k(f)$. Note that

$$\prod_{i=1}^{t-1} (1 - m^k(A_i)) = P(T \geqq kt) .$$

Conversely, given T in $S_k(f)$, the sequence $(A_t)_{t=1}^\infty$ satisfying (4.1) and (4.2) is easy to build. The Von Neumann example of T in $S_2(f)$ seen in the introduction was corresponding to the constant sequence $A_t = A$ with

$$A = \{(x_1, x_2); bx_1 \leqq f(x_2)\} \subset [0, 1]^2 .$$

Let us give now another example of T in $S_2(f)$ when $(I, \mathscr{B}, m)$ is $[0, 1]$ with Lebesgue measure (f in $\mathscr{L}$ is not necessarily bounded). To do this, we define $H: \mathscr{L} \to \mathscr{L}$ by

$$\begin{aligned} Hg &= g && \text{if } \; g = 1 \; \text{ almost surely,} \\ &= (g-1)^+ / \textstyle\int_0^1 (g-1)^+ \, dm , && \text{otherwise.} \end{aligned}$$

Graphs of functions $H^0 f = f$, Hf, $H^2 f$, $\cdots$, are easy to draw. It will be a consequence of the proof of Theorem 3 below (stated for general $(I, \mathscr{B}, m)$), that the sequence:

$$A_t = \{(x_1, x_2); x_1 \leqq H^{t-1}f(x_2)\}$$

defines an element T of $S_2(f)$ and that $E(T) = 2\|f\|_\infty$.

THEOREM 3. *If f is in $\mathscr{L}$ and m has no atoms, then there exists T in $S_2(f)$ such that $\|f_{2t}\|_\infty = P[T \geqq 2t]$ for all t and, when f is bounded, $2\|f\|_\infty = E(T)$.*

PROOF. Define by induction on t a sequence $(g_t)_{t=0}^\infty$ in $L^1(m)$ as follows:

$$g_0 = f \tag{4.3}$$
$$g_{t+1}(x) = (g_t(x) - \textstyle\int_I g_t\, dm)^+ .$$

We apply the lemma of Section 3 to $(Y, \mathscr{Y}, Q) = (I, \mathscr{B}, m)$. The $A(\alpha)$ are the sets built in the lemma. Now we define the subsets $(A_t)_{t\geqq 1}$ of I^2:

$$A_t = \textstyle\bigcup_{x\in I} A[g_{t-1}(x)/\int_I g_{t-1}(x)] \times \{x\} ,$$

with $A_t = \emptyset$ if $g_{t-1} = 0$ a.e. Hence $f_{A_t}(x) = \inf\{1, g_{t-1}(x)/\int_I g_{t-1}\, dm\}$,

$$1 - m^2(A_t) = \textstyle\int_I g_t\, dm/\int_I g_{t-1}\, dm \tag{4.4}$$
$$\textstyle\prod_{i=1}^{t-1}(1 - m^2(A_i)) = \int_I g_{t-1}\, dm .$$

Let $\lambda(x)$ be the limit of the decreasing sequence $(g_t(x))_{t=0}^\infty$. From (4.3) and monotone convergence:

$$0 \leqq \lambda(x) = [\lambda(x) - \textstyle\int \lambda\, dm]^+ ,$$

and we get $\lambda = 0$. Now define $T = \inf\{2t; (X_{2t-1}, X_{2t}) \in A_t\}$. (4.4) implies that $P[T \geqq 2t] = \int_I g_{t-1}\, dm \to_{t\infty} 0$,

$$f_{2t}(x) = f_{A_t}(x) \textstyle\prod_{i=1}^{t-1}(1 - m^2(A_i)) = g_t(x) - g_{t-1}(x)$$

and we have $f(x) = \sum_{t=1}^\infty f_{2t}(x)$ a.e. Hence T is in $S_2(f)$, and it is easy to check that $P(T \geqq 2t) = \|f_{2t}\|_\infty$.

REMARK. If m has atoms, minor modifications to this proof show that $S_2(f)$ is not empty.

5. An element of $S_1(f)$. Notations and remarks at the beginning of Section 4 about $S_k(f)$ are still in force. In order to define an element of $S_1(f)$, we have to find a sequence $(A_t)_{t=1}^\infty$ of $\mathscr{B}$ such that

(5.1) (i) $\alpha_t = \prod_{i=1}^{t-1}(1 - m(A_i))$,

with $\alpha_1 = 1$, is such that $\alpha_t \to 0$ as $t \to \infty$,

(ii) $f = \sum_{t=1}^\infty \alpha_t \mathbf{1}_{A_t}$ a.e.

Note that (5.1) implies $A_1 \subset \{f \geqq 1\}$ and

$$A_t \subset \left\{\frac{1}{\alpha_t}[f - (\mathbf{1}_{A_1} + \cdots + \alpha_{t-1}\mathbf{1}_{A_t})] \geqq 1\right\} \qquad \text{for all } t > 1 .$$

To try to build an element of $S_1(f)$ we can replace these inclusions by equalities.

We shall see in Theorem 4 that this procedure really works, except for a very small class of f. We need some definitions: First we denote, if g is in $\mathscr{L}$, $A(g) = \{g \geqq 1\}$. Clearly $m(A(g)) > 0$, and $m(A(g)) = 1$ only if $g = 1$ a.e. Next define $G: \mathscr{L} \to \mathscr{L}$ by

$$Gg = g \quad \text{if} \quad g = 1 \quad \text{a.e.}$$
$$Gg = (g - \mathbf{1}_{A(g)})/(1 - m(A(g)))\,.$$

DEFINITION. f in $\mathscr{L}$ will be said a *resisting* function if there exist a sequence

$$(\beta_t)_{t\geqq 1}^{\infty} \text{ with } \beta_1 = 1\,, \quad 0 < \beta_t \leqq 1\,, \quad \beta_t \beta_{t+2} \leqq \beta_{t+1}^2 \quad \text{and}$$
$$\lim_{t\infty} \beta_t = B > 0\,,$$

and a random variable N on $(I, \mathscr{B}, m)$, valued in $\{0, 1, 2, \cdots\}$, whose distribution is given by:

$$m(N < t) = \beta_{t+1}/\beta_t\,,$$

such that $f(x) = B + \sum_{t=1}^{N(x)} \beta_t$ (with $f(x) = B$ when $N(x) = 0$).

EXAMPLE 5.0. $(I, \mathscr{B}, m)$ is $[0, 1]$ with Lebesgue measure. Consider a sequence $A_1 \supset A_2 \supset \cdots \supset A_t \supset \cdots$ with A_t in $\mathscr{B}$ and $m(A_t) = (t+1)^{-2}$. Denote $N(x) = \sup\{t; x \in A_t\}$, with $N(x) = 0$ if this set is empty. Then f defined by

$$f(x) = \tfrac{1}{2} \qquad \text{if} \quad N(x) = 0$$
$$f(x) = \frac{1}{2}\left[1 + N(x) + 1 + \tfrac{1}{2} + \cdots + \frac{1}{N(x)}\right] \qquad \text{if} \quad N(x) > 0$$

is resisting (take $\beta_t = (t+1)/2t$, $t = 1, 2, \cdots$).

PROPOSITION. *Gf is resisting when f is resisting. Furthermore*

$$\textstyle\sum_{t=0}^{\infty} m(A(G^t f)) < \infty\,.$$

PROOF. The function f being defined by $(\beta_t)_{\geqq t}$ and the rv N, it is easy to check that

$$Gf(x) = \frac{B}{\beta_2} + \sum_{t=2}^{N(x)} \frac{\beta_t}{\beta_2}$$

(with $Gf(x) = B/\beta_2$ when $N(x) \leqq 1$). This shows that Gf is resisting and defined by $(\beta_t')_{t\geqq 1}$ and the rv N, with $\beta_t' = \beta_{t+1}/\beta_2$ and $N' = (N-1)^+$. Trivially $1 - m(A(G^t f)) = \beta_{t+2}/\beta_{t+1}$ and $\prod_{t=0}^{\infty}(1 - m(A(G^t f))) = B$.

THEOREM 4. *Let f in $\mathscr{L}$, $A_1 = A[f]$, $\alpha_1 = 1$, $A_t = A[G^{t-1}f]$ and $\alpha_t = \prod_{i=1}^{t-1}(1 - m(A_i))$ if $t > 1$. Then $\lim_{t\infty} \alpha_t = 0$ and $f = \sum_{t=1}^{\infty} \alpha_t \mathbf{1}_{A_t}$ a.e. if and only if for any $t \geqq 0$, $G^t f$ is not resisting.*

PROOF. The "if" part has been proved by the preceding proposition. We show the "only if" part. Clearly $\lim_{t\infty} \alpha_t = 0$ implies $f = \sum_{t=1}^{\infty} \alpha_t \mathbf{1}_{A_t}$ a.e.

Suppose that $\lim_{t\infty} \alpha_t = K > 0$. This implies that $\sum_{t=1}^{\infty} m(A_t) < \infty$ and $\sum_{t=1}^{\infty} \mathbf{1}_{A_t} < \infty$ a.e. We can introduce the rv M by $M(x) = \sup\{t; x \in A_t\}$ $M(x) = 0$ if this set is empty. We have $M < \infty$ a.e. From the definition of G:

$$f = \alpha_1 \mathbf{1}_{A_t} + \cdots + \alpha_t \mathbf{1}_{A_t} + \alpha_{t+1} G^t f\,. \tag{5.2}$$

Since $\mathbf{1}_{A_t}(x) = 0$ if $t > M(x)$, one deduces that $G^t f(x) < 1$ if $t > M(x)$ and $\alpha_{M+1} G^M f = \alpha_{t+1} G^t f$ if $t \geqq M(x)$. Since $\lim_{t\infty} \alpha_t = K > 0$, we get $G^M f \leqq K/\alpha_{M+1}$. Using the fact that $G^t f$ is in $\mathscr{L}$, we write now

$$1 = \int_{M \leqq t} G^t f \, dm + \int_{M > t} G^t f \, dm . \tag{5.3}$$

But $G^t f \leqq f/K$ (from (5.2)). Hence $(G^t f)_{t=1}^{\infty}$ is uniformly integrable and $\int_{M>t} G^t f \, dm \to_{t\infty} 0$. Replacing $G^t f$ by $(\alpha_{M+1}/\alpha_{t+1}) G^M f \, dm$ in the first integral of (5.3) and doing $t \to +\infty$ we get from (5.3):

$$1 = \int_I \frac{\alpha_{M+1}}{K} G^M f \, dm .$$

The inequality $G^M f \leqq K/\alpha_{M+1}$ implies now that $\alpha_{M+1} G^M f = K$ and $\alpha_{t+1} G^t f = K$ if $t \geqq M$. Doing $t \to +\infty$ in (5.2) we get:

$$f = K + \sum_{i=1}^{\infty} \alpha_i \mathbf{1}_{A_i} \qquad \text{and} \qquad G^t f = \frac{K}{\alpha_{t+1}} + \sum_{i=t+1}^{\infty} \frac{\alpha_i}{\alpha_{t+1}} \mathbf{1}_{A_i} \qquad \text{if } t > 0 .$$

From the definition of G: $\mathbf{1}_{\mathscr{C} A_{t+1}} G^t f < 1$. The last equality gives:

$$\frac{K}{\alpha_{t+1}} + \sum_{i=t+1}^{\infty} \frac{\alpha_i}{\alpha_{i+1}} \mathbf{1}_{A_i \cap \mathscr{C} A_{t+1}} < 1 . \tag{5.4}$$

We deduce from (5.4) that there exists t_0 such that $A_{t_0+1} \supset A_{t_0+2} \supset \cdots$. To see this, we can take t_0 with $\alpha_{t_0+1} \leqq 2K$. Then if $t_0 \leqq t < i$ we get $K/\alpha_{t+1} \geqq \frac{1}{2}$ and $\alpha_i/\alpha_{t+1} \geqq \frac{1}{2}$; then (5.4) implies that $A_{t+1} \supset A_i$.

We claim now that $G^{t_0} f$ is a resisting function. Define $\beta_t = \alpha_{t_0+t}/\alpha_{t_0+1}$ and $N = \sup\{t; x \in A_{t_0+t}\}$, with $N = 0$ if this set is empty. A simple computation shows that $G^{t_0} f$ is the resisting function associated to $(\beta_t)_{t \geqq 1}$ and N.

REMARK 5.1. The class $\mathscr{L}_r$ of functions f in $\mathscr{L}$ such that $G^t f$ is resisting for some t is small: f in $\mathscr{L}_r$ implies that the distribution of the real rv f is discrete and unbounded.

REMARK 5.2. The method of proof of Theorem 4 can be used to show that if f is in $\mathscr{L} \backslash \mathscr{L}_r$, and if N is the number of visits to $[1, +\infty)$ by the process $(G^t f)_{t \geqq 0}$, then $\{N < \infty\} = \bigcup_{t=0}^{\infty} \{G_t f = 0\}$.

REMARK 5.3. $S_1(f)$ is never empty. Even if f is in $\mathscr{L}_r$, we could take $A_1 \in \{f \geqq 1\}$ such that $(f - \mathbf{1}_{A_1})(1 - m(A_1))^{-1} \notin \mathscr{L}_r$.

REMARK 5.4. One can study the class of functions f such that $\sum_{t=1}^{\infty} \mathbf{1}_{A(G^t f)} < \infty$ a.e. We shall say that f in $\mathscr{L}$ is *absorbable* if there exist a sequence $(\beta_t)_{t \geqq 1}$, with $\beta_1 = 1$, $0 < \beta_t \leqq 1$, $\beta_t \beta_{t+2} \leqq \beta_{t+1}^2$ and $\lim_{t\infty} \beta_t = 0$, and an rv N valued in $\{0, 1, 2, \cdots\}$, whose the distribution is given by $m(N < t) = \beta_{t+1}/\beta_t$, such that

$$f(x) = \sum_{t=1}^{N(x)} \beta_t$$

(with $f(x) = 0$ if $N(x) = 0$).

EXAMPLES. $(I, \mathscr{B}, m)$ is $[0, 1]$ with Lebesgue measure. Consider a sequence

$A_1 \supset A_2 \supset \cdots \supset A_t \supset \cdots$ with A_t in $\mathscr{B}$ and $m(A_t) = 1/(t+1)(t+2)$. Denote $N = \sup\{t; x \in A_t\}$ $N = 0$ is empty. Then $f = \sum_{t=1}^{N(x)} 1/t$ is absorbable (take $\beta_t = 1/t$, $t = 1, 2, \cdots$). With $\beta_t = \exp -(t-1)^\alpha$ with α in $(0, 1)$, we could get a bounded absorbable function.

One can prove that Gf is absorbable when f is absorbable. In that case $\sum_{t=0}^\infty \mathbf{1}_{A(G^t f)} < \infty$ a.e. Conversely if f is not in $\mathscr{L}_r$ and if there exists t_0 such that $A(G^t f) \supset A(G^{t+1} f)$ for $t \geqq t_0$, then $\sum_{t=0}^\infty \mathbf{1}_{A(G^t f)} < \infty$ a.e. imply that $G^{t_0} f$ is absorbable. The method of proof is the same as that in Theorem 4.

REMARK 5.5. We do not know whether $E(T) = \sum_{t=1}^\infty \alpha_t$ is finite when f is bounded (notations of Theorem 4, f not in $\mathscr{L}_r$). Trivially $\|f\|_\infty = E(T)$ if and only if $m(A_1 \cap A_2 \cap \cdots \cap A_t) > 0$ for all t.

REMARK 5.6. If f is bounded, $G^t f$ is bounded, but not uniformly with respect to t: in the examples of Remark 5.4, take f absorbable and $\beta_t = \exp -(t-1)^{\frac{1}{2}}$: we get $\|G^t f\|_\infty = K \exp t^{\frac{1}{2}}$ (K being a constant).

REMARK 5.7. The sets $(A_t)_{t=1}$ can be independent: if $(I, \mathscr{B}, m)$ is $[0, 1]$ with the Lebesgue measure, we take $f(x) = 2x$.

REMARK 5.8. If $I = \{0, 1\}$ and $m(0) = m(1) = \frac{1}{2}$, we take $f(0) = 2q$ and $f(1) = 2p$ with $p + q = 1$ and suppose that the binary expansion of p, $p = \sum_{i=1}^\infty \varepsilon_i / 2^i$, with $\varepsilon_i \in \{0, 1\}$, contains an infinite number of 0 and 1. Then

$$G^t f(1) = 2 \sum_{i=t+1}^\infty \frac{\varepsilon_i}{2^{i-t}} \quad \text{and} \quad G^t f(0) = 2 \sum_{i=t+1}^\infty \frac{1 - \varepsilon_i}{2^{i-t}}.$$

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