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A STOPPED BROWNIAN MOTION FORMULA[1]

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We determine $E[\exp(\alpha X(T) - \beta T)]$ where $X(t)$ is a Brownian motion having arbitrary drift and variance and T is the first time the process drops a specified amount below its maximum to date. From this result, the moments of $X(T)$ and T and some asymptotic distributions may be found. Applications in process control and financial management are mentioned.

1. Introduction and summary. Let $M(t) = \max\{X(u); 0 \leqq u \leqq t\}$ be the maximum process associated with a Brownian motion $X(t)$ starting from $X(0) = 0$ and having drift parameter μ and variance parameter σ^2. For a fixed $a > 0$ we are interested in the Markov time

$$T = T_a = \inf\{t \geqq 0 : M(t) - X(t) \geqq a\},$$

the first time the process drops a units below its maximum to date. Our main result is the identity

$$E[\exp(\alpha X(T) - \beta T)] = \frac{\delta \exp(-(\alpha + \gamma)a)}{\delta \cosh(\delta a) - (\alpha + \gamma)\sinh(\delta a)} \tag{1.1}$$

which holds for $\beta > 0$ (and even $\beta = 0$ when $\mu \neq 0$) and $\alpha < \theta$ where

$$\theta = \delta \coth(\delta a) - \gamma > 0$$

and

$$\gamma = \mu/\sigma^2$$
$$\delta = [(\mu/\sigma^2)^2 + 2\beta/\sigma^2]^{\frac{1}{2}}.$$

This is, of course, the joint moment generating function and Laplace transform for $X(T)$ and T. Using conventional techniques, we can determine moments and some asymptotic distributions from it. Some of this is done in Section 3.

2. Derivation of the formula. Because the processes are continuous, $X(T) = M(T) - a$ and the quantity we seek differs from $E[\exp(\alpha M(T) - \beta T)]$, which we will study first, only by the factor $e^{\alpha a}$. Introduce the random lifetime ζ, independent of the Brownian motion and having the exponential distribution

$$\Pr\{\zeta > t\} = e^{-\beta t}, \qquad t \geqq 0.$$

Because of the assumed independence,

$$\Pr\{T < \zeta\} = E[\Pr\{T < \zeta \mid T\}]$$
$$= E[e^{-\beta T}].$$

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In view of the independence of ζ and the Brownian motion, and the nonnegativity of $e^{\alpha M(T)}$, we may write[2]

$$\begin{aligned}
E[\exp(\alpha M(T) - \beta T)] &= E[e^{\alpha M(T)} \Pr\{T < \zeta \mid T\}] \\
&= E[e^{\alpha M(T)} E[\mathbf{1}_{\{T<\zeta\}} \mid X(\cdot)]] \\
&= E[E[e^{\alpha M(T)} \mathbf{1}_{\{T<\zeta\}} \mid X(\cdot)]] \\
&= E[e^{\alpha M(T)} \mathbf{1}_{\{T<\zeta\}}] \\
&= E[e^{\alpha M(T)}; T < \zeta] \\
&= \int_{\{T<\zeta\}} e^{\alpha M(T)} \Pr\{d\omega\} \\
&= \int_{\{T<\zeta\}} \int_0^\infty \mathbf{1}(z \leqq e^{\alpha M(T)})\, dz \Pr\{d\omega\} \\
&= \int_0^\infty \int \mathbf{1}(z \leqq e^{\alpha M(T)}; T < \zeta) \Pr\{d\omega\}\, dz \\
&= \int_0^\infty \Pr\{e^{\alpha M(T)} \geqq z; T < \zeta\}\, dz\,.
\end{aligned}$$

Observe from the proof that changing "$e^{\alpha M(T)} \geqq z$" to a strict inequality in the last expression will not change that expression's value.

In a moment we will look at the other case, but first suppose $\alpha > 0$. Then $e^{\alpha M(T)} \geqq 1$ and we continue in the manner

$$\begin{aligned}
E[\exp(\alpha M(T) - \beta T)] &= \int_0^1 \Pr\{T < \zeta\}\, dz + \int_1^\infty \Pr\{e^{\alpha M(T)} \geqq z; T < \zeta\}\, dz \\
&= E[e^{-\beta T}] + \int_0^\infty \Pr\{e^{\alpha M(T)} \geqq e^{\alpha x}; T < \zeta\}\, d\{e^{\alpha x}\} \\
&= E[e^{-\beta T}] + \int_0^\infty \Pr\{M(T) \geqq x; T < \zeta\} \alpha e^{\alpha x}\, dx\,.
\end{aligned}$$

Now go back and suppose $\alpha < 0$ so that $e^{\alpha M(T)} \leqq 1$. Then

$$\begin{aligned}
E[\exp(\alpha M(T) - \beta T)] &= \int_0^1 \Pr\{e^{\alpha M(T)} \geqq z, T < \zeta\}\, dz \\
&= \int_0^1 [\Pr\{T < \zeta\} - \Pr\{e^{\alpha M(T)} < z, T < \zeta\}]\, dz \\
&= E[e^{-\beta T}] + \int_0^\infty \Pr\{e^{\alpha M(T)} < e^{\alpha x}, T < \zeta\}\, d\{e^{\alpha x}\} \\
&= E[e^{-\beta T}] + \int_0^\infty \Pr\{M(T) > x, T < \zeta\} \alpha e^{\alpha x}\, dx\,,
\end{aligned}$$

and we obtain the same expression as before, except "$M(T) \geqq x$" has become a strict inequality. As mentioned, the value remains unchanged.

For $x \geqq 0$, introduce the Markov time

$$\tau = \tau(x) = \inf\{t \geqq 0 : M(t) = x\}\,.$$

Because $M(\cdot)$ is monotonic, $M(t) \geqq x$ if and only if $\tau(x) \leqq t$. This holds for all t and, indeed, holds for the random T as well. Thus, $M(T) \geqq x$ if and only if $\tau(x) \leqq T$ and our expression becomes

$$E[\exp(\alpha M(T) - \beta T)] = E[e^{-\beta T}] + \int_0^\infty \Pr\{\tau(x) \leqq T, T < \zeta\} \alpha e^{\alpha x}\, dx\,.$$

Now the bivariate process $(M(t), M(t) - X(t))$ is Markov and application of the strong Markov property at the random point $(M(\tau(x)), M(\tau(x)) - X(\tau(x))) = (x, 0)$

[2] Notation. $\mathbf{1}_A = \mathbf{1}(A)$ denotes the indicator random variable associated with an event A and $E[\cdot; A] = \int_A \cdot \Pr\{d\omega\}$ is expectation restricted to the event A.

leads to

$$\begin{aligned}\Pr\{\tau(x) \leqq T < \zeta\} &= \Pr\{T < \zeta \mid \tau(x) \leqq T, \tau(x) < \zeta\} \Pr\{\tau(x) \leqq T, \tau(x) < \zeta\} \\ &= \Pr\{T < \zeta\} \Pr\{\tau(x) \leqq T, \tau(x) < \zeta\} \\ &= E[e^{-\beta T}] f(x)\end{aligned}$$

where

$$f(x) = \Pr\{\tau(x) \leqq T, \tau(x) < \zeta\}\,.$$

These results simplify our expression to

$$E[\exp(\alpha M(T) - \beta T)] = E[e^{-\beta T}]\{1 + \textstyle\int_0^\infty f(x) \alpha e^{\alpha x}\, dx\}\,.$$

The problem has been reduced to that of determining $E[e^{-\beta T}]$ and $f(x) = \Pr\{\tau(x) \leqq T, \tau(x) < \zeta\}$. Begin with the latter. Again the strong Markov property applied at the point $(M(\tau(x)), M(\tau(x)) - X(\tau(x))) = (x, 0)$ together with the spatial homogeneity of the Brownian motion and the memoryless nature of ζ's exponential distribution show

$$\begin{aligned}f(x + y) &= \Pr\{\tau(x + y) \leqq T, \tau(x + y) < \zeta\} \\ &= \Pr\{\tau(x + y) \leqq T, \tau(x + y) < \zeta \mid \tau(x) \leqq T, \tau(x) < \zeta\} \\ &\qquad \times \Pr\{\tau(x) \leqq T, \tau(x) < \zeta\} \\ &= \Pr\{\tau(y) \leqq T, \tau(y) < \zeta\} \Pr\{\tau(x) \leqq T, \tau(x) < \zeta\} \\ &= f(x) f(y) \qquad \text{for } x, y \geqq 0\,.\end{aligned}$$

Cognizant of the bounded nature of $f(\cdot)$, we deduce

$$f(x) = e^{-\theta x}\,, \qquad x \geqq 0$$

for some constant $\theta \geqq 0$, and so

$$\begin{aligned}E[\exp(\alpha M(T) - \beta T)] &= E[e^{-\beta T}]\{1 + \textstyle\int_0^\infty f(x) \alpha e^{\alpha x}\, dx\} \\ &= E[e^{-\beta T}]\{1 + \alpha \textstyle\int_0^\infty \exp(-(\theta - \alpha)x)\, dx\} \qquad (2.1) \\ &= \left(\frac{\theta}{\theta - \alpha}\right) E[e^{-\beta T}] \qquad \text{for } \alpha < \theta\,, \\ &= \infty \qquad \text{for } \alpha \geqq \theta\,.\end{aligned}$$

We turn to the problem of determining θ. With

$$\lambda = \lambda_\pm = \frac{1}{\alpha}\left[\left(\frac{\mu}{\sigma^2}\right) \pm \left(\left(\frac{\mu}{\sigma^2}\right)^2 + \frac{2\beta}{\sigma^2}\right)^{\frac{1}{2}}\right] \qquad (2.2)$$

bring in the martingales

$$Z(t) = \exp(-\alpha\lambda X(t) - \beta t)\,.$$

We check the martingale property:

$$\begin{aligned}E[Z(t + s) \mid X(u); 0 \leqq u \leqq t] &= Z(t) E[Z(t + s)/Z(t) \mid X(u); 0 \leqq u \leqq t] \\ &= Z(t) E[Z(s)] \\ &= Z(t)\end{aligned}$$

since

$$E[Z(s)] = \exp(-\alpha\lambda\mu s + \tfrac{1}{2}\alpha^2\lambda^2\sigma^2 s - \beta s)$$
$$= 1$$

because $\frac{1}{2}\alpha^2\lambda^2\sigma^2 - \alpha\lambda\mu - \beta = 0$ when $\lambda = \lambda_\pm$ is given by (2.2). Apply the martingale optional stopping theorem to the Markov time $T \wedge \tau(x) = \min\{T, \tau(x)\}$. The conditions for this theorem are easily met since $X(t)$ is constrained to $-a \leqq X(t) \leqq x$ when $t < T \wedge \tau(x)$, and so $Z(t)$ also is bounded in this range. We note too that $T \wedge \tau(x) \leqq \tau(-a, x)$, the first time $X(t)$ reaches $-a$ or x.

Then, applying the theorem,

$$\begin{aligned} 1 &= E[Z(T \wedge \tau(x))] \\ &= E[\exp(-\alpha\lambda X(\tau(x)) - \beta\tau(x)); \tau(x) \leqq T] \\ &\qquad + E[\exp(-\alpha\lambda X(T) - \beta T); T < \tau(x)] \\ &= e^{-\alpha\lambda x}E[e^{-\beta\tau(x)}; \tau(x) \leqq T] + e^{\alpha\lambda a}E[\exp(-\alpha\lambda M(T) - \beta T); T < \tau(x)] \\ &= e^{-\alpha\lambda x}f(x) + e^{\alpha\lambda a}E[\exp(-\alpha\lambda M(T) - \beta T); T < \tau(x)]\,. \end{aligned}$$

Insert $f(x) = e^{-\theta x}$ and rearrange to read

$$e^{-\alpha\lambda a}[1 - \exp(-(\alpha\lambda + \theta)x)] = E[\exp(-\alpha\lambda M(T) - \beta T); T < \tau(x)]\,. \tag{2.3}$$

If we divide on the left by $x > 0$ and let $x \downarrow 0$ the limit obtained on the left is

$$(\alpha\lambda + \theta)e^{-\alpha\lambda a} = \lim_{x\downarrow 0}\frac{1}{x} \text{ R.H.S. of (2.3)}\,, \tag{2.4}$$

and this holds whether we take $\lambda = \lambda_+$ or $\lambda = \lambda_-$. We begin a similar evaluation of the right-hand side, up to terms of order x as $x \downarrow 0$. On the portion of the sample space where $\{T < \tau(x)\}$ we have $0 \leqq M(T) \leqq x$ so that $e^{-\alpha\lambda M(T)}$ is trapped between 1 and $e^{-\alpha\lambda x}$. That is $e^{-\alpha\lambda M(T)}$ differs from 1 by terms of order x as $x \downarrow 0$ and so

$$E[\exp(-\alpha\lambda M(T) - \beta T); T < \tau(x)] = (1 + O(x))E[e^{-\beta T}; T < \tau(x)]\,.$$

We again divide by $x > 0$, let $x \downarrow 0$, and obtain

$$\lim_{x\downarrow 0}\frac{1}{x} \text{ R.H.S. of (2.3)} = \lim_{x\downarrow 0}\frac{1}{x}E[e^{-\beta T}; T < \tau(x)]\,. \tag{2.5}$$

At this point we may either continue what is a long and tedious direct evaluation of the limit on the right, or we may do as Harry Kesten suggested and note from (2.5) that this limit does not depend on λ, and that therefore we may equate the left side of (2.4) when $\lambda = \lambda_+$ with itself when $\lambda = \lambda_-$ and deduce

$$(\alpha\lambda_+ + \theta)e^{-\alpha\lambda_+ a} = (\alpha\lambda_- + \theta)e^{-\alpha\lambda_- a}\,.$$

We solve this for θ (or make the long direct evaluation) and obtain

$$\theta = \frac{\alpha\lambda_- e^{-\alpha\lambda_- a} - \alpha\lambda_+ e^{-\alpha\lambda_+ a}}{e^{-\alpha\lambda_+ a} - e^{-\alpha\lambda_- a}}\,.$$

Abbreviate $\gamma = \mu/\sigma^2$ and $\delta = [(\mu/\sigma^2)^2 + 2\beta/\sigma^2]^{\frac{1}{2}}$ so that $\alpha\lambda_\pm = \gamma \pm \delta$. Then we may simplify the expression for θ to

$$\begin{aligned}\theta &= \frac{(\gamma - \delta)e^{\delta a} - (\gamma + \delta)e^{-\delta a}}{e^{-\delta a} - e^{\delta a}} \\ &= \delta \coth(\delta a) - \gamma .\end{aligned} \tag{2.6}$$

For later use, note that (2.4) and (2.5) state

$$\begin{aligned}\lim_{x\downarrow 0} \frac{1}{x} E[e^{-\beta T}; T < \tau(x)] &= (\alpha\lambda_+ + \theta)e^{-\alpha\lambda_+ a} \\ &= \delta[\coth(\delta a) + 1]\exp(-(\gamma + \delta)a) \\ &= \delta\left[\frac{e^{\delta a} + e^{-\delta a}}{e^{\delta a} - e^{-\delta a}} + 1\right]e^{-(\gamma+\delta)a} \\ &= \delta\left[\frac{2e^{\delta a}}{e^{\delta a} - e^{-\delta a}}\right]e^{-(\gamma+\delta)a} \\ &= \frac{\delta e^{-\gamma a}}{\sinh(\delta a)} .\end{aligned} \tag{2.7}$$

We have determined θ. Equation (2.1), our reduced formula for $E[\exp(\alpha M(T) - \beta T)]$, contains only one factor remaining unknown, namely $E[e^{-\beta T}]$, and we turn to determining it. Again invoke the strong Markov property at the point $(M(\tau(x)), M(\tau(x)) - X(\tau(x))) = (x, 0)$ and use the spatial homogeneity to see

$$E[\exp(-\beta(T - \tau(x)) - \beta\tau(x)); \tau(x) < T] = E[e^{-\beta T}]E[e^{-\beta\tau(x)}; \tau(x) < T] .$$

Insert this into the decomposition

$$\begin{aligned}E[e^{-\beta T}] &= E[e^{-\beta T}; \tau(x) < T] + E[e^{-\beta T}; \tau(x) \geqq T] \\ &= E[\exp(-\beta(T - \tau(x)))e^{-\beta\tau(x)}; \tau(x) < T] + E[e^{-\beta T}; \tau(x) \geqq T] \\ &= E[e^{-\beta\tau(x)}; \tau(x) < T]E[e^{-\beta T}] + E[e^{-\beta T}; \tau(x) \geqq T]\end{aligned}$$

and solve for $E[e^{-\beta T}]$ to obtain

$$E[e^{-\beta T}] = \frac{E[e^{-\beta T}; \tau(x) \geqq T]}{1 - E[e^{-\beta\tau(x)}; \tau(x) < T]} . \tag{2.8}$$

Recognize the denominator on the right as $1 - f(x) = 1 - e^{-\theta x}$, so that

$$E[e^{-\beta T}] = \frac{E[e^{-\beta T}; T < \tau(x)]}{1 - e^{-\theta x}} .$$

As this holds for any $x > 0$, it holds in the limit as $x \downarrow 0$. Divide numerator and denominator by $x > 0$, let x vanish and refer to (2.6) and (2.7) to see

$$\begin{aligned}E[e^{-\beta T}] &= \lim_{x\downarrow 0} \frac{E[e^{-\beta T}; T < \tau(x)]}{1 - e^{-\theta x}} \\ &= \frac{\lim_{x\downarrow 0} x^{-1}E[e^{-\beta T}; T < \tau(x)]}{\lim_{x\downarrow 0} x^{-1}(1 - e^{-\theta x})}\end{aligned}$$

$$(2.9) \qquad = \frac{1}{\theta}(\alpha\lambda + \theta)e^{-\alpha\lambda a}$$

$$= \frac{\delta e^{-\gamma a}}{\sinh(\delta a)[\delta \coth(\delta a) - \gamma]} \qquad \text{(Using (2.6) and (2.7))}$$

$$= \frac{\delta e^{-\gamma a}}{\delta \cosh(\delta a) - \gamma \sinh(\delta a)}.$$

From the third equality above and using (2.7)

$$(2.10) \qquad \theta E[e^{-\beta T}] = (\alpha\lambda_+ + \theta)e^{-\alpha\lambda_+ a}$$

$$= \frac{\delta e^{-\gamma a}}{\sinh(\delta a)}.$$

Return to (2.1) which reads, when $\alpha < \theta = \delta \coth(\delta a) - \gamma$,

$$E[\exp(\alpha M(T) - \beta T)]$$

$$= \frac{\theta E[e^{-\beta T}]}{\theta - \alpha}$$

$$= \frac{\delta e^{-\gamma a}}{\sinh(\delta a)[\delta \coth(\delta a) - (\alpha + \gamma)]} \qquad \text{(From (2.6) and (2.10))}$$

$$= \frac{\delta e^{-\gamma a}}{\delta \cosh(\delta a) - (\alpha + \gamma)\sinh(\delta a)}.$$

Finally,

$$E[\exp(\alpha X(T) - \beta T)] = e^{-\alpha a}E[\exp(\alpha M(T) - \beta T)]$$

$$= \frac{\delta e^{-(\alpha+\gamma)a}}{\delta \cosh(\delta a) - (\alpha + \gamma)\sinh(\delta a)}, \qquad \alpha < \theta.$$

This completes the derivation of (1.1).

3. Moments and asymptotics. Using standard techniques we can extract the moments, marginal transforms and asymptotic distributions from the identity just derived. Here are some sample results, stated in terms of $\gamma = \mu/\sigma^2$ and $\delta = [(\mu/\sigma^2)^2 + 2\beta/\sigma^2]^{\frac{1}{2}}$. The proofs, of less interest than the results themselves, follow.

(a)

$$(3.1) \qquad E[e^{\alpha X(T)}] = \frac{\gamma e^{-(\alpha+\gamma)a}}{\gamma \cosh(\gamma a) - (\alpha + \gamma)\sinh(\gamma a)} \quad \text{for } \alpha < \theta \quad \text{if } \mu \neq 0,$$

$$= \frac{e^{-\alpha a}}{1 - \alpha a} \quad \text{for } \alpha < \frac{1}{a} \quad \text{if } \mu = 0;$$

(Recall (2.6): $\theta = \delta \coth(\delta a) - \gamma$.)

(b)

$$(3.2) \qquad E[X(T)] = \frac{1}{2\gamma}[e^{2\gamma a} - 1] - a \quad \text{if } \mu \neq 0$$

$$= 0 \quad \text{if } \mu = 0;$$

(c)

(3.3)
$$\text{Var}\,[X(T)] = \left[\frac{e^{2\gamma a}-1}{2\gamma}\right]^2 \quad \text{if} \quad \mu \neq 0$$
$$= a^2 \quad \text{if} \quad \mu = 0\,;$$

(d) When $\mu < 0$,

(3.4)
$$\lim_{a\to\infty} \Pr\{X(T) + a > x\} = e^{-2|\gamma|x}\,, \qquad x > 0\,;$$

(e) When $\mu > 0$,

(3.5)
$$\lim_{a\to\infty} \Pr\{e^{-2\gamma a}X(T) > x\} = e^{-2\gamma x}\,, \qquad x > 0\,;$$

(f)

(3.6)
$$\lim_{a\downarrow 0} \Pr\{[X(T) + a]/a > x\} = e^{-x}\,, \qquad x > 0\,;$$

(g)

(3.7)
$$E[e^{-\beta T}] = \frac{\delta e^{-\gamma a}}{\delta \cosh(\delta a) - \gamma \sinh(\delta a)}\,;$$

(h) When $\mu = 0$ this reduces to

(3.8)
$$E[e^{-\beta T}] = \frac{1}{\cosh((2\beta/\sigma^2)^{\frac{1}{2}}a)}\,,$$

(i)
$$E[T] = \frac{1}{\mu}\left\{\frac{1}{2\gamma}[e^{2\gamma a} - 1] - a\right\} \quad \text{if} \quad \mu \neq 0\,,$$
$$= \frac{a^2}{\sigma^2} \quad \text{if} \quad \mu = 0\,;$$

(j) If $\mu > 0$,
$$\lim_{a\to\infty} \Pr\{e^{-2\gamma a}T > t\} = e^{-2\gamma^2\sigma^2 t}\,;$$

Finally,

(k) If $\mu < 0$ then asymptotically for large a, $U = (T - a/|\mu|)/(a\sigma^2/|\mu|^3)^{\frac{1}{2}}$ is normally distributed with zero mean and unit variance.

The proofs of these results use standard techniques learned in a first course in probability. First suppose $\mu \neq 0$. Then, when $\beta = 0$, we have $\delta = ((\mu/\sigma^2)^2)^{\frac{1}{2}} = |\gamma|$ and the basic identity becomes

$$\phi(\alpha) = E[e^{\alpha X(T)}] = \frac{|\gamma| e^{-(\alpha+\gamma)a}}{|\gamma| \cosh(|\gamma| a) - (\alpha + \gamma)\sinh(|\gamma| a)}\,.$$

This gives (3.1) immediately when $\gamma = \mu/\sigma^2 > 0$. When $\gamma = \mu/\sigma^2 < 0$ we substitute $|\gamma| = -\gamma$ and obtain

$$\phi(\alpha) = \frac{-\gamma e^{-(\alpha+\gamma)a}}{-\gamma \cosh(-\gamma a) - (\alpha+\gamma)\sinh(-\gamma a)}$$
$$= \frac{\gamma e^{-(\alpha+\gamma)a}}{\gamma \cosh(\gamma a) - (\alpha+\gamma)\sinh(\gamma a)}$$

because cosh (•) is an even function while sinh (•) is odd. This shows (3.1) when $\mu \neq 0$.

Continue to suppose $\mu \neq 0$, and differentiate $\phi(\alpha) = E[e^{\alpha X(T)}]$ to get

$$\phi'(\alpha) = \phi(\alpha)\left\{\frac{\sinh(\gamma a)}{\gamma\cosh(\gamma a) - (\alpha+\gamma)\sinh(\gamma a)} - a\right\}.$$

Then $\phi(0) = 1$ and

$$\begin{aligned}\phi'(0) &= \frac{\sinh(\gamma a)}{\gamma\cosh(\gamma a) - \gamma\sinh(\gamma a)} - a \\ &= \frac{1}{2\gamma}[e^{2\gamma a} - 1] - a\,.\end{aligned}$$

Since $E[X(T)] = \phi'(0)$, this verifies (3.2) when $\mu \neq 0$.

Again differentiate $\phi'(\alpha)$ and set $\alpha = 0$ to get

$$\phi''(0) = [\phi'(0)]^2 + \left\{\frac{\sinh(\gamma a)}{\gamma[\cosh(\gamma a) - \sinh(\gamma a)]}\right\}^2$$

so that

$$\begin{aligned}\operatorname{Var}[X(T)] &= \phi''(0) - [\phi'(0)]^2 \\ &= \left\{\frac{\sinh(\gamma a)}{\gamma[\cosh(\gamma a) - \sinh(\gamma a)]}\right\}^2 \\ &= \frac{1}{\gamma^2}[e^{\gamma a}\sinh(\gamma a)]^2 \\ &= \left[\frac{e^{2\gamma a} - 1}{2\gamma}\right]^2,\end{aligned}$$

which verifies (3.3) when $\mu \neq 0$. Now let us suppose $\mu = 0$. When $\beta > 0$, $\lambda_+ \neq \lambda_-$ which is enough to obtain the basic identity with $\gamma = 0$ and $\delta = (2\beta/\sigma^2)^{\frac{1}{2}}$. That is

$$E[\exp(\alpha X(T) - \beta T)] = \frac{\delta e^{-\alpha a}}{\delta\cosh(\delta a) - \alpha\sinh(\delta a)}\,.$$

As $\beta \downarrow 0$, the left side increases to $E[e^{\alpha X(T)}]$ by monotone convergence. Use $\cosh(\delta a) = 1 + o(\delta a)$ and $\sinh(\delta a) = \delta a + o(\delta a)$ to see

$$\begin{aligned}E[e^{\alpha X(T)}] &= \lim_{\delta\downarrow 0}\frac{\delta e^{-\alpha a}}{\delta(1 + o(\delta a)) - \alpha(\delta a + o(\delta a))} \\ &= \frac{e^{-\alpha a}}{1 - \alpha a}\,, \qquad \alpha < \frac{1}{a}\end{aligned}$$

as claimed. We repeat the derivation of moments with $\phi(\alpha) = e^{-\alpha a}/(1 - \alpha a)$. Then $\phi'(\alpha) = \alpha a^2\phi(\alpha)/(1 - \alpha a)$, and, easily

$$\begin{aligned}E[X(T)] &= \phi'(0) = 0\,, \\ E[X(T)^2] &= \phi''(0) = a^2\,.\end{aligned}$$

Equation (3.4) states that when $\mu < 0$, asymptotically $X(T) + a$ is exponentially distributed with parameter $\nu = 2|\mu|/\sigma^2$. The corresponding moment generating function is

$$\int_0^\infty e^{\alpha x}\nu e^{-\nu x}\,dx = \frac{\nu}{\nu - \alpha}, \qquad \alpha < \nu .$$

Thus, to verify the asymptotic distribution, we need only show

$$\lim_{a\to\infty} E[\exp(\alpha(X(T) + a))] = \nu/(\nu - \alpha)$$

and invoke Lévy's theorem on the convergence of generating functions. We have

$$\begin{aligned} E[\exp(\alpha(X(T) + a))] &= e^{+\alpha a}\phi(\alpha) \\ &= \frac{\gamma e^{-\gamma a}}{\gamma \cosh(\gamma a) - (\alpha + \gamma)\sinh(\gamma a)} \\ &= \frac{\gamma e^{-\gamma a}}{\gamma\left\{\dfrac{e^{\gamma a} + e^{-\gamma a}}{2}\right\} - (\alpha + \gamma)\left\{\dfrac{e^{\gamma a} - e^{-\gamma a}}{2}\right\}} \\ &= \frac{2\gamma}{\gamma\{e^{2\gamma a} + 1\} - (\alpha + \gamma)\{e^{2\gamma a} - 1\}} . \end{aligned}$$

When $\gamma = \mu/\sigma^2 < 0$, then $e^{2\gamma a} \to 0$ as $a \to \infty$ so

$$\begin{aligned} \lim_{a\to\infty} E[\exp(\alpha(X(T) + a))] &= \frac{2\gamma}{\gamma + \alpha + \gamma} \\ &= \frac{2\gamma}{2\gamma + \alpha} \\ &= \frac{\nu}{\nu - \alpha} \end{aligned}$$

since $\nu = 2|\gamma| = -2\gamma$ when $\gamma < 0$.

When $\mu > 0$ we still arrive at an asymptotic exponential distribution, but the normalization changes. Let $U = e^{-2\gamma a}X(T)$. Then

$$\begin{aligned} E[e^{\alpha U}] &= \frac{2\gamma e^{-\gamma a}\exp(-\alpha a e^{-2\gamma a})}{\gamma(e^{\gamma a} + e^{-\gamma a}) - (\alpha e^{-2\gamma a} + \gamma)(e^{\gamma a} - e^{-\gamma a})} \\ &= \frac{2\gamma e^{-\gamma a}\exp(-\alpha a e^{-2\gamma a})}{(2\gamma + \alpha e^{-2\gamma a})e^{-\gamma a} - \alpha e^{-2\gamma a}e^{\gamma a}} \\ &= \frac{2\gamma \exp(-\alpha a e^{-2\gamma a})}{(2\gamma + \alpha e^{-2\gamma a}) - \alpha} . \end{aligned}$$

Now let $a \to \infty$. Since $\gamma > 0$ we have

$$\lim_{a\to\infty} E[e^{\alpha U}] = \frac{2\gamma}{2\gamma - \alpha}$$

which shows that the limiting distribution of $U = e^{-2\gamma a}X(T)$ is exponential with parameter $2\gamma = 2\mu/\sigma^2$. This proves (3.5).

Next, let $U = (X(T) + a)/a$. Then

$$E[e^{\alpha U}] = \frac{e^{\alpha}\gamma \exp(-(\alpha/a + \gamma)a)}{\gamma \cosh(\gamma a) - (\alpha/a + \gamma)\sinh(\gamma a)}$$
$$= \frac{\gamma e^{-\gamma a}}{\gamma(1 + o(\gamma a)) - (\alpha/a + \gamma)(\gamma a + o(\gamma a))},$$

and

$$\lim_{a \downarrow 0} E[e^{\alpha U}] = \frac{1}{1 - \alpha}.$$

This is enough to prove (3.6).

Equation (3.7), the Laplace transform of the distribution of T, is an immediate consequence of the basic identity under $\alpha = 0$, and (3.8), the simpler form when $\mu = 0$ follows easily.

One may obtain $E[T]$ when $\mu \neq 0$ from Wald's identity $E[X(T)] = \mu E[T]$, or one may take the longer route and compute $E[T] = -\partial E[e^{-\beta T}]/\partial\beta|_{\beta=0}$. We omit the derivation. When $\mu = 0$, one may use the identity $E[X(T)^2] = \sigma^2 E[T]$, or differentiate. Again, this is omitted.

The asymptotic exponential distribution of T when $\mu > 0$ follows from Lévy's theorem and the convergence of the appropriately normalized Laplace transform. From the Taylor series expansion, note that $\delta = \gamma + \beta/\gamma\sigma^2 + o(\beta)$ as $\beta \to 0$. Then

$$E[e^{-\beta T}] = \frac{2\delta e^{-\gamma a}}{\delta(e^{\delta a} + e^{-\delta a}) - \gamma(e^{\delta a} - e^{-\delta a})}$$
$$= \frac{2\delta}{(\delta - \gamma)e^{(\delta+\gamma)a} + (\delta + \gamma)e^{-(\delta-\gamma)a}}$$
$$= \frac{2(\gamma + O(\beta))}{\left[\frac{\beta}{\gamma\sigma^2} + o(\beta)\right]\exp[[2\gamma + O(\beta)]a] + [2\gamma + O(\beta)]e^{-O(\beta)a}}$$
$$= [1 + O(\beta)]\frac{2\gamma}{2\gamma + \left(\frac{\beta}{\gamma\sigma^2}\right)e^{2\gamma a}},$$

and with $U = e^{-2\gamma a}T$,

$$E[e^{-\beta U}] = E[\exp\{-(\beta e^{-2\gamma a})T\}]$$
$$= [1 + O(e^{-2\gamma a})]\left\{\frac{2\gamma}{2\gamma + \beta/\gamma\sigma^2}\right\}$$
$$\to \frac{2\gamma^2\sigma^2}{2\gamma^2\sigma^2 + \beta} \qquad \text{as } a \to \infty.$$

This is the Laplace transform of the claimed limiting exponential distribution.

Our last claim is the asymptotic normal distribution of $(T + a/\mu)/(a\sigma^2/|\mu|^3)^{\frac{1}{2}}$

when $\mu < 0$. We will use the expansion

$$\begin{aligned}\delta &= \left(|\gamma|^2 + \frac{2\beta}{\sigma^2}\right)^{\frac{1}{2}} \\ &= |\gamma| + \frac{\beta}{|\gamma|\sigma^2} - \frac{\beta^2}{2|\gamma|^3\sigma^4} + \cdots \\ &= |\gamma| + O(\beta)\,.\end{aligned}$$

Then

$$\begin{aligned}E[e^{-\beta T}] &= \frac{2\delta e^{-\gamma a}}{\delta(e^{\delta a} + e^{-\delta a}) - \gamma(e^{\delta a} - e^{-\delta a})} \\ &= \frac{2\delta}{(\delta - \gamma)e^{(\delta+\gamma)a} + (\delta + \gamma)e^{-(\delta-\gamma)a}} \\ &= \frac{2[|\gamma| + O(\beta)]}{[2|\gamma| + O(\beta)]e^{(\delta+\gamma)a} + O(\beta)\exp[-[2|\gamma| + O(\beta)]a]} \\ &= [1 + O(\beta)]e^{-(\delta+\gamma)a} \\ &= [1 + O(\beta)]\exp\left\{-\frac{\beta a}{|\gamma|\sigma^2} + \frac{\beta^2 a}{2|\gamma|^3\sigma^4}\right\}\end{aligned}$$

and

$$\begin{aligned}E[e^{-\beta U}] &= E\left[\exp\left\{-\beta\left(T - \frac{a}{|\gamma|\sigma^2}\right)\Big/\left(\frac{a}{|\gamma|\mu^2}\right)^{\frac{1}{2}}\right\}\right] \\ &= e^{\beta(a|\gamma|)^{\frac{1}{2}}}E\left[\exp\left\{-\frac{\beta}{(a/|\gamma|\mu^2)^{\frac{1}{2}}}\,T\right\}\right] \\ &= e^{\beta(a|\gamma|)^{\frac{1}{2}}}\left[1 + O\left(\frac{1}{a^{\frac{1}{2}}}\right)\right]\exp\left\{-\frac{\beta}{(a/|\gamma|\mu^2)^{\frac{1}{2}}}\frac{a}{|\gamma|\sigma^2} + \left(\frac{\beta}{(a/|\gamma|\mu^2)^{\frac{1}{2}}}\right)^2\frac{a}{2|\gamma|^3\sigma^4}\right\} \\ &= \left[1 + O\left(\frac{1}{a^{\frac{1}{2}}}\right)\right]e^{\beta^2/2}\,.\end{aligned}$$

As $a \to \infty$ this converges to the (symmetric) moment generating function of the standard normal distribution. This completes the proof of convergence.

4. Examples. Here are two applications of the formula of Section 3. The second problem is the one that motivated the study.

Quickest detection. A process starts as a Brownian motion with drift $\nu_0 > 0$. At some random and unobservable time Θ the drift drops to $\nu_1 < 0$. We want to detect the shift time Θ as soon as possible after it occurs. The Markov time T is a natural candidate.

This model is further motivated by an analog in discrete time process control. Here $U_1, U_2, \cdots$ are observations on a process. While "in-control," these observations are independent and identically distributed with mean μ_0. At the random time Θ the process goes "out-of-control" and the mean lowers to $\mu_1 < \mu_0$. The well-known cumulative sum control scheme for monitoring such a process, introduced by Page [2], looks at the partial sum $S_n = U_1 + \cdots + U_n - nh$.

Here h is a constant, often chosen to be $h = (\mu_0 + \mu_1)/2$. The method signals detection when S_n first drops a specified amount below its previous maximum. Our Brownian motion is the continuous time analog to S_n, and $T = T(a)$ compares to the one-sided cumulative sum detection time.

Returning to the continuous case, of interest in evaluating quickest detection schemes are the average run lengths in-control, ARL_0 and out-of-control, ARL_1, defined to be the mean times

$$\text{ARL}_0 = E[T \mid \mu = \nu_0 > 0]$$

and

$$\text{ARL}_1 = E[T \mid \mu = \nu_1 < 0]\,.$$

The exponential role of $\gamma = \mu/\sigma^2$ in

$$E[T] = \frac{1}{\mu}\left\{\frac{1}{2\gamma}[e^{2\gamma a} - 1] - a\right\} \tag{4.1}$$

shows that, as desired, the ARL is high when in-control ($\mu > 0$) and small when out-of-control ($\mu < 0$). These continuous time average run lengths might well approximate the discrete time situation and it would be of interest to compare the two numerically or seek weak convergence justification.

Also of interest here is the probability of false alarm, $\Pr\{T < \Theta\}$. In the common case where Θ is assumed exponentially distributed with parameter β, we have a ready formula in (3.7):

$$\begin{aligned} \Pr\{T < \Theta\} &= E[e^{-\beta T}] \\ &= \frac{\delta e^{-\gamma a}}{\delta \cosh(\delta a) - \gamma \sinh(\delta a)}\,. \end{aligned} \tag{4.2}$$

The random walk model for stock prices. In 1900 Bachelier [1] proposed that prices of securities traded in a perfect market should fluctuate randomly, and from this he derived, at least informally, the Brownian motion process. A better model, because it maintains nonnegative prices and exhibits the observed long run exponential growth, is the geometric or multiplicative Brownian motion $W(t) = e^{X(t)}$ (see e.g. Samuelson (1973)). Let us suppose we are holding a stock whose price $W(t)$ fluctuates according to this model, where $X(t)$ is a Brownian motion with drift μ and variance σ^2. If we sell at time $T = T(a)$ our discounted mean return is

$$E[e^{-\beta T} W(T)] = \frac{\delta e^{-(1+\gamma)a}}{\delta \cosh(\delta a) - (1 + \gamma)\sinh(\delta a)} \tag{4.3}$$

where $\beta > 0$ is the discount rate.

The selling time T embodies the Wall Street maxim "Ride your winners and cut your losses." One would hope that such a rule would maintain an investor in the better stocks longer than in the poorer ones. Out of a group of n stocks having parameters $(\mu_1, \sigma_1^2), \cdots, (\mu_n, \sigma_n^2)$, pick a stock at random and sell it at time T, then pick another stock at random and hold it until its T, and so on.

The rate of growth of a portfolio managed under this strategy is found by solving for β in $E[e^{-\beta T}W(T)] = 1$, where now we consider (μ, σ^2) to be random. According to (4.3), with $\gamma_i = \mu_i/\sigma_i^2$ and $\delta_i = (\gamma_i^2 + 2\beta/\sigma_1^2)^{\frac{1}{2}}$ we solve for β in

$$1 = \frac{1}{n} \sum_{i=1}^{n} \frac{\delta_i \exp\{-(1 + \gamma_i)a\}}{\delta_i \cosh(\delta_i a) - (1 + \gamma_i) \sinh(\delta_i a)} .$$

Whether or not this rate is higher than the growth rate of the group as a whole depends on how μ_i varies with σ_i^2 in the group, and must be checked numerically under any given assumptions.

5. Related results. Stated in our notation, Brockwell (1974) has shown

$$\Pr\{\tau(x) \leqq T\} = \exp\{-2\gamma x(e^{2\gamma a} - 1)^{-1}\} .$$

This coincides with the special case $\beta = 0$ in our $f(x) = e^{-\theta x}$. Brockwell, in addition, points out applications of his formula in cell growth models and heavy traffic queues.

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