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# FIRST PASSAGE DISTRIBUTIONS OF PROCESSES WITH INDEPENDENT INCREMENTS[1]

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Let $\{X_t, t \geqq 0\}$ be a process with stationary independent increments taking values in $d$-dimensional Euclidean space. Let $S$ be a set in $R^d$, and let $T = \inf\{t > 0 : X_t \notin S\}$. For a reasonably wide class of processes and sets $S$, criteria are given for deciding when $P\{X_T \in B\} > 0$ and when $P\{X_T \in B\} = 0$, where $B \subset \partial S$.

**1. Introduction.** This paper studies certain properties of the first passage distributions for $d$-dimensional stochastic processes with stationary independent increments. Specifically, if the process $X$ starts in an open connected set $S$ containing 0 we wish to decide whether or not $X$ hits the boundary of $S$ upon first leaving $S$, and if so which parts of the boundary are hit in this manner. The main results of this paper give such criteria under various regularity assumptions on $S$ and $X$. Roughly speaking, if $T_r$ is the time to leave a sphere of radius $r$, and if $A$ is a cone in $R^d$ with vertex at 0 and with sufficiently small vertex angle, then upon first leaving $S$, $X$ will not hit the part of $\partial S$ falling in $A$ if $\int_{y \in A} E^0 T_{|y|} \nu(dy) = +\infty$; here $\nu$ is the Lévy measure of $X$. There is also a converse to the effect that if the vertex angle of $A$ is rather large, and if $\int_{y \in A} E^0 T_{|y|} \nu(dy) < \infty$, then upon leaving $S$, $X$ will hit $\partial S \cap A$. See Section 3 for the precise results, and Section 4 for some examples. The conditions used in formulating the basic criteria are slightly involved; this is perhaps unavoidable (see the example near the end of this section) if one is to have a result that applies to a reasonably wide selection of regions and processes. Nevertheless, several interesting and easily formulated results emerge; for example, it is reasonably easy to deduce from these criteria that an isotropic process will hit $\partial S$ upon first leaving $S$ if and only if this process has a Gaussian component.

The way in which a process exits from open sets has a close connection with the local growth of the process. One aspect of this relationship is discussed in Section 5 in order to make clear the probabilistic content of the criteria developed in Section 3.

The exit problem described above has been completely solved in [8] for the case when the process is real and the set $S$ is an interval of the form $(-\infty, x]$, $x > 0$. The methods there depended heavily on the use of local time, a tool which is not available in two or more dimensions. There are a number of other

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qualitative differences between the one-dimensional case and the general case that can perhaps best be illustrated by a simple example. Let $X^1$, $X^2$ be independent, real stable processes of indices $\alpha_i > 1$ respectively, where (say) $X^1$ has its Lévy measure concentrated entirely on $(-\infty, 0)$ but $X^2$ does not. Let $X = (X^1, X^2)$. Then $X$ falls into what one would ordinarily describe as a very nice class of processes. Let $S$ be a square, centered at 0 with sides parallel to the coordinate axes in the plane. Then using the independence of the $X^i$ and the hypothesis that $X^1$ is completely asymmetric it is not difficult to show directly that (starting from any point in $S$) $X$ will with positive probability hit the right-hand boundary of $S$ at its first exit. However, if $S$ is tilted slightly (so that its center is still 0 but its sides no longer parallel to the axes) then the process will never hit $\partial S$ on its first passage from $S$. For if so, then at least one of the projections of $X$ onto axes parallel to the sides of $S$ will be a real process that hits the endpoint of an interval on first leaving it; but each such projection is a stable process that is not completely asymmetric and this behavior is impossible for such a process (see [8]). It is now perhaps not immediately obvious how the process leaves a sphere centered at 0 and the reader may wish to amuse himself at this point by trying to settle the issue.

This example suggests several "bad" features of the general $d$-dimensional problem that are worth noting. First, the example shows that it is entirely possible to approximate one region with very smooth boundary arbitrarily closely by other regions with very smooth boundaries, yet the exit behavior relative to the approximating regions will give no indication of the result to be expected of the limiting region. In particular the method of proving an exit property relative to (say) certain polygonal regions and then extending the results to "arbitrary" regions via a limiting procedure seems destined to fail most of the time. This is regrettable since the problem for polygonal regions can often be solved immediately by the results of [8]. Second, in the one-dimensional case, if a process has a Lévy measure entirely supported by the negative $x$-axis, then upon first leaving $(-\infty, a)$ $(a > 0)$, the process will evidently hit $\{a\}$. In the $d$-dimensional case the fact that the Lévy measure is 0 in various directions does not guarantee that for all reasonable $S$ certain sections of the boundary will be hit. The end result will depend on the specific shape and orientation of $S$, and on how the regions in which $\nu$ is 0 are arranged. (For the process above, the Lévy measure gives 0 mass to all four open quadrants and to the open right half-plane.) The final point is merely the observation that there are quite a few more varieties of connected open sets in multidimensional space than there are in one dimension. In particular exit behavior depends not only on the shape of the region but on its orientation, and even minute changes in orientation change the exit behavior completely.

**2. Local isotropy and its variants.** In this paper $X = \{X(t), t \geqq 0\}$ will denote a stochastic process with stationary independent increments, taking values in

$d$-dimensional Euclidean space $R^d$. As usual $X$ is assumed to be a Hunt process. Of course

$$Ee^{i(u,X(t))} = \exp\{-t\psi(u)\} \tag{2.1}$$

where

$$\psi(u) = i(a, u) + \tfrac{1}{2}(\sigma u, u) + \int [1 - e^{i(u,y)} + i(u, y)/(1 + |y|^2)]\nu(dy)\,.$$

If $(\sigma x, x) = 0$ for all $x$, then we will say that $X$ has no Gaussian component; we will call $\nu$ the Lévy measure. If there is no Gaussian component, we will always assume $\nu(R^d) = +\infty$. Under these assumptions, if $T_r = \inf\{t > 0: |X_t| \geqq r\}$, then $T_r$ converges to 0 in probability as $r \downarrow 0$. Terminology and notation belonging to the theory of Markov processes will be that of [1]. In particular $P^x$, $E^x$ denote probability and expectation for the process starting at $x$; if $x = 0$, then the superscript will be omitted.

Let $f$ be a nonnegative Borel function on $R^d \times R^d$ vanishing on the diagonal. It is now well known (see [6], [10]) that if $\lambda \geqq 0$,

$$\sum_{s\leqq t} e^{-\lambda s}f(X_{s-}, X_s) - \int_{s=0}^{t} \int_{y\in R^d} e^{-\lambda s}f(X_s, X_s + y)\nu(dy)\, ds \tag{2.2}$$

is a martingale, provided the expectation of each term is finite. We use this fact to obtain our first preliminary result. If $A$ is a set, let $A^c$ denote the complement of $A$.

PROPOSITION 2.1. *Let $S$ be an open set containing 0, and let $T = \inf\{t > 0: X_t \notin S\}$. Assume $\partial S$ has potential 0. Then $P\{X_{T-} \in S, X_T \in \partial S\} = 0$. Whether $\partial S$ has potential 0 or not $P\{X_{T-} \in \partial S, X_T \in (S^c - \partial S)\} = 0$.*

PROOF. Choose $f(u, v) = 1$ if $u \in S$, $v \in \partial S$; $f(u, v) = 0$ otherwise, and take $\lambda = 0$ in (2.2). A simple martingale argument gives

$$\begin{aligned} P\{X_{T-} \in S, X_T \in \partial S\} &= E \int_0^T \int f(X_s, X_s + y)\nu(dy)\, ds \\ &\leqq \int_{y\in R^d} E \int_0^\infty I_{S\cap(\partial S - y)}(X_s)\, ds\, \nu(dy)\,, \end{aligned} \tag{2.3}$$

where $I_A$ is the indicator of the set $A$. However, if $\partial S$ is of potential 0, so evidently is $S \cap (\partial S - y)$, whence the conclusion. The second part of the proposition can be proved in the same way, if it is known that $\partial S$ has potential 0; otherwise this is an immediate consequence of a general fact about Hunt processes [3].

It will be convenient from time to time to impose various hypotheses on $X$. These hypotheses involve cones in $R^d$, which we shall define as follows. A cone $A$ in $R^d$, $d \geqq 2$ (with vertex at 0) will be any set that can be constructed in this way: take a closed ball of radius $r$ whose center is a distance $r_0 > r$ from 0; then $A$ will consist of all the points that lie on the rays that pass from 0 through this ball. The axis of $A$ is the ray from 0 through the center of the ball; the vertex angle is $2\sin^{-1}(r/r_0)$. The direction of the cone is the direction of its axis. A cone with vertex at $x \in R^d$ is a set of the form $x + A$, where $A$ is a cone with vertex at 0; if the position of the vertex is not specified, it is automatically assumed to be 0. An open cone is the interior of a cone as just defined. In $R^1$, a cone will be either $[0, \infty)$ or $(-\infty, 0]$. We shall also call complements of the sets just described (improper) cones, and shall for convenience call $R^d$ itself a cone. Finally, a truncated (proper) cone (with vertex at 0) is formed in the obvious way using a hyperplane perpendicular to the axis of the cone. We can now formulate our hypotheses. Let $T_r = \inf\{t > 0: |X_t| \geqq r\}$. Let $A$ be a cone and $\alpha$ some angle.

$(H_1; A, \alpha)$. The process is said to satisfy hypothesis $(H_1; A, \alpha)$ if there is $\delta(\alpha) > 0$ such that for all cones $A' \subset A$ of vertex angle $\alpha$

$$P\{X_{T_r} \in A'\} \geqq \delta(\alpha) \qquad \text{for all small } r . \tag{2.4}$$

$(H_2; A, \alpha)$. The process is said to satisfy $(H_2; A, \alpha)$ if there is $\varepsilon(\alpha) > 0$ such that for all cones $A' \subset A$ of vertex angle $\alpha$

$$E \int_0^{T_r} I_{A'}(X_s)\, ds \geqq \varepsilon(\alpha) E T_r \qquad \text{for all small } r . \tag{2.5}$$

The main point of these hypotheses is that they are to hold for all cones (within $A$) with the same vertex angle, *independent of the direction of the cone.* If $X$ satisfies $(H_1; R^d, \alpha)$ and $(H_2; R^d, \alpha)$ for all $\alpha \leqq \alpha_0$, then it will be convenient to call $X$ locally isotropic. Of course, any isotropic process is locally isotropic, but a surprising number of other processes share this property as well; examples include processes with genuinely $d$-dimensional Gaussian component, the type $A$ stable processes, and many others (see Section 4). We make no attempt here to delineate all such processes. Easy examples of processes that satisfy $H_1$ and $H_2$ but are not locally isotropic are the real processes with increasing paths (subordinators); however, for processes in $R^2$ whose components are subordinators, $H_1$ and $H_2$ (with $A$ as the upper right quadrant) impose a real restriction. We record now several of the properties of processes satisfying $H_1$, $H_2$ that will be used later.

PROPOSITION 2.2. *Let $S$ be an open set, and let $X$ satisfy either $(H_1, A, \alpha)$ or $(H_2, A, \alpha)$ for all $\alpha \leqq \alpha_0$. Assume that $x \in \partial S$ has the property that there is a small (truncated) cone $A' \subset A$ such that $x + A' \subset S^c$. Then $x$ is regular for $S^c$.*

The proof is an easy consequence of the Blumenthal zero-one law.

Let $S$ be a connected open set. Call a point $y \in S$ "possible" if, starting at any other point $x \in S$, there is positive probability that $\{X_t\}$ will come within $\varepsilon$ of $y$ before leaving $S$ (all small $\varepsilon$).

PROPOSITION 2.3. *Let $X$ satisfy $(H_2, R^d, \alpha)$ for all $\alpha \leqq \alpha_0$, and let $S$ be an open connected set. Then every point in $S$ is possible.*

Since the complete proof of this proposition appears to be a bit tedious in detail, we give only a rough sketch. Draw a curve (lying entirely within $S$) from $x$ to $y$. By polygonal approximation and strong Markov, we may assume that this curve is a straight line. At each point on the curve, construct a sphere of radius $h$ having this point as center; $h$ is chosen so small that all of these spheres lie well within $S$. Let $\alpha$ denote a small angle. For any sphere $S_h$ of radius $h$ centered at 0 there is a concentric sphere $S_{ah}$ of radius $ah$ ($a$ is small, possibly depending on $h$) so that if $A$ is a cone of angle $\alpha$ then $P\{X \text{ hits } (S_h - S_{ah}) \cap A \text{ before leaving } S_h\} \geqq \varepsilon/2$, where $\varepsilon$ is the quantity appearing in $(H_2)$. Indeed, choose $a$ so small that if $T_r = \inf\{t > 0 : |X_t| \geqq r\}$, then $E \int_{T_{ah}}^{T_h} I_A(X_s)\, ds \geqq \varepsilon/2\; ET_h$ (this can be done independently of the orientation of $A$); *a fortiori*, $P\{\text{hit } (S_h - S_{ah})A \text{ before leaving } S_h\}ET_h \geqq \varepsilon/2\; ET_h$. For convenience call the region $(S_h - S_{ah})A$ an $A - \varepsilon$ sector. Now starting at $x$, choose a cone $A$ of angle $\alpha$ with vertex at $x$ and axis on the line from $x$ to $y$. With probability at least $\varepsilon/2$ the process will hit the $A - \varepsilon$ sector of the $h$-sphere about $x$ before leaving it; if it does, stop the process at this point, and construct a sphere of radius $h$ about the point just reached. Choose a cone $A$ for the new circle so that its $A - \varepsilon$ sector is roughly centered on the line between $x$ and $y$ and farther along the line in the direction of $y$ than was the first $A - \varepsilon$ sector. Let the process run and, stop it if it hits the new $A - \varepsilon$ sector before leaving the new circle—this it will do with probability at least $\varepsilon/2$. Continue in this manner. If we call a "success" the event of hitting the distinguished $A - \varepsilon$ sector before leaving its sphere, then if a sequence of $N$ successes (starting with the first ball) are experienced, the process will certainly deviate from the $x - y$ line during those successes by no more than $h$; if $\alpha$ is small then we will need roughly no more than $l/a$ successes in a row to put us within $h$ of $y$ ($l$ = length of line between $x$ and $y$). By strong Markov the probability of $l/a$ successes in a row is $\geqq (\varepsilon/2)^{l/a}$, whence the result.

REMARK. If the $a$ used in this proof can be chosen independently of $h$ then a (simpler) version of this argument shows that $(H_2; A, \alpha)$ implies $(H_1; A, \alpha)$. This will be true for example if the process is stable.

PROPOSITION 2.4. *Let $X$ be any $d$-dimensional process with stationary independent increments (not identically zero), and let $T_r = \inf\{t > 0 : |X_t| \geqq r\}$. If $1 > a > 0$ then there is a constant $K = K(a) > 0$ such that $ET_{ax} \geqq K(a)ET_x$ for all $x > 0$.*

PROOF. Let $\alpha$ be a very small angle. Any sphere centered at 0 can be covered by a finite number, say $1/\delta(\alpha)$, of cones of angle $\alpha$ and vertex at 0. For each $r$, there then exists such a cone, say $A_r$, such that $P\{X_{T_r} \in A_r\} \geqq \delta(\alpha)$. Define a sequence of random variables $T^n$, $n \geqq 1$, by $T^1 = T_{ax}$, $T^{n+1} = \inf\{t > 0 : |X_{t+T^n} - X_{T^n}| > ax\}$. Let us say that there is a success at the $n$th try if $X(\sum^{n+1} T^i) - X(\sum^n T^i) \in A_{xa}$. Since $\alpha$ is small, there is a fixed number $N$ depending only on $a$ and $\alpha$ such that if the process experiences $N$ successes in a row then it surely will have traveled a net distance at least $2x$ from the point at which the success run began. (For example, if $\alpha$ is nearly 0, then $N$ is roughly $2/a$.) Let $U$ be the waiting time for $N$ successes in a row. Evidently $\sum_{i=1}^{U} T^i \geqq T_x$ so Wald's equation implies that (since the $T_i$ are independent, identically distributed) $ET_x \leqq ET^1EU$, proving the theorem with $K(a) = [EU]^{-1} > 0$.

One final point: Since by our assumptions $T_r \to 0$ as $r \downarrow 0$, $(H_1;\ A,\ \alpha)$ continues to hold (with the $\delta(\alpha)$ there replaced by any smaller one—e.g., $\delta(\alpha)/2$) if all the cones $A'$ appearing in the definition are truncated at the same height. This fact will be used in the proof of Theorem 3.2.

**3. Main results.** Throughout this section $S$ will be a connected open set in $R^d$, containing 0. It will be assumed that the boundary of $S$ is smooth in the sense that there is an angle $\alpha$ and a positive number $h$ such that at each $x \in \partial S$ it is possible to construct a (truncated) cone with vertex angle $\alpha$, height $h$, and vertex at $x$, which lies entirely in $S^c$. In particular if $X$ is locally isotropic, every point of $\partial S$ is regular for $S^c$. Throughout this section the phrase "smooth boundary" will be used precisely in this sense.

Let $A$ be a cone, and let $T = \inf\{t \geqq 0 : X_t \notin S\}$. Our first task will be to establish a criterion that guarantees $P\{X_T \in (\partial S)A\} = 0$; i.e., for $X$ to "leave $S$ via $A$ without ever hitting $\partial S$." To do this we first formulate a condition for $S$ which we shall call the translation property relative to $A$:

(3.1) For every cone $A' \subset A$ with the same axis and strictly smaller vertex angle, $(\partial S \cap A') - y \subset A \cap \bar{S}$ for all sufficiently small $y \in A$, and there is a positive constant $c$ such that for all sufficiently small $y \in A$, the distance between $(\partial S \cap A) - y$ and $A \cap \partial S$ is at least $c|y|$.

The "translation property" is a stronger requirement than what is actually needed for the proof. Typically (depending on the shape of $S$) it is a restriction on the cones $A$ that can be considered, usually forcing consideration only of "small" cones. For example if $S$ is a disk in the plane, then $S$ has the translation property relative to $A$ if the vertex angle of $A$ is, say, less than 45°; the translation property will fail, on the other hand, if angles close to 180° are used. Or, let $S$ be a square in the plane; if $A$ has vertex angle of 90° then the translation property will be satisfied for $A$ in some directions but not in others; if angles less than (say) 45° are used, the translation property will be satisfied relative to any such $A$, regardless of direction. The translation property evidently fails (in certain directions) for tori; the methods here can be extended to such surfaces, but it is tedious and offers no particular gain in insight.

THEOREM 3.1. *Let $S$ be an open connected set containing* 0 *and having smooth boundary. Let $A$ be an open cone so that $S$ has the translation property relative to $A$. Let $X$ satisfy $(H_2, R^d, \alpha)$ $\alpha \leqq \alpha_0$ and suppose that all $\lambda$-excessive functions are lower semicontinuous. Let $T = \inf\{t > 0 : X_t \notin S\}$, $T_r = \inf\{t > 0 : |X_t| \geqq r\}$. If $\int_{y \in A \cap \{|y|<1\}} ET_{|y|}\nu(dy) = +\infty$ then*

$$P\{X_T \in (\partial S) \cap A\} = 0 .$$

REMARKS (i) Regularity of the $\lambda$-excessive functions is assumed only to guarantee the lower semicontinuity at 0 of the function in (3.5), *under hypothesis* (3.2). For real processes leaving a half line, this condition is always satisfied

([2], [5]). Perhaps the hypothesis is superfluous in a large number of other cases as well.

(ii) The full strength of $(H_2, R^d, \alpha)$, $\alpha \leqq \alpha_0$ is not used in the proof. If the section of the boundary of interest (namely $\partial S \cap A$) has the property that on this section all of the truncated cones used in the definition of smooth boundary can be chosen to have roughly the same orientation (i.e., these cones when translated so that the vertex falls at 0 all lie within a single cone $A_1$), then one need impose only $(H_2; A_1, \alpha)$, $\alpha \leqq \alpha_0$. This would be the case if, for example, $S$ were a convex set in the plane, and the boundary of interest fell in the upper right quadrant. The exit of a real process from a half line is another obvious instance. Also, if the vertex angles of the cones used in the definition of smooth boundary can be taken to be $\beta$ for $x \in \partial S \cap A$, then we need impose only $(H_2; R^d, \alpha_0)$ (or $(H_2; A_1, \alpha_0)$ if circumstances are as in the first part of this paragraph) for some fixed $\alpha_0 \leqq \beta$.

(iii) It is evident from the proof that only $\partial S \cap A$ need be "smooth."

(iv) If $\int_{A \cap \{|y|<1\}} ET_{|y|}\nu(dy) = +\infty$ for some $A$, then evidently the integral would be infinite for any cone containing $A$; the conclusion of the theorem, of course, need not hold for the larger cone because the translation property may not continue to hold. The example of the introduction illustrates this point. However, the conclusion holds for any cone containing the given cone, provided only that the larger cone has the translation property. Evidently, then, $X$ will never hit $(\bigcup A_\alpha) \cap \partial S$ where the union is over all cones containing $A$ which have the translation property. Of course the translation property need not hold for $\bigcup A_\alpha$ (even if this is a cone which it need not be). To illustrate, if $S$ is a disk in the plane and the hypothesis holds for some small cone $A$ then it is possible by this means to extend the part of the boundary not hit from $\partial S \cap A$ to something nearly 180°.

PROOF. We will assume

$$P\{X_T \in (\partial S) \cap A\} \geqq \delta > 0 \tag{3.2}$$

and derive a contradiction. First take $f$ in the martingale (2.2) to be of the form $f(u, v) = 1$ if $u \in S$, $v \in (S^c - \partial S) \cap A$; $f(u, v) = 0$ otherwise. Then using Proposition 2.1, a simple martingale argument yields

$$P\{X_T \in (S^c - \partial S) \cap A\} = \int_{y \in R^d} E \int_{s=0}^{T} f(X_s, X_s + y)\, ds\, \nu(dy)\,. \tag{3.3}$$

Let $A'$ be a cone contained in $A$ having the same axis but slightly smaller vertex angle and such that $P\{X_T \in \partial S \cap A'\} \geqq \delta/2$. For fixed $y \in A$, let $U_y = \inf\{t > 0 : X_t \in [\partial S \cap A'] - y\}$ and $D_y = S \cap [(S^c - \partial S)A - y]$. If $x \in (\partial S \cap A') - y$ then the smoothness of $\partial S$ and the translation property relative to $A$ guarantee that a ball of radius $c|y|$ centered at $x$ will lie in $S$ and have the property that for some open truncated cone $B$ of angle $\beta$ ($\beta$ independent of $x$), the intersection of this ball with $B + x$ will lie entirely in $D_y$, provided $y \in A$ is small. Using this fact, the strong Markov property and $H_2$ we find that

$$E \int_{s=0}^{T} f(X_s, X_s + y)\, ds \geqq E^0 I\{U_y < T\} \int_{U_y}^{T} I_{D_y}(X_s)\, ds$$
$$\geqq P^0\{U_y < T\}\varepsilon(\beta) E^0 T_{c|y|} .$$

Let $\Delta > 0$ be small, and let $\partial S_\Delta$ be the collection of points whose distance from $\partial S$ is less than $\Delta$. Define a new region $S'$ by $S' = (SA') \cup (S - \partial S_\Delta)[(A')^c]$. If $y \in A$ is sufficiently small, then $S' - y \subset S$; moreover if $T'$ is the time it takes to leave $S'$, then $P\{X_{T'} \in \partial S' \cap A'\} \geqq \delta/4$ by quasi left continuity, provided $\Delta$ is sufficiently small. If $C$ is a set let $T(C) = \inf\{t > 0 : X_t \notin C\}$. Then for small $y$ (say $|y| < \eta$)

$$
\begin{aligned}
P\{U_y < T\} &\geqq P\{U_y = T(S' - y)\} \\
&= P^y\{X \text{ hits } \partial S' \cap A' \text{ while leaving } S'\} \qquad (3.4)\\
&\geqq \frac{\delta}{8}, \qquad \text{all small } y \in A ,
\end{aligned}
$$

provided that the function $f$

$$f : y \to P^y\{X_{T'} \in \partial S' \cap A'\} \tag{3.5}$$

is lower semicontinuous at 0 ($f(0) \geqq \delta/4$). Assuming (3.5), we then find, using Proposition 2.4 and setting $A_\eta = A \cap \{y : |y| < \eta\}$,

$$
\begin{aligned}
1 \geqq P\{X_T \in (S^c - \partial S) \cap A\} &\geqq (\delta/8)\varepsilon(\beta) \int_{A_\eta} ET_{\varepsilon|y|}\nu(dy) \\
&\geqq (\delta/8)\varepsilon(\beta)K(\varepsilon) \int_{A_\eta} ET_{|y|}\nu(dy) ,
\end{aligned}
$$

the required contradiction.

We now verify (3.5); in fact we shall verify that $f : y \to P^x\{X(T(C)) \in D\}$ is l.s.c. at 0 for an open set $C$ containing 0 and $D \subset \partial C$. Since all excessive functions are assumed lower semicontinuous, evidently

$$\lambda U^\lambda f(x) = \lambda \int_0^\infty e^{-\lambda t} P_t f(x)\, dt$$

is l.s.c. Moreover, using the explicit form of $f$,

$$|P_t f(x) - f(x)| \leqq 2P^x\{T(C) \leqq t\}$$

so that

$$
\begin{aligned}
|\lambda U^\lambda f(x) - f(x)| &\leqq 2\lambda \int e^{-\lambda t} P^x\{T(C) \leqq t\}\, dt \\
&\leqq 2P^x\{T(C) \leqq M\} + 2e^{-\lambda M} .
\end{aligned}
$$

Let $N$ be a neighborhood of 0 which is a positive distance from $\partial C$. Choose $M$ so small that $P^x\{T(C) \leqq M\} < \varepsilon/2$ for all $x \in N$ (possible since $N$ is a positive distance from $\partial C$). With $M$ chosen, pick $\lambda$ so large that $2e^{-\lambda M} < \varepsilon/2$. Thus $f$ is approximated (uniformly in $N$) by a l.s.c. function, whence the result.

REMARK. If the resolvent of $X$ is strong Feller, then evidently $\lambda U^\lambda f$ is continuous, so we have actually proved continuity of $f$ in a neighborhood of 0. In particular we recover some of the results of [8], Section 3 without the use of local time. As a matter of fact, if excessive functions are l.s.c. it may be shown (for any $R^d$-valued process with stationary independent increments) that the resolvent is strong Feller, so we have actually proved continuity of the function $x \to P^x\{X_{T(C)} \in D\}$ on $C$.

We turn now to formulating a criterion for hitting $\partial S$ at the first passage time from $S$. First, some more notation. Let $A$ be a cone as usual. For $\Delta > 0$, define

$$C(A, \Delta) = \{y : 0 < |y| < \Delta,\, S \cap [(S^c - \partial S)A - y] \neq \phi\}\,.$$

Even if $\Delta$ is very small, $C(A, \Delta)$ will be very much larger, in general, than $\{y : |y| < \Delta\} \cap A$. For example, in the plane, if $S$ is a disk and $A$ is a cone centered on the $x$-axis, then $C(A, \Delta)$ will contain all $y$, $|y| < \Delta$, lying in the closed right half-plane, and more; if $S$ is a square with sides parallel to the axes, and if $A$ is the cone formed by the origin, the northeast and southeast corners, then $C(A, \Delta)$ consists of all $y$, $|y| < \Delta$, lying in the open right half-plane. In the case of a real process with $S = (-\infty, a)$, $a > 0$ $C(A, \Delta) = (0, \Delta)$. Evidently if $\Delta_1 < \Delta_2$, $C(A, \Delta_1) \subset C(A, \Delta_2)$.

THEOREM 3.2. *Let $S$ be open connected, with smooth boundary, $0 \in S$. Let $X$ satisfy $(H_1, R^d, \alpha)$, $\alpha \leqq \alpha_0$; suppose all points of $S$ are possible and that if $x \in S$, $x \to z \in \partial S$ then $E^x e^{-\lambda T} \to 1$, where $T = \inf\{t > 0 : X_t \notin S\}$. Let $A$ be a cone. If for some (hence all smaller) $\Delta$*

$$\int_{C(A,\Delta)} ET_{|y|}\,\nu(dy) < \infty$$

*then $P\{X_T \in \partial S \cap A\} > 0$.*

REMARKS. (a) Evidently if $A'$ is a cone, $A' \subset A$, then $C(A', \Delta) \subset C(A, \Delta)$, so the conclusion holds for $A'$ as well. Hence if every ray from 0 lying in $A$ hits $\partial S \cap A$ in just one point then the points of $\partial S \cap A$ that are ever hit on first passage by some sample function are dense in $\partial S \cap A$.

(b) A consequence of the theorem is, in particular, that the process indeed hits the set $\partial S \cap A$; thus the method here provides a technique (in addition to the usual ones—capacity, Hausdorff dimension, etc.) that is occasionally effective in proving that certain sets are hit.

(c) The proof also reveals that as $x \to z \in \partial S \cap A$ $(x \in S)$, $P^x\{X_T \in \partial S \cap A\} \to 1$, providing a generalization of a property already observed for the exit of real processes from an interval [8].

(d) If all the points of $\partial S \cap A$ have their corresponding cones (in the definition of smooth boundary) contained in some cone $A_1$ after translating the vertex to 0, then we need require only $(H_1, A_1, \alpha)$, $\alpha \leqq \alpha_0$. If all the cones in the definition of smooth boundary have vertex angle $\beta$, then we need only require $(H_1, R^d, \alpha_0)$ for some fixed $\alpha_0 \leqq \beta$.

(e) The condition $E^x e^{-\lambda T} \to 1$ is evidently satisfied if all excessive functions are lower semicontinuous. Indeed since all points of $\partial S$ are regular for $S^c$ (Proposition 2.2)

$$\liminf_{x \to z \in \partial S} E^x e^{-\lambda T} \geqq E^z e^{-\lambda T} = 1\,.$$

PROOF. As in the proof of Theorem 3.1 (except now $\lambda > 0$)

$$E^x e^{-\lambda T} I\{X_T \in (S^c - \partial S)A\} = \int_{y \in R^d} E^x \int_0^T e^{-\lambda s} f(X_s, X_s + y)\, ds\, \nu(dy) \tag{3.6}$$

where $f(u, v) = 1$ if $u \in S$, $v \in (S^c - \partial S)A$; $f(u, v) = 0$ otherwise. For $\Delta'$ as described in the statement of the theorem, the expression in (3.6) is bounded above by

$$[1 - E^x e^{-\lambda T}]\nu\{y : |y| \geqq \Delta\} + \int_{y \in C(A,\Delta)} E^x \int_0^T f(X_s, X_s + y)\, ds\, \nu(dy)\,. \tag{3.7}$$

For fixed $y \in C(A, \Delta)$ define

$$U_y = \inf\{t > 0 : X_t \in D_y\}$$

where $D_y = S \cap [(S^c - \partial S)A - y]$. Then

$$\begin{aligned} E^x \int_0^T f(X_s, X_s + y)\, ds &\leqq E^x I\{U_y < T\} E^{X_{U_y}} \int_0^T I_{D_y}(X_s)\, ds \\ &\leqq \sup_{x \in \bar{D}_y} E^x \int_0^T I_{D_y}(X_s)\, ds\,. \end{aligned} \tag{3.8}$$

However if $x \in \bar{D}_y$, the fact that $\partial S$ is smooth implies that an open ball $M(x)$ of radius $2|y|$ centered at $x$ has this property: there is a truncated cone $B$ having vertex angle $\beta$ ($\beta$ and the height independent of $x$, but direction depending on $x$) such that $[M(x)]^c \cap [x + B] \subset (S^c - \partial S)$. This being so, we may bound the last expression in (3.8) as follows. If, starting at $x$, the process leaves the $2|y|$-ball centered at $x$ via the distinguished cone, then the process will have left $S$ and, *a fortiori*, $D_y$. The probability of doing this is at least $\delta(\beta) > 0$. The total time in $D_y$ would, in this case, be less than the time spent leaving the $2|y|$-ball at $x$. If the process does not leave the $2|y|$-ball at $x$ via the distinguished cone, then its position on first leaving the ball may or may not be in $D_y$. If not in $D_y$ wait until it again (if ever) enters $D_y$ and then let it run from the entry point until it leaves a $2|y|$-ball centered at the entry point; if on the other hand, the position of the process was in $D_y$, draw a $2|y|$-ball about the point just reached, watch the process until it leaves the new ball. If the process leaves the new ball via its distinguished cone, then the process has left $S$; if not, repeat the procedure just described. The total time in $D_y$ until leaving $S$ is then bounded by the total time spent in these $2|y|$-balls until the process finally leaves one via its distinguished cone. The probability of "success" with any given ball is $\geqq \delta(\beta)$, and by strong Markov, the successive "tries" are independent. Therefore, by Wald's equation, if $z \in \bar{D}_y$

$$E^z \int_0^T I_{D_y}(X_s)\, ds \leqq [\delta(\beta)]^{-1} E^0 T_{2|y|}\,.$$

Moreover, according to Proposition 2.2, $ET_{2|y|} \leqq KET_{|y|}$, where $K$ does not depend on $y$. Putting all this together

$$\begin{aligned} &E^x e^{-\lambda T} I\{X_T \in (S^c - \partial S)A\} \\ &\qquad \leqq [1 - E^x e^{-\lambda T}]\nu\{y : |y| \geqq \Delta\} + K \int_{C(A,\Delta)} E^0 T_{|y|}\nu(dy)\,. \end{aligned}$$

Choose $\Delta$ so small that the second term on the right is less than $\varepsilon/2$; with $\Delta$ so chosen, the hypothesis that $E^x e^{-\lambda T} \to 1$ as $x \to z \in \partial S \cap A$ implies that for $x \in S$

close to $z \in \partial S$ the first term is less than $\varepsilon/2$. Thus the process will hit $(\partial S)A$ if it starts close to $(\partial S)A$. That it will hit $\partial S$ with positive probability, starting from 0 follows from the hypothesis that all points of $S$ are possible.

**4. Special processes.** In this section we apply the criteria of Section 3 to obtain results of interest for several important classes of stochastic processes with independent increments.

(a) *Isotropic processes.* The isotropic processes satisfy all of the regularity hypotheses on $X$ given in Theorems 3.1, 3.2. A collection of basic facts about these processes may be found in [7]. It follows from the isotropy that either $\int_{A \cap \{|y|<1\}} ET_{|y|}\nu(dy) = +\infty$ for all cones, or else it is finite for all cones. Suppose that in addition to isotropy the process has no Gaussian component (general processes with Gaussian component will be treated separately later). If $S$ is a cube then it follows from Theorems 3.1 and 3.2 that if $T = \inf\{t > 0 : X_t \notin S\}$ then

$$P\{X_T \in \partial S\} = 0 \qquad \text{if and only if} \quad \int_{A\{|y|<1\}} ET_{|y|}\nu(dy) = \infty .$$

The argument in Section 1 together with Theorem 3.7 of [8] shows that if there is no Gaussian component, then $P\{X_T \in \partial S\} = 0$. Therefore $\int_{A\{|y|<1\}} ET_{|y|}\nu(dy) = +\infty$ for all cones $A$. This proves

PROPOSITION 4.1. *Let $X$ be an isotropic process in $R^d$, $d \geqq 2$, having no Gaussian component. Let $S$ be a connected open set with smooth boundary such that for each $x \in \partial S$ there is a cone $A$ containing $x$ such that the translation property holds relative to $A$. Then $P\{X_T \in \partial S\} = 0$.*

Notice that a good deal less than isotropy is needed to carry through the proof. Criteria for hitting $\partial S$ eventually may be found in [7] under regularity on $\partial S$. Taking $S$ to be a sphere this result together with Proposition 4.2 gives a description of the exit of the radial process $R_t = |X_t|$ from the intervals $(0, a)$, $(a, \infty)$.

H. Kesten has shown me (private communication) a proof that if $\int_{|x|<1} |x|\nu(dx) = \infty$ and if $\{X_t\}$ has no Gaussian part, then $\int_{|y|<1} ET_{|y|}\nu(dy) = \infty$. By Theorem 3.1 there is then a cone $A$ such that for any nice $S$, such a process will never hit $\partial S \cap A$ upon first leaving $S$. Together with Proposition 4.3 below this gives an analogue of Theorem 3.5 of [8], to the effect that the "only" processes having $\int_{|x|<1} |x|\nu(dx) = \infty$ and continuous passages in all directions out of a region $S$ are the ones with Gaussian component.

(b) *Processes with Gaussian component.* If $\{X_t\}$ has a Gaussian component, then we may write

$$X_t = G_t + Y_t$$

where $\{G_t\}$ and $\{Y_t\}$ are independent processes, $\{G_t\}$ being a Gaussian process with stationary independent increments, and $\{Y_t\}$ having characteristic function (2.1) with $\sigma \equiv 0$. We assume that $\{G_t\}$ is genuinely $d$-dimensional. Since for small values of the time parameter $Y$ is almost negligible in comparison with $G$, it is plausible that $X$ will be locally isotropic. We verify this formally in

PROPOSITION 4.2. *If $X$ has a genuinely d-dimensional Gaussian component, then $X$ is locally isotropic.*

PROOF. We verify $(H_1)$, which is all that is needed for the application we have in mind; $(H_2)$ can be verified by similar arguments. Let $A$ be any cone having vertex angle $\alpha$, let $T_r = \inf\{t > 0 : |X_t| \geqq r\}$. It is a known fact that

$$\sup_{0\leqq s\leqq t} |Y_s|/\sqrt{t} \to 0 \quad \text{in probability}$$

as $t \to 0$. Let $\varepsilon > 0$, $\eta > 0$ be given, and let $K(\varepsilon)$ be a number depending on $\varepsilon$ only that will be specified in a moment. Then

$$\begin{aligned} P\{T_r \leqq K(\varepsilon)r^2, \sup_{0\leqq s\leqq T_r} |Y_s| < \eta r\} & \\ &\geqq P^0\{T_r \leqq K(\varepsilon)r^2, \sup_{0\leqq s\leqq K(\varepsilon)r^2} |Y_s| < \eta r\} \\ &= P\{\sup_{0\leqq s\leqq K(\varepsilon)r^2} |X_s| > r, \sup_{0\leqq s\leqq K(\varepsilon)r^2} |Y_s| < \eta r\} \\ &\geqq P\{\sup_{0\leqq s\leqq K(\varepsilon)r^2} |G_s| > (1-\eta)r, \sup_{0\leqq s\leqq K(\varepsilon)r^2} |Y_s| < \eta r\} \\ &= P\left\{\sup_{0\leqq s\leqq 1} |G_s| > \frac{1-\eta}{(K(\varepsilon))^{\frac{1}{2}}}\right\} P\{\sup_{0\leqq s\leqq K(\varepsilon)r^2} |Y_s| < \eta r\} \\ &\geqq 1-\varepsilon \end{aligned}$$

if $K(\varepsilon)$ is chosen so large that

$$P\left\{\sup_{0\leqq s\leqq 1} |G_s| > \frac{1}{(K(\varepsilon))^{\frac{1}{2}}}\right\} > 1 - \frac{\varepsilon}{2}$$

and then $r$ is chosen so small (depending on $\varepsilon$, $\eta$) that

$$P\{\sup_{0\leqq s\leqq K(\varepsilon)r^2} |Y_s| < \eta r\} > 1 - \frac{\varepsilon}{2}.$$

If $R_z = \inf\{t > 0 : |G_t| > z\}$ then on the set

$$\{\sup_{0\leqq s\leqq T_r} |Y_s| < \eta r\}, \qquad R_{(1-\eta)} \leqq T_r \leqq R_{(1+\eta)r}.$$

Let $A'$, $A''$ be cones with the same axis as $A$ but vertex angles one-fifth and three-fifths that of $A$, respectively. Assuming $\eta$ chosen so that $\eta r < \alpha r/5$, if $G(R_{(1-\eta)r}) \in A'$ and $G(R_{(1-\eta)r} + t) \in A''$ for $0 \leqq t \leqq R_{(1+\eta)r} - R_{(1-\eta)r}$, then on the set $\{\sup_{0\leqq s\leqq T_r} |Y_s| < \eta r\}$ we will have $X_{T_r} \in A$. By the scaling property it is possible (given $\varepsilon$) to make

$$P\{R_{(1+\eta)r} - R_{(1-\eta)r} \leqq r^2\} \geqq 1 - \varepsilon \tag{4.1}$$

for all $r$ merely by taking $\eta$ small. Hence by strong Markov, the fact that $G$ is itself locally isotropic, and an elementary approximation

$$\begin{aligned} P\{G(R_{(1-\eta)r}) \in A', G(R_{(1-\eta)r} + t) \in A'', 0 \leqq t \leqq R_{(1+\eta)r} - R_{(1-\eta)r}\} & \\ &\geqq \tfrac{1}{2} P\{G(R_{(1-\eta)r}) \in A', G(R_{(1-\eta)r} + t) \in A'', 0 \leqq t \leqq r^2\} \\ &\geqq K \frac{\alpha}{10} P\{|G(t)| \leqq \alpha r/5, 0 \leqq t \leqq r^2\} \end{aligned}$$

where $K$ is a constant independent of $\alpha$, $r$ (the $\varepsilon$ in (4.1) will have to be chosen less than $\frac{1}{2}P\{|G(t)| < \alpha/5, 0 \leqq t \leqq 1\}$). Putting this together

$$\begin{aligned} P\{X_{T_r} \in A\} &\geqq P\{T_r \leqq K(\varepsilon)r^2, \sup_{0\leqq s\leqq T_r} |Y_s| < \eta r, G(R_{(1-\eta)r}) \in A', \\ &\qquad G(R_{(1-\eta)r} + t) \in A'', 0 \leqq t \leqq R_{(1+\eta)r} - R_{(1-\eta)r}\} \\ &\geqq K \frac{\alpha}{20} P\left\{\sup_{0\leqq s\leqq 1} |G(s)| \leqq \frac{\alpha}{5}\right\} \equiv \delta(\alpha) . \end{aligned}$$

This completes the proof of local isotropy. If $T_r = \inf\{t > 0 : |X_t| > r\}$ then it is not too hard to show that there is a constant $K$ depending on $X$ such that

$$ET_r \leqq Kr^2 \tag{4.2}$$

for all small $r$. In particular, for any cone $A$,

$$\int_{C(A,\Delta)} ET_{|y|}\nu(dy) \leqq K \int_{|y|<\Delta} |y|^2\nu(dy) < \infty .$$

This, together with the local isotropy proves

PROPOSITION 4.3. *Let $S$ be any connected open set with smooth boundary. If $X$ has a genuinely $d$-dimensional Gaussian component, and $A$ is any cone, then* $P\{X_T \in \partial S \cap A\} > 0$.

(c) *Real processes.* If one considers the exit problem from an interval $(-\infty, x)$, $x > 0$ for a real process, then all the regularity properties that the main criteria impose on $S$ are evidently satisfied (see the remarks after Theorems 3.1 and 3.2). The criteria for the exit from $S = (-\infty, x)$ then become:

Suppose $X$ is a real process such that for all small $r > 0$

(i) $P\{X$ exceeds $r$ before $-r\} \geqq \delta > 0$

(ii) if $U_r = \inf\{t > 0 : |X_t| > r\}$, then

$$E \int_0^{U_r} I_{(0,r)}(X_s)\, ds \geqq \varepsilon E U_r$$

for some positive $\delta$, $\varepsilon$ depending only on $X$. Let $T_x = \inf\{t : X_t > x\}$. Then $P\{X_{T_x} = x\} = 0$ if and only if $\int_0^1 EU_r\nu(dr) = +\infty$.

This should be compared to the necessary and sufficient condition established in [8] without hypotheses (i), (ii). Hypotheses (i), (ii) are obviously satisfied by symmetric processes and subordinators; it is an interesting problem in its own right to characterize those processes satisfying (i) and (ii). Since for a subordinator $P\{X_{T_x} = x\} > 0$ if and only if the subordinator has positive drift [5], one obtains the following amusing result (no doubt well known but I have no reference) to be mentioned again in Section 5.

PROPOSITION 4.4. *A subordinator has positive drift if and only if* $\int_0^1 EU_r\nu(dr) < \infty$.

(d) *Stable processes.* Let $X$ now be a transient stable process of type $A$ having the scaling property and index $\alpha$ (see [9] for terminology). Of course we assume $X$ genuinely $d$-dimensional. Let us verify that such a process is locally isotropic. By the remark at the end of Section 2, it will suffice to verify that if $A$ is a cone with small vertex angle $\beta$, then $E\int_0^{T_r} I_A(X_s)\,ds \geqq \delta ET_r$, where $T_r = \inf\{t > 0: |X_t| \geqq r\}$. By the scaling property it will be enough to prove this for $r = 1$. Let $\varepsilon$ be a small positive number (to be specified in a moment), and let $B$ be a sphere of radius (say) $\varepsilon\beta/2$ whose center is on the axis of $A$ a distance $\varepsilon$ from 0. Then $B \subset S \cap A$, where $S = \{y: |y| < 1\}$. It will then be enough to verify $E\int_0^{T_1} I_B(X_s)\,ds \geqq \delta ET_1$. If $u(\cdot)$ is the potential kernel for $X$, then the first passage relation yields

$$\int_B u(y)\,dy - \int_{y\in B}\int_{S^c} u(y-z)P^0\{X_T \in dz\}\,dy = E\int_0^T I_B(X_s)\,ds\,.$$

If $T_B = \inf\{t > 0: X_t \in B\}$, and if $|z| \geqq 1$, then by the strong Markov property

$$\begin{aligned}\int_B u(y-z)\,dy &\leqq P^z\{T_B < \infty\}\int_{\{x:|x|<\varepsilon\beta\}} u(x)\,dx \\ &= \text{const. } P^z\{T_B < \infty\}(\varepsilon\beta)^\alpha \\ &= K_1 f_1(\varepsilon\beta)(\varepsilon\beta)^\alpha\end{aligned}$$

where $K_1$ is a constant independent of $\beta$, $\varepsilon$, and where $f_1(x)$ is either $x^{1-\alpha}$, $x^{1+\alpha}$ or $x^{d-\alpha}[1 + \log x^{-1}]$ depending on the index and the dimension of the space (see [9] Theorem 3).

On the other hand, if $T_{B'} = \inf\{t > 0: X_t \in B'\}$ where $B'$ is a sphere with the same center as $B$ but half the radius, and if $T_0 = \inf\{t > 0: |X_t| > \varepsilon\beta/4\}$

$$\int_B u(y)\,dy \geqq P\{T_{B'} < \infty\}ET_0 \geqq K_2 f_2(\beta)(\varepsilon\beta)^\alpha\,,$$

where $K_2$ is a constant and where $f_2(\beta) = \beta^{d-\alpha}$ (see [9] again; this is the only place where we have used the hypothesis that the process is of type $A$). Hence

$$E\int_0^{T_1} I_B(X_s)\,ds \geqq [K_2 f_2(\beta) - K_1 f_1(\varepsilon\beta)](\varepsilon\beta)^\alpha\,.$$

For sufficiently small $\varepsilon$, the form of $f_1$ guarantees that the bracketed expression on the right will be positive. This completes the proof of local isotropy; it is evident that stable processes not of type $A$ need not be locally isotropic. It is also now clear that type $A$ stable processes satisfy all the regularity hypotheses in the theorems of Section 3.

The Lévy measure of a stable process of index $\alpha$ has the form

$$\nu(dx) = \frac{dr}{r^{1+\alpha}}\,\mu(d\theta)\,,$$

where $x = (r, \theta)$ with $r \geqq 0$ and $\theta$ a point on $\partial S$ where $S$ is the unit sphere in $R^d$, and $\mu$ a finite Borel measure on $\partial S$. If $A$ is a cone, it is evident that if $\mu$ puts positive mass on $\partial S \cap A$, then $\int_{A\{|y|<1\}} ET_{|y|}\nu(dy) = +\infty$; otherwise this integral is 0. To illustrate, let $S$ be the unit disk in the plane. From the facts just mentioned and the criteria of Section 3, it is not difficult to see that if $x \in \partial S$ is a "point of increase" for $\mu$ then $X$ will never hit (on first leaving $S$) any part of $\partial S$ falling in the hemisphere containing $x$ and determined by the line through the origin perpendicular to the ray through $x$. In particular if $X$ is ever to hit $\partial S$ (at even a small arc) on first passage, $\mu$ can assign no mass to at least a closed hemisphere. On the other hand, if $Q$ is a square, in order to hit one side $l$ of $Q$, $\mu$ must assign no mass to the open hemisphere of $S$ determined by $l$. Returning to the disk again one can see that $\partial S$ will consist of two disjoint arcs: one arc (spanning at least 180 degrees) is never hit on first exit; the other is always hit at first passage if the process leaves in the direction determined by this arc. We spare the reader the details of this as well as the formulation for higher dimensions.

**5. Local growth.** In this section we examine one of the connections between local growth and the manner in which a process leaves open sets. Let $A$ be a cone and suppose

$$\int_{A \cap \{|y| \leq 1\}} ET_{|y|}\nu(dy) = +\infty . \tag{5.1}$$

Recall in addition that if $a > 0$

$$ET_{ar} \geqq K(a)ET_r , \qquad \text{all small } r \tag{5.2}$$

(see Proposition 2.4). Let $M$ be a positive number and, if $0 < a < b$, write $(a, b]$ for $\{y \in R^d : a < |y| \leqq b\}$. The following theorem is the main result of this section.

THEOREM 5.1. *Let $A$ be a cone, and suppose $X$ satisfies* (5.1). *Let $X_{t-}^* = \sup_{s<t} |X_s|$. Then*

$$\limsup_{t\to 0} \frac{|X_t - X_{t-}| I_A(X_t - X_{t-})}{X_{t-}^*} = +\infty . \tag{5.3}$$

The result gives some intuitive content to the hypothesis of Theorem 3.1. Roughly, the result says that, for small values of $t$, the position of the process at time $t$ is essentially due to one large jump in the direction $A$. This is one formulation of a general kind of behavior that has been observed in various contexts (see, for example, Fristedt [4]). Alternatively, the result says that the process exits from small spheres by jumps that are arbitrarily large compared with the radius of the sphere. Analytic criteria for (5.1) in the special case $A = R^d$ are given at the end of this section.

Of course, subordinators with no drift satisfy the hypothesis of this theorem; using Proposition 4.4, and taking $A = (0, \infty)$ in Theorem 5.1, one obtains the following interesting result first established with different proof by Kesten (private communication).

COROLLARY 5.1. *A subordinator $\{X_t\}$ has drift or no drift according as*

$$\limsup_{t\to 0} (X_t - X_{t-})/X_{t-} = 0 \quad \text{or} \quad = +\infty \quad \textit{respectively}. \tag{5.4}$$

PROOF. Because of (5.2), (5.1) is equivalent to

$$\sum ET_{2^{-k}}\nu\{(M2^{-k}, M2^{-k+1}] \cap A\} = +\infty . \tag{5.5}$$

Let $J_k = \inf\{t > 0 : X_t - X_{t-} \in (M2^{-k}, \infty)\}$. Then if $M > 2$,

$$\begin{aligned} ET_{2^{-k}}\nu\{(M2^{-k}, M2^{-k+1}] \cap A\} \\ = P\{T_{2^{-k}} = J_k, X(T_{2^{-k}}) - X(T_{2^{-k}}-) \in (M2^{-k}, M2^{-k+1}] \cap A\} . \end{aligned} \tag{5.6}$$

Indeed, let $X_t^k$ be the process obtained from $X$ by removing all jumps of size $(M2^{-k}, \infty)$ from $X$. The probability in (5.6) is

$$(5.7) \qquad P\{\max_{0\leq s<J_k} |X_s^k| < 2^{-k}, X(J_k) - X(J_k-) \in (M2^{-k}, M2^{-k+1}] \cap A\}$$

since $M > 2$. By the independence of jumps bigger than $M2^{-k}$ and those less, and by known relations between the Lévy measure and the jump size, the probability in (5.7) is equal to

$$(5.8) \qquad \begin{aligned} &\int P\{\max_{0\leq s<t} |X_s^k| < 2^{-k}\}P\{J_k \in dt\}\nu\{(M2^k, M2^{-k+1}]A\}/\nu\{(M2^{-k}, \infty)\} \\ &\quad = \int P\{\max_{0\leq s<t} |X_s^k| < 2^{-k}\} \\ &\qquad \times \nu\{(M2^{-k}, M2^{-k+1}]A\} \exp(-t\nu\{(M2^{-k}, \infty)\})\, dt\,. \end{aligned}$$

But

$$\begin{aligned} P\{\max_{0\leq s<t} |X_s| < 2^{-k}\} &= P\{\max_{0\leq s<t} |X_s| < 2^{-k}, J_k \geqq t\} \qquad \text{(since } M > 2) \\ &= P\{\max_{0\leq s<t} |X_s^k| < 2^{-k}, J_k \geqq t\} \\ &= P\{\max_{0\leq s<t} |X_s^k| < 2^{-k}\} \exp(-t\nu\{(M2^{-k}, \infty)\})\,. \end{aligned}$$

Therefore the second expression in (5.8) is equal to

$$\begin{aligned} &\int P\{\max_{0\leq s<t} |X_s| < 2^{-k}\}\, dt\, \nu\{(M2^{-k}, M2^{-k+1}] \cap A\} \\ &\qquad = ET_{2^{-k}}\nu\{(M2^{-k}, M2^{-k+1}]A\}\,, \end{aligned}$$

proving (5.6).

If $B_k = \{T_{2^{-k}} = J_k, X(T_{2^{-k}}) - X(T_{2^{-k}}-) \in (M2^{-k}, M2^{-k+1})A\}$, then according to (5.5), $\sum P(B_k) = +\infty$. Since the $B_k$ are not independent we use the well known extension of the Borel-Cantelli lemma:

$$\begin{aligned} &P(B_n \text{ i.o.}) > 0 \quad \text{if} \quad \sum P(B_n) = \infty \qquad \text{and} \\ &P(B_i B_j) \leqq cP(B_i)P(B_j)\,, \qquad\qquad i \neq j\,. \end{aligned}$$

By the Blumenthal zero-one law, it will then follow that $P(B_n \text{ i.o.}) = 1$. Let $\{\mathscr{F}_t, t \geqq 0\}$ be the usual sigma fields associated with a Markov process. If $k > j$, then since $B_k$ is $\mathscr{F}_{J_k}$-measurable, the strong Markov property implies

$$\begin{aligned} P(B_k B_j) &= EI_{B_k} I\{|X_{J_k}| < 2^{-j}\} \\ &\quad \times P^{X_{J_k}}\{\max_{s<J_j} |X_s^j| < 2^{-j}, X(J_j) - X(J_j-) \in (M2^{-j}, M2^{-j+1}]A\}\,. \end{aligned}$$

But if $|x| < 2^{-j}$,

$$\begin{aligned} &P^x\{\max_{s<J_j} |X_s^j| < 2^{-j}, X(J_j) - X(J_j-) \in (M2^{-j}, M2^{-j+1}]A\} \\ &\quad \leqq P\{\max_{s<J_j} |X_s^j| < 2 \cdot 2^{-j}, X(J_j) - X(J_j-) \in (M2^{-j}, M2^{-j+1}]A\} \\ &\quad = ET_{2\cdot 2^j}\nu\{(M2^{-j}, M2^{-j+1}]A\}\,, \end{aligned}$$

provided $M > 4$. The last equality is established essentially the same way as (5.6). Hence

$$\begin{aligned} P(B_j B_k) &\leqq P(B_k)ET_{2\cdot 2^{-j}}\nu\{(M2^{-j}, M2^{-j+1}]A\} \\ &\leqq cP(B_k)ET_{2^{-j}}\nu\{(M2^{-j}, M2^{-j+1}]A\} \\ &= cP(B_k)P(B_j) \end{aligned}$$

where $c$ comes from (5.2).

Since

$$P(B_k \text{ i.o.}) = 1,$$

$$\limsup_{k\to\infty} \frac{|X(T_{2^{-k}}) - X(T_{2^{-k}}-)| I_A[X(T_{2^{-k}}) - X(T_{2^{-k}}-)]}{X(T_{2^{-k}}-)}$$

$$\geqq \lim_{k\to\infty} \frac{(M-1)2^{-k}}{\cdot 2^{-k}} = M - 1 .$$

Since $M$ may be taken arbitrarily large,

$$\limsup_{p\to\infty} \frac{|X(T_{2^{-k}}) - X(T_{2^{-k}}-)| I_A[X(T_{2^{-k}}) - X(T_{2^{-k}}-)]}{X(T_{2^{-k}}-)} = +\infty \tag{5.9}$$

which is stronger than the result to be proved.

REMARK. Let $X$ be a real process, and put $M_t = \sup_{s\leq t} X_s$. Kesten pointed out upon proving his result for subordinators (mentioned above as Corollary 5.1) that by a certain transformation (described in [8] Section 3), the subordinator result immediately yields: $X$ hits the point $a > 0$ upon first leaving $(-\infty, a)$ if and only if

$$\limsup_{t\downarrow 0} \frac{M_t - M_{t-}}{M_{t-}} < \infty .$$

Unfortunately such transformations (which involve local time) do not seem to be available in the $d$-dimensional case.

We conclude this section with analytic criteria for $\int_{|y|<1} ET_{|y|}\nu(dy) = \infty$; the result (c) below is due to Kesten (private communication). As usual, if $\int_{|x|<1} |x|\nu(dx) < \infty$, we will assume that the exponent $\psi$ (see (2.1)) is written in the form $\psi(u) = i(b, u) + (1/2)(\sigma u, u) + \int \{1 - e^{i(u,y)}\}\nu(dy)$. Recall that we continue to assume $\nu(R^d) = +\infty$ if $\sigma = 0$.

THEOREM 5.2. (a) *If* $\int_{|x|<1} |x|\nu(dx) < \infty$, $\sigma = 0$, $b = 0$, *then* $\int_{|y|<1} ET_{|y|}\nu(dy) = \infty$.
(b) *If* $\int_{|x|<1} |x|\nu(dx) < \infty$, $\sigma = 0$, $b \neq 0$, *then* $\int_{|y|<1} ET_{|y|}\nu(dy) < \infty$.
(c) *If* $\int_{|x|<1} |x|\nu(dx) = \infty$, $\sigma = 0$, *then* $\int_{|y|<1} ET_{|y|}\nu(dy) = \infty$.
(d) *If* $\sigma \neq 0$, $\int_{|x|<1} ET_{|x|}\nu(dx) < \infty$.

The following corollary is then immediate from Theorem 5.1 and known facts.

COROLLARY 5.2. *If* (a) *or* (c) *holds, then*

$$\limsup_{t\to 0} \frac{|X_t - X_{t-}|}{X^*_{t-}} = \infty .$$

*If, on the other hand,* (b) *or* (d) *holds, then* $\lim_{t\to 0} |X_t - X_{t-}|/X^*_{t-} = 0$ a.s.

PROOF. Cases (b) and (d) of Theorem 5.2 follow from the estimates $ET_{|y|} \leqq K|y|$, $ET_{|y|} \leqq K|y|^2$ as $|y| \to 0$; the not particularly difficult proof is omitted. Here is a sketch of a proof of (a).

If $\{X_t\}$ is a subordinator without drift, then, as already noted, $\int_0^1 ET_r\nu(dr) = \infty$ by Proposition 4.4. If $\{X_t\}$ is a real process satisfying the hypotheses of (a), then it may be written as the difference of two subordinators; the exit time $T_r$ of $\{X_t\}$ from a sphere of radius $r$ is evidently greater than the exit time $\tilde{T}_r$ of the process obtained by adding the two subordinators. If $\nu_1, \nu_2$ are $\nu$ restricted to $(0, \infty)$ and $(-\infty, 0)$ respectively, then the Lévy measure of the sum of subordinators is given by $\tilde{\nu}(A) = \nu_1(A) + \nu_2(-A)$, $A \subset (0, \infty)$, so by the case already treated,

$$\infty = \int_0^1 E\tilde{T}_r \tilde{\nu}(dr) \leqq \int_1^0 ET_r \tilde{\nu}(dr) = \int_0^1 ET_r \nu(dr)\,,$$

proving the general one-dimensional case satisfying (a). The $d$-dimensional case is similar; for ease of exposition, we treat only two-dimensional processes. Let $\nu_i$, $i = 1, 2, 3, 4$ be $\nu$ restricted to the $i$th quadrant. Then $\{X_t\}$ may be written as the sum of four independent processes, the $i$th such process having Lévy measure $\nu_i$. If $A$ is a set in the first quadrant, define $\tilde{\nu}_1 = \nu_1$, and $\tilde{\nu}_i$, $i = 2, 3, 4$ by $\tilde{\nu}_i(A) = \nu_i(A_i)$ where $A_i$ is $A$ reflected into the $i$th quadrant. Let $\tilde{X}$ be a process that is the sum of four independent processes $\tilde{X}_i$ having Lévy measures $\tilde{\nu}_i$; $\tilde{X}$ then has Lévy measure $\tilde{\nu} = \sum \tilde{\nu}_i$, and is a sum of four two-dimensional subordinators. The time $\tilde{T}_r$ for $\tilde{X}$ to leave a sphere of radius $r$ is less than $T_r$. Thus if we show $\int_0^1 E\tilde{T}_r \tilde{\nu}(dr) = \infty$, then clearly $\int_0^1 ET_r \nu(dr) = \infty$. Consider now the process obtained from $\tilde{X}_i$ by adding the co-ordinates together. The Lévy measure $n_i$ of this process is given by

$$n_i\{(a, b)\} = \tilde{\nu}_i\{(x, y) : a < x + y < b\}\,.$$

Let $\{N_t\}$ be the subordinator obtained by adding together the coordinates of all the processes $X_i$, $i = 1, 2, 3, 4$. The Lévy measure of $N_t$ is then $n = \sum n_i$, and if $S_r$ is the time for $N_t$ to leave $[0, r]$ then $ES_r \leqq E\tilde{T}_r$. From the subordinator case already treated and Proposition 2.4 we then find

$$\begin{aligned}\infty = \int_0^1 ES_r n(dr) &\leqq \int_0^1 E\tilde{T}_r n(dr)\\ &\leqq \textstyle\sum_{k \geqq 1} E\tilde{T}_{2^{-k+1}} n\{(2^{-k}, 2^{-k+1}]\}\\ &\leqq \textstyle\sum_{k \geqq 1} ET_{2^{-k+1}} \tilde{\nu}\{(x, y) : 2^{-k} < x + y \leqq 2^{-k+1}\}\\ &\leqq \textstyle\sum_{k \geqq 1} ET_{2^{-k+1}} \tilde{\nu}\{(x, y) : 2^{-k-1} < (x^2 + y^2)^{\frac{1}{2}} \leqq 2^{-k+1}\}\\ &\leqq \text{const.} \textstyle\sum ET_{2^{-k-1}} \tilde{\nu}\{(x, y) : 2^{-k-1} < r \leqq 2^{-k}\} \qquad \text{(Prop. 2.4)}\\ &\leqq \text{const.} \int_0^1 ET_r \tilde{\nu}(dr)\,.\end{aligned}$$

Here is a sketch of Kesten's proof of the case (c). Let $J^r(t)$ be the sum of the jumps of $\{X_s\}$ up to time $t$, having magnitude at least $r$; and let $X_t{}^r$ be the process with exponent

$$\psi^r(u) = \int_{|y|<r} \{1 - e^{i(u,x)} + i(u, x)\}\nu(dx)$$

so that

$$X(t) = X^r(t) + J^r(t) + tb_r$$

with $|b_r| \leqq \mu(r) \equiv |a| + \int_{|y| \geqq r} |y|/(1 + |y|^2)\nu(dy) + 1$, $r \downarrow 0$. Let $\rho(r) = \nu\{y : |y| \geqq r\}$, $\sigma^2(r) = \int_{|y| \leqq r} |y|^2 \nu(dy)$. Then, whenever $t|b_r| < r/4$,

$$\begin{aligned}P\{T_r > t\} &\geqq P\{\text{no jumps bigger than } r \text{ before } t\} P\{\sup_{s \leqq t} |X^r(s)| < r/2\}\\ &\geqq e^{-t\rho(r)}(1 - 4\,dt\,\sigma^2(r) r^{-2})\,.\end{aligned}$$

Hence if $t = \min\{r/4|b_r|, 1/(2\rho(r)), r^2/(8d\sigma^2(r))\}$, then

$$ET_r \geqq K \min\{r/\mu(r), 1/\rho(r), r^2/\sigma^2(r)\} = K \min\{r/\mu(r), r^2/\sigma^2(r)\}$$

since $r/\mu(r) \leqq 1/\rho(r)$. We must show that (in evident notation) $\int_{0<r<1} \min\{r/\mu(r), r^2/\sigma^2(r)\}\nu(dr) = \infty$ and we know that $\int_{0<r<1} r/\mu(r)\nu(dr) = \infty$, $\int_{0<r<1} r^2/\sigma^2(r)\nu(dr) = \infty$. Kesten has obtained the desired equality from the latter two by an unpleasant calculation which we omit.

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