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TRANSLATED RENEWAL PROCESSES AND THE EXISTENCE OF A LIMITING DISTRIBUTION FOR THE QUEUE LENGTH OF THE GI/G/s QUEUE

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Some ideas from the theory of weak convergence of probability measures on function spaces are modified and extended to show that the queue-length of the GI/G/s system converges in distribution as time passes, for the case of atomless interarrival and service distributions. The key to this result is the concept of the uniform σ-additivity of certain sets of renewal measures on a space endowed with incompatible topology and σ-field.

1. Introduction and summary. The main result of this paper is that the queue-length of a general multi-server queue (GI/G/s) converges in distribution as time passes. We restrict ourselves to atomless interarrival and service times, but no additional assumptions. This limit is known to exist whenever there exists an embedded persistent aperiodic renewal process, see [29]; our object is to go beyond the regenerative case. Other quantities such as actual waiting time are known to possess limiting distributions; see [6] page 63, [15], [17], and [18].

Our method of attack is to consider the queue-length as a function of the input of the system, where the input is initiated at times receding to $-\infty$. Let μ_t be the measure of an input process commencing at time $-t$ and let h_t map the input during $[-t, 0]$ into the queue length at time 0. We first show, without great difficulty, that μ_t and h_t have limits as t approaches infinity. We then prove a "continuity theorem" similar to Billingsley's (1968) Theorem 5.5, which shows that the induced measures (distribution of queue-length) converge, i.e., $\mu_t h_t^{-1} \to \mu h^{-1}$ as $t \to \infty$. Weak convergence is widely used in queueing theory (see Iglehart (1973)) but usually to derive central-limit-type results arising from scale changes. (See [14] and [31] for some different applications.) In our case we use a location change.

A problem in applying the usual theory of weak convergence is that h is far from continuous in the usual product topology. (Note that this context is different than [14] and [31] where queue-length is almost surely continuous.) Metric topologies fine enough to make h continuous (even almost surely) are nonseparable. In such a topology the measures in which we are interested have empty support, consequently it is impossible (if we assume the continuum

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hypothesis) to extend these measures to the Borel sets induced by this topology. See the discussion in Dudley (1967) and appendix III of Billingsley (1968). Pyke (1969) reviews some of the theories of weak convergence on nonseparable spaces, including [7], [24], and [32]. These do not seem to apply directly to our problem because in these theories the limit measure is supposed to be Borel, which is not the case for us. However, the idea of incompatible topology and σ-field is fruitful and we use it. Roughly speaking, we derive a continuity theorem, but use the usual product topology to define measurable sets and another finer topology to define continuity. We exploit special properties of the input of the queue and also the functions which map input into queue-length. It is conceivable that the results here could be obtained as a consequence of a general theory of convergence of probability measures on topological spaces with incompatible topologies and σ-fields, but this general theory does not appear to exist. Perhaps this paper can help serve as a motivation for the development of such a theory.

The paper is structured as follows: Section 2 presents renewal processes and the processes which we call "input processes;" a function space approach is used. Section 3 investigates the functions which map input processes into queue-length. Section 4 presents a "uniformity" theorem which corresponds to uniform σ-additivity of certain families of renewal measures on a non-separable space with incompatible topology and σ-field. Section 5 gives a "continuity" theorem (in the spirit of Theorem 5.5 of Billingsley (1968)). In Section 6 all the preceding culminates in a limit theorem for the GI/G/s queue. In Section 7 we try to relate our work to the ideas of general topological measure theory.

2. Renewal processes and input processes. Renewal processes arise frequently; see Feller (1966) and Smith (1958). The classical approach is to consider various related processes such as the renewal counting process or backwards recurrence time process as a family of random variables on some probability space. Actually renewal processes are a special case of point processes on the line. See Daley and Vere-Jones (1973) for a survey of this area. A useful approach for studying point processes is to consider them as probability measures on certain function spaces. Whitt (1973) and others take this approach. We shall use it for renewal processes.

Let ν be an atomless Borel measure on the positive real numbers, R_+, with finite first moment, m_1; ν will be the interarrival measure of the renewal process. Let R_+^∞ be a countable product of copies of R_+, indexed by $\{0, \pm 1, \pm 2, \cdots\}$. Give R_+ the usual topology and R_+^∞ the product topology. The measurable sets of R_+^∞ will be the product σ-field, which because R_+ is separable, coincides with the Borel σ-field (for the product topology) of R_+^∞. Let ν^∞ be the product measure on R_+^∞: $\nu^\infty = \prod_{i=-\infty}^{\infty} \nu_i$, where $\nu_i = \nu$.

For the product R^∞ of real lines over $\{0, \pm 1, \pm 2, \cdots\}$, let $\pi_i \colon R^\infty \to R$ be the projection map onto the ith coordinate. Now define the partial sum space $S^\infty \subset R^\infty$: $S^\infty = \{x \in R^\infty: \pi_i(x) < \pi_j(x),$ for $i < j$; $\pi_{-1}(x) < 0 \leqq \pi_0(x)$; $\lim_{n\to-\infty} \pi_n(x) = -\infty$; and $\lim_{n\to+\infty} \pi_n(x) = +\infty\}$. We shall put 2 topologies on S^∞:

(i) a relativization of the product topology on R^∞ (denoted $\mathscr{P}$-topology), and
(ii) a shift topology (denoted $\mathscr{S}$-topology).

The $\mathscr{P}$-topology is metrizable, a metric being

$$\rho_{\mathscr{P}}(x, y) = \sum_{n=-\infty}^{\infty} 2^{-|n|} \frac{|x_n - y_n|}{1 + |x_n - y_n|}\,.$$

where $x_n = \pi_n(x)$ and $y_n = \pi_n(y)$ for $x, y \in S^\infty$; this is the obvious generalization of a metric which appears in [3], [14], and [30], and is the usual product metric.

Now let us consider the shift (or $\mathscr{S}$) topology: For $x, y \in R^\infty$ say $x \sim y$ if and only if there exist i and δ such that $x_n = y_{n+i} + \delta$ for all n. (If more than one such δ exists, take the δ with the smallest magnitude.) This is an equivalence relation on R^∞. If $x \not\sim y$ define $\rho_{\mathscr{S}}(x, y) = 1$. If $x \sim y$ define $\rho_{\mathscr{S}}(x, y) = |\delta|/(1 + |\delta|)$. The function $\rho_{\mathscr{S}}$ is a pseudo-metric on R^∞, each equivalence class is $\rho_{\mathscr{S}}$-open and closed and R^∞ with the induced $\rho_{\mathscr{S}}$-topology is non-separable since there are uncountably many equivalence classes. The restriction of $\rho_{\mathscr{S}}$ to S^∞ is easily seen to be a metric. The definition of points in S^∞ yields the implication that: if $\rho_{\mathscr{S}}(x, y) < \min(-x_{-1}, x_0)/(1 + \min(-x_{-1}, x_0))$ and if $x_0 > 0$, then $x_n = y_n + \delta$ with $|\delta| = (1 - \rho_{\mathscr{S}}(x, y))/\rho_{\mathscr{S}}(x, y)$. From this it follows that there are essentially only two kinds of sequences in S^∞ $\rho_{\mathscr{S}}$-convergent to x, namely, $\{x \pm \delta_n\}$. This conclusion also holds if $x_0 = 0$, but is verified separately.

We shall call the induced $\rho_{\mathscr{S}}$-topology on S^∞, the $\mathscr{S}$-topology (or shift topology); $(S^\infty, \mathscr{S})$ is also non-separable since again there are uncountably many non-equivalent points in S^∞. The product topology $\mathscr{P}$ and the shift topology $\mathscr{S}$ are not in general comparable, but, given a point $x \in S^\infty$ with $x_0 > 0$ the above implication shows that $\mathscr{S}$ is finer than $\mathscr{P}$ at x. The metrizability of $(S^\infty, \mathscr{S})$ is not used in this paper. The metric is introduced because it provides a convenient way of visualizing what the space looks like.

The $\mathscr{P}$-topology will be used to generate measurable sets and the $\mathscr{S}$-topology to define continuity. Let $\sigma(\mathscr{P})$ be the Borel field generated by $\mathscr{P}$. Let $\bar{\sigma}(\mathscr{P})$ be its universal completion. In this paper, "measurability" will refer to the universal completions of the appropriate topological σ-fields, e.g., sets $\bar{\sigma}(\mathscr{P})$ will be called $\mathscr{P}$-measurable. In this connection, the following easily-proved lemma is useful, [19] page 23.

LEMMA 2.1. *Let X and Y be sets with σ-fields $\mathscr{A}$ and $\mathscr{B}$ and universal completions $\bar{\mathscr{A}}$ and $\bar{\mathscr{B}}$, respectively. If $f: (X, \mathscr{A}) \to (Y, \mathscr{B})$ is measurable, then $f: (X, \bar{\mathscr{A}}) \to (Y, \bar{\mathscr{B}})$ is also measurable.*

We shall now induce a renewal measure μ_0 on S^∞, the space of renewal sequences. Let $g_0: R_+{}^\infty \to S^\infty$ be such that $g_0(x) = s$ where $s_{-n} = \sum_{i=-(n-1)}^{0} x_i$, $s_0 = 0$, and $s_n = \sum_{i=1}^{n} x_i$, $n > 0$ ($\pi_j(s) = s_j$, and $\pi_j(x) = x_j$). In renewal-theoretic terms we are mapping the interarrival times into a renewal sequence with the 0th renewal occurring at epoch 0. The function $g_0: (R_+^\infty, \sigma(\mathscr{P})) \to (S^\infty, \sigma(\mathscr{P}))$ is measurable and therefore induces a measure $\mu_0 = \nu^\infty g_0^{-1}$ on $(S^\infty, \bar{\sigma}(\mathscr{P}))$. This is the measure of a renewal process with a renewal at epoch 0.

Let us return to the $\mathscr{S}$-topology: since S^∞ has the cardinality of the continuum it follows from the continuum hypothesis that any subset of S^∞ has cardinal of measure zero, hence that any Borel measure on S^∞ has non-empty $\mathscr{S}$-separable support; see [28] or [3] page 235. It can be shown that μ_0 gives measure 0 to any $\mathscr{S}$-neighborhood of diameter less than 1. Thus from the above comments, μ_0 cannot be extended to the Borel σ-field generated by $\mathscr{S}$.

We shall not restrict ourselves completely to the function-space view of renewal processes discussed above. It will occasionally be easier to think of subsets of S^∞ in terms of the traditional processes, such as renewal counting process $\{N(t), -\infty < t < \infty\}$ or backwards recurrence-time process $\{B(t), -\infty < t < \infty\}$, (see Feller (1966) or Smith (1958)). For example, $\{x \in S^\infty : t - \delta < \pi_i(x) \leqq t$ for some $i\} = \{B(t) < \delta\}$. Whitt (1973) discusses this idea of equivalent representations for point processes.

Now consider the shift-transformations, $T_t: S^\infty \to S^\infty$. Define the following functions: $f_t: S^\infty \to R^\infty$, where $\pi_i(f_t(s)) = \pi_i(s) - t$; $g_t: f_t(S^\infty) \to R$, where $g_t(y) = \min\{n : \pi_n(y) \geqq 0\}$; and $h_n: R^\infty \to R^\infty$, where $\pi_i(h_n(r)) = \pi_{i+n}(r)$. Define $T_t(x) = h_n \circ f_t(x)$, for $x \in S^\infty$ such that $g_t \circ f_t(x) = n$. Verbally, T_t shifts the partial sum sequence a distance t to the left, so the renewal which occurred at 0 now occurs at $-t$. From the above constructive definition of T_t, it is clearly $(\sigma(\mathscr{S}), \sigma(\mathscr{S}))$-measurable. Lemma 2.1 implies that it is also measurable with respect to the universal completions. We define $\mu_t = \mu_0 T_t^{-1}$, the renewal measure of a process with a renewal at epoch $-t$. Define $S_0 \subset S^\infty$ as the subset for which $\pi_0(s) = 0$ and $S_t^\infty \subset S$ as the subset for which $\pi_i(s) = -t$ from some $i = 1, 2, \cdots$. Then $\mu_0(S_0^\infty) = 1$ and $\mu_t(S_t^\infty) = 1$.

We now define a measure μ on S^∞ which will correspond to a stationary renewal process (in the sense that the backwards recurrence time process is strictly stationary). Let $\pi_{-0,1}: S^\infty \to R^2$ be the projection onto the -1st and 0th coordinates of a partial sum sequence; $\pi_{-1,0}(s) = (s_{-1}, s_0)$. We call s_0 the forward recurrence time from epoch 0 and $-s_{-1}$ the backward recurrence time from epoch 0. Let $F(x) = \nu([0, x])$ be the interarrival cdf. Recall that m_1 is the expected interarrival time. Define ν_f, forward recurrence measure on $[0, \infty)$, by $\nu_f([0, 1]) = m_1^{-1} \int_0^x (1 - F(s))\, ds$ and define ν_b, backwards recurrence measure on $[0, \infty)$ similarly. Define the joint measure $\nu_{b,f}$ on $\pi_{-1,0}(S^\infty)$ by

$$\nu_{b,f}(\{(-\infty, -c) \times (b, \infty)\}) = \frac{1}{m_1} \int_{c+b}^\infty (1 - F(s))\, ds\,.$$

Put the product measure $\nu_{b,f} \times \nu^\infty$ on $\pi_{-1,0}(S^\infty) \times R_0^\infty$ where R_0^∞ is a countable product of copies of R, this time indexed by $\{\pm 1, \pm 2, \cdots\}$. Consider the function $g: \pi_{-1,0}(S^\infty) \times R_0^\infty \to S^\infty$ such that $g(s'_{-1}, s_0', r) = s$ where $\pi_{-1}(s) = s'_{-1}$,

$\pi_0(s) = s_0'$, $\pi_n(s) = s_0' + \sum_{i=1}^{n} \pi_i(r)$ for $n > 0$, and $\pi_n(s) = s_{-1} - \sum_{i=n+1}^{-1} \pi_i(r)$ for $n < -1$. We define $\mu = (\nu_{b,f} \times \nu^\infty)g^{-1}$.

Essentially the same construction is presented in [20] and [21] where it is verified that μ indeed corresponds to a renewal process with an associated stationary backwards recurrence time process $\{B(t), -\infty < t < \infty\}$. In that case one is dealing with regenerative processes but the same results apply here. See Beutler and Leneman (1966) for a discussion of several equivalent ways to define stationarity of a point process.

LEMMA 2.2. *As $t \to \infty$, $\mu_t \to_w \mu$ in the $\mathscr{P}$-topology, if ν is a non-lattice measure.* ($\to_w$ *denotes weak convergence.*)

PROOF. It is well known that the finite dimensional distributions of a renewal process converge. It follows from [20] that μ is the limit. The lemma then follows from the fact that, for the $\mathscr{P}$-topology on R^∞, finite dimensional sets are a convergence-determining class ([3], page 19).

Now we turn to "input processes": these are renewal processes which have a random quantity attached to each renewal point. For example, for a queueing process we will have the arrival sequence plus a service time associated with each arrival. For a shot-noise process (Takács (1956)), the input is a series of "shots" of random magnitude arriving at random times. The space of sample paths will be (S^∞, R_+^∞). As before we will put two topologies on this space: the product ($\mathscr{P}$) topology on (S^∞, R_+^∞) will be the product of the product topologies on each component; the shift ($\mathscr{S}$) topology on (S^∞, R_+^∞) will be the product of the shift topology on S^∞ and the discrete topology on R_+^∞. We use the same $\mathscr{S}$ and $\mathscr{P}$ notation for this space; it will be clear from the context whether we are considering S^∞ or (S^∞, R_+^∞). As before we let $\mathscr{P}$ generate the measurable sets $\bar{\sigma}(\mathscr{P})$ and use $\mathscr{S}$ to define continuity. Let λ be an atomless Borel measure on R_+ (the service time measure), then define μ_t' on (S^∞, R_+^∞) by $\mu_t \times \lambda^\infty$ where $\lambda^\infty = \prod_{i=-\infty}^{\infty} \lambda_i$, $\lambda_i = \lambda$; similarly define μ'. (We are assuming i.i.d. service times.)

Before we proceed we present some notation:

(i) in the future we will drop the prime ($'$) notation of these measures and refer to μ_t and μ_t' both as μ_t; it will be clear from the context which space we are working with;

(ii) define π_i': $(S^\infty, R_+^\infty) \to R$ as the projection onto the ith coordinate of R_+^∞; as before π_i is the projection onto the ith coordinate of S^∞;

(iii) for $x \in (S^\infty, R_+^\infty)$, $y = x + \delta$ is defined as $\pi_i(y) = \pi_i(x) + \delta$ and $\pi_i'(y) = \pi_i'(x)$ for all i; except that the coordinates of y may be re-indexed to satisfy $\pi_{-1}(y) < 0 \leqq \pi_0(y)$.

LEMMA 2.3. *The statement of Lemma 2.2 is also true for μ_t' and μ'.*

Proof. Use Lemma 2.2, the independence of service times and Billingsley's (1968) Theorem 3.2 for product measures.

LEMMA 2.4. *Let $A \subset (S^\infty, R_+^\infty)$ be a $\mathscr{P}$-Borel set, then $A^{-\mathscr{S}} \in \bar{\sigma}(\mathscr{P})$. ($A^{-\mathscr{S}}$ is the $\mathscr{S}$-closure of A.)*

PROOF. For $\delta \in R$, let $A+\delta = \{x+\delta : x \in A\}$, then $A^{-\mathscr{S}} = \bigcap_{n=1}^{\infty} \bigcup_{|\delta| \leq n^{-1}} (A+\delta)$. Let ϕ be a measure on $((S^\infty, R_+^\infty), \sigma(\mathscr{P}))$. Let $X = (S^\infty, R_+^\infty) \times [0, 1]$ and π the projection of X onto (S^∞, R_+^∞). Let $f: X \to X$ be defined by $f(x, y, z) = (x + z, y, z)$. Let $A_n = f(A \times [0, n^{-1}])$. Since f^{-1} is continuous, A_n is a Borel set in X and $\bigcup_{|\delta| \leq n^{-1}} (A + \delta) = \pi(A_n)$. Since X is homeomorphic to a complete separable metric space (in the product topology), π is continuous and A_n is Borel, it follows from Theorem 2.2.13 of Federer (1969) that ϕ extends to $\pi(A_n) = \bigcup_{|\delta| \leq n^{-1}} (A + \delta)$. Consequently ϕ also extends to $A^{-\mathscr{S}}$.

LEMMA 2.5. *Let A be a $\mathscr{P}$-measurable subset of (S^∞, R_+^∞) and $R_t = \{x \in (S^\infty, R_+^\infty) : -t < \pi_i(x) \leq 0$ for some $i\}$ then*

$$\mu(A) = \frac{1}{m_1} \int_0^\infty \mu_s(A \mid \bar{R}_s)(1 - F(s))\, ds .$$

PROOF. The proof in [21] for regenerative processes holds in this setting.

LEMMA 2.6. *Let A be a $\mathscr{P}$-measurable subset of (S^∞, R_+^∞) and suppose $\mu_t(A) = 0$ for almost all (Lebesgue) t, then $\mu(A) = 0$.*

PROOF. The hypothesis implies that $\mu_t(A \mid \bar{R}_t) = 0$ for almost all (Lebesgue) t for which the conditional measure is defined, i.e., t such that $F(t) < 1$. Lemma 2.5 then proves the lemma.

3. **Queue-length as a function of input.** Consider the general multi-server (GI/G/s) queueing system. Starting at epoch 0, customers arrive at random times, requiring random amounts of service. The interarrival times are i.i.d. with finite expected value. The service times are i.i.d. Interarrival and service times are independent of one another. In short, the input to the system is an "input process" with no renewal before epoch 0 and the first renewal at epoch 0 (or equivalently, an input process on $(-\infty, \infty)$ with renewals in $(-\infty, 0)$ ignored). There are s servers who dispense service on a first-come-first-served basis (one server per customer). If all servers are busy a customer waits. The total number of customers in the system at any time (waiting and being served) in the queue-length. See Prabhu (1965) or DeSmit (1971). See Feller, (1966) page xv, for discussion on the term "epoch."

We are interested in the stochastic behavior of queue-length after much time passes. It will be convenient to approach this by letting the starting epoch recede further into the past; for example, the queue-length at epoch t of a system which began accepting customers at epoch 0 has the same distribution as the queue-length at epoch 0 of a system which began at epoch $-t$. Thus we shall be interested in the queue-length at epoch 0, but we shall start the system at epochs which recede into the distant past.

Consider a point x in the input space (S^∞, R_+^∞); the ith customer arrives at

$s_i = \pi_i(x)$, (π_i is the projection defined in Section 2). We shall say that the ith customer is excluded from the queueing process if he is sent away without being allowed to enter the system (or equivalently, that he is serviced instantaneously); in either interpretation he does not affect the queue-length. The ith customer is *admitted* to the system if he is allowed to wait and obtain service.

Now consider the family of functions $Q_s: (S^\infty, R_+^\infty) \times R \to R$, $s \geqq 0$; where $Q_s(x, t) = 0$, for $t < -s$; and for $t \geqq -s$, $Q_s(x, t)$ equals the queue-length at epoch t which results from input x when customers arriving during $(-\infty, -s)$ are excluded and customers arriving during $[-s, \infty)$ are admitted, i.e. the system starts at epoch $-s$. (This verbal definition of Q_s will suffice for our presentation. See [31] for a rigorous mathematical approach to defining queue-length, which includes treatment of measurability questions.) We have a family of stochastic processes $\{Q_s(\cdot, t), -\infty < t < \infty\}$, $s \geqq 0$, on the space (S^∞, R_+^∞) for each measure μ_t on this space.

In the interest of brevity we shall state Lemmas 3.1 through 3.4 without proof. The proofs are straightforward but tedious and similar results already appear in [13], [14], and [31]. Proofs of 3.5 and 3.6 are sketched.

LEMMA 3.1. *For fixed x and t, $Q_s(x, t)$ is monotonically non-decreasing in s.*

This lemma guarantees the existence (possibly infinite) of a limiting function $Q(x, t) = \lim_{s\to\infty} Q_s(x, t)$. The stochastic process induced by Q and μ is stationary because μ is the measure of a stationary input process. (In general if $\{X(t), -\infty < t < \infty\}$ is a stationary stochastic process and f is $\sigma(\{X(t), -\infty < t < \infty\})$-measurable then $\{f(X(s + \cdot)), -\infty < s > \infty\}$ is also a stationary process. See [14], page 105, for the discrete time case.) This process can be interpreted as a stationary queueing process which admits all customers. Its 1-dimensional distribution is $\mu Q(\cdot, 0)^{-1}$.

In this paper our methods of proof only consider queue-length at epoch 0. Therefore, we introduce the following notation: define $h_s: (S^\infty, R_+^\infty) \to R$ by $h_s(x) = Q_s(x, 0)$ and $h: (S^\infty, R_+^\infty) \to R$ by $h(x) = Q(x, 0)$. Also define $D_{s,i}(x)$ to equal the departure epoch of the ith customer in the Q_s-process for input x; if $\pi_i(x) = s_i < -s$ then $D_{s,i}(x) = s_i$. Define the following subsets of $(S^\infty, R^+{}_\infty)$: $A_0 = \{x : \pi_0(x) = 0\}$; $A_t = \{x : \pi_i(x) = -t \text{ for some } i\}$; $D_{0,t,i} = \{x : D_{t,i}(x) = 0\}$; $D_{0,t} = \bigcup_{i=-\infty}^{\infty} D_{0,t,i}$; and $D_{0,i} = \{x : \lim_{s\to\infty} D_{s,i}(x) = 0\}$.

LEMMA 3.2. *The $\mathscr{P}$-discontinuities of h_t are contained in $A_0 \cup A_t \cup D_{0,t}$.*

LEMMA 3.3. *If the interarrival and service measures (ν and λ) are atomless then*

(i) $\mu_s(A_0) = 0$ *for* $s \neq 0$,
(ii) $\mu_s(A_t) = 0$ *for* $s \neq t$,
(iii) $\mu_s(D_{0,t,i}) = 0$ *for all* s,
(iv) $\mu_s(D_{0,i}) = 0$ *for all* s, *and*
(v) $\mu_s(D_{0,t}) = 0$ *for all* s.

LEMMA 3.4. *If the interarrival and service measures (ν and λ) are atomless then*

h_t is $\mathscr{P}$-continuous a.s. (μ_s) *for* $s \neq 0, t$. *Furthermore, under these conditions h_t is also $\mathscr{S}$-continuous* a.s. (μ_s) *for* $s \neq 0, t$.

LEMMA 3.5. *If h is $\mathscr{S}$-discontinuous at x then there exists a T such that for $t \geqq T$ there exists a net $\{x_t\}$, $x_t \to_{\mathscr{S}} x$, such that h_t is $\mathscr{S}$-discontinuous at x_t.*

PROOF. S^∞ is locally arcwise connected in the $\mathscr{S}$-topology. Suppose h has an $\mathscr{S}$-discontinuity at x and $N(x)$ is the $\mathscr{S}$-closure of an $\mathscr{S}$-connected $\mathscr{S}$-neighborhood of x, then there exists a $y \in N(x)$ such that $h(x) \neq h(y)$. Suppose $h(x) < \infty$ and $h(y) < \infty$ then since the h_t's are integer-valued there exists a $T_{x,y}$ such that $h_t(x) = h(x)$ and $h_t(y) = h(y)$ for $t \geqq T_{x,y}$. Let $A_t = \{z \in N(x) : h_t(z) = h(x)\}$, $B_t = \{z \in N(x) : h_t(z) \neq h(x)\}$. Since $N(x)$ is connected, $A_t^{-\mathscr{S}} \cap B_t^{-\mathscr{S}} \neq \varnothing$, and $x_t \in A_t^{-\mathscr{S}} \cap B_t^{-\mathscr{S}}$ is a point of discontinuity of h_t. Thus any neighborhood of x contains discontinuities of h_t for all $t > T$ for some T which depends on the neighborhood. Suppose either $h(x) = \infty$ and $h(y) < \infty$, or vice versa, then there exists a $T_{x,y}$ such that $h_t(x) > h(y)$ and $h_t(y) = h(y)$ for all $t \geqq T$ and we can proceed as before.

There exist points of $\mathscr{S}$-discontinuity for h at which h_t is $\mathscr{S}$-continuous for all t. One such point, x_0, is defined as follows: $\pi_0(x_0) > 0$, $\pi'_{-i}(x_0) = -(i+1)$, and $\pi'_{-i}(x_0) = 1 + 2^{-i}$, for $i = 1, 2, 3, \cdots$. Thus we need Lemma 3.5 in its stated form.

LEMMA 3.6. *If ν and λ are atomless then h is $\mathscr{S}$-continuous* a.s. (μ_s), $s > 0$.

PROOF. As mentioned in Section 2, the $\mathscr{S}$-topology is a refinement of the $\mathscr{P}$-topology at all points of (S^∞, R_+^∞) except those x for which $\pi_0(x) = 0$. This fact and Lemma 3.2 imply that the $\mathscr{S}$-discontinuities of h_t are contained in $A_0 \cup A_t \cup D_{0,t}$. Suppose h has an $\mathscr{S}$-discontinuity at x. Then by Lemma 3.5, there exists $\{x_t, t > T\}$ such that $x_t \to_{\mathscr{S}} x$ as $t \to \infty$ and h_t is $\mathscr{S}$-discontinuous at x_t.

First we examine the possibility that for some T, $x_t \in A_t$ for $t > T$. Suppose for $t \geqq T_\delta > T$, $x_t = x + \varepsilon_t$, $|\varepsilon_t| \leqq \delta$. Consider i such that $\pi_i(x) \leqq -T_\delta \leqq \pi_{i+1}(x)$, then for $j < i$, $|\pi_j(x) - \pi_{j+1}(x)| < 2\delta$. Let $U_j(x)$ equal $\pi_{j-1}(x) - \pi_j(x)$, which equals the jth interarrival time. Since δ can be made arbitrarily small this implies that $U_j(x) \to 0$ as $j \to -\infty$ for such x. But for x such that $\pi_j(x) < -s$, the random variables U_j are i.i.d. with measure ν, so we know that $U_j \to 0$ with μ_s-measure 0. Thus if we exclude a set of μ_s-measure 0 we can assume that $x_t \notin A_t$ for some arbitrarily large t.

Suppose there is a subsequence of $\{x_t\}$; call it $\{x_{t_i}\}$ such that $x_{t_i} \in A_0$. Since A_0 is an $\mathscr{S}$-closed subset of (S^∞, R_+^∞) and $x_{t_i} \to_{\mathscr{S}} x$, then $x \in A_0$. Thus by Lemma 3.3 if we throw out another set of μ_s-measure zero we can avoid this possibility.

The only other possibility is that there exists a subsequence $\{x_{t_i}\}$ of $\{x_t\}$ such that $x_{t_i} \in D_{0,t_i} \cap \bar{A}_0 \cap \bar{A}_{t_i}$; $x_{t_i} \to_{\mathscr{S}} x$ thus $x_{t_i} = x + \delta_i$ where $\delta_i \to 0$ as $i \to \infty$. We can guarantee that $t_i + \delta_i$ is a monotone increasing sequence (if necessary by taking a subsequence). Consider the Q_{t_i}-process: for input x_{t_i} there is a departure at epoch 0;

$$Q_{t_i}(x_{t_i}, t) = Q_{t_i-\delta_i}(x_{t_i} + \delta_i, t + \delta_i) = Q_{t_i-\delta_i}(x, t + \delta_i) .$$

Therefore, the $Q_{t_i-\delta_i}$-process has a departure at epoch δ_i when the input is x. As in Lemma 3.1 it follows that for fixed k and x, $D_{t,k}(x)$ is monotonically non-decreasing in t. Consequently, if $\delta_i > \delta_{i+1}$ then the departure from the $Q_{t_i-\delta_i}$-process (with input x) at epoch δ_i must be a later customer (higher index number) than the customer who departs at δ_{i+1} from the $Q_{t_{i+1}-\delta_{i+1}}$-process (with input x). We can restrict $\{\delta_i\}$ (i.e. take a subsequence) such that $|\delta_i| \searrow 0$ (strictly) and then split it into a positive and negative subsequence $\{\delta_i^+\}$ and $\{\delta_i^-\}$. If $\{\delta_i^+\}$ has an infinite number of elements then since the sequence is strictly decreasing each element must correspond to the departure of an earlier customer, i.e., the kth term corresponds to departure of the $-k$th or earlier customer. Thus at epoch 0, as $t \to \infty$, the Q_t-process has arbitrarily long queue-length and $h(x) = \infty$. We shall return to this possibility later. If $\{\delta_i^+\}$ consists of only a finite number of elements we may ignore it. Now consider $\{\delta_i^-\}$: we know that the $Q_{t_i-\delta_i}$-process has a departure at epoch δ_i^- for input x; suppose it is the $k(i, x)$th customer. Consider the limit points of $\{k(i, x), i = 1, 2, \cdots\}$. If k_0 is a limit point then (because it is an integer-valued sequence) $k(i, x) = k_0$ for a subsequence of i's. This implies that x belongs to D_{0,k_0} but by Lemma 3.3, $\mu_s(D_{0,k_0}) = 0$. If $\{k(i, x)\}$ has no limit point then for i sufficiently large, the departure corresponds to a customer with arbitrarily low index number and again the queue-length at epoch δ_i^- is arbitrarily large as $i \to \infty$, thus $h(x) = \infty$.

We have shown that the set of $\mathscr{S}$-discontinuities for which h is finite-valued has μ_s-measure zero. Now we show that almost all (μ_s) x for which $h(x) = \infty$ are points of $\mathscr{S}$-continuity. Consider μ_s: let $\{h_{t_i}\}$ be a subsequence of $\{h_t\}$, $t_i \to \infty$, $t_i \neq s$. Let $D_i = \{x : h_{t_i}$ has an $\mathscr{S}$-discontinuity at $x\}$, then from before $\mu_s(\bigcup_{i=1}^\infty D_i) = 0$. Suppose $h(x) = \infty$ for $x \notin \bigcup_{i=1}^\infty D_i$, then given any M there exists T such that $h_{t_i}(x) > M$, for $t_i \geqq T$. Because $x \notin D_i$ there exists a neighborhood $N_i(x)$ such that $h_{t_i}(y) > M$ for all $y \in N_i(x)$. Since $h \geqq h_t$, $h(y) > M$ for all $y \in N_i(x)$. Thus if $x_n \to_{\mathscr{S}} x$, then $\liminf_{n\to\infty} h(x_n) \geqq M$, but M is arbitrary so $\lim_{n\to\infty} h(x_n) = \infty$, and h is $\mathscr{S}$-continuous at x.

COUNTEREXAMPLE 3.7. The function h is not a.s. $\mathscr{P}$-continuous. Consider the special case G/D/1, where the interarrival distribution is uniform on [1, 2] and unit service times. Consider $x \in (S^\infty, R_+^\infty)$, $\pi_i(x) = s_i$ and $\pi_i'(x) = 1$, for all i, and a product neighborhood $N_{\delta,n}(x) = \{y \in (S^\infty, R_+^\infty) : |\pi_i(y) - \pi_i(x)| < \delta$ and $|\pi_i'(y) - \pi_i'(x)| < \delta$ for $|i| \leqq n\}$. Suppose z_n is identical to x in all coordinates except $\pi_i'(z_n) = 3$ for $i < -n$. Then $z_n \in N_{\delta,n}(x)$, but $h_t(z_n) \to \infty$ and $z_n \to_{\mathscr{P}} x$ but $h(x) = 0$ or 1. Thus h is not $\mathscr{P}$-continuous at x, but μ_s is concentrated on such x; thus almost all (μ_s) $x \in (S^\infty, R_+^\infty)$ are $\mathscr{P}$-discontinuity points. In fact, h is $\mathscr{P}$-discontinuous everywhere.

4. A uniformity theorem. We shall obtain convergence of sequences of induced measures, such as $\mu_t h_t^{-1}$, essentially by showing that the limit of $\mu_t h_s^{-1}$ is independent of the rates with which t and s approach ∞. For this we will need some kind of "uniformity," i.e. Theorem 4.5. We proceed with some preliminary lemmas.

LEMMA 4.1. *If A is an $\mathscr{S}$-closed, $\mathscr{P}$-measurable subset of S^∞, μ_t a renewal measure on S^∞, and $\{\delta_n\}$ a sequence of real numbers which converge to* 0 *as $n \to \infty$, then* $\limsup_{n\to\infty} \mu_t(A - \delta_n) \leqq \mu_t(A)$.

PROOF. First we verify that $\bigcap_{m=1}^\infty [\bigcup_{n\geq m} (A - \delta_n)]^{-\mathscr{S}} = A$. Clearly $[\bigcup_{n\geq m} (A - \delta_n)]^{-\mathscr{S}} \supset A$ because any point $x \in A$ has a sequence $\{x - \delta_n, n \geqq m\}$ in $\bigcup_{n\geq m}(A - \delta_n)$ which $\mathscr{S}$-converges to x, thus it remains to show the reverse inclusion. Suppose $x \in [\bigcup_{n\geq m} (A - \delta_n)]^{-\mathscr{S}}$ for all m, then two possibilities exist: either

(i) $x \in A - \delta_n$ for infinitely many n, which implies $x + \delta_n \in A$ for infinitely many n, which in turn implies that $x \in A$ because A is $\mathscr{S}$-closed and x is an $\mathscr{S}$-limit point of $\{x + \delta_n\}$; or

(ii) $x \in A - \delta_n$ for at most finitely many n, which implies that there exists m_0 such that $x \in [\bigcup_{n\geq m} (A - \delta_n)]^{-\mathscr{S}} \backslash \bigcup_{n\geq m} (A - \delta_n)$, for $m \geqq m_0$. Thus for each $m \geqq m_0$ there is a sequence $\{x_{m,i}, i = 1, 2, 3, \cdots\} \subset \bigcup_{n\geq m} (A - \delta_n)$ such that $x_{m,i} \to_{\mathscr{S}} x$, as $i \to \infty$. By a diagonal argument, we can get a sequence $x_m' = x_{m,k(m)}$ where $k(m) = \min(k: |x_{m,k} - x| < 1/m)$, $x_m' \in A - \delta_{n(m)}$ for some $n(m) \geqq m$. Thus $x_m' + \delta_{n(m)} \in A$. But $x_m' \to_{\mathscr{S}} x$ and $\delta_{n(m)} \to 0$, as $m \to \infty$. Therefore since A is $\mathscr{S}$-closed, $x \in A$.

Suppose $\limsup_{n\to\infty} \mu_t(A - \delta_n) = c$. Given any $\varepsilon > 0$, there exists a subsequence $n' \to \infty$ such that $\mu_t(A - \delta_{n'}) > c - \varepsilon$. Thus $\mu_t(\bigcup_{n\geq m} (A - \delta)) > c - \varepsilon$ and by monotonicity $\mu_t(\bigcap_m \bigcup_{n\geq m} (A - \delta_n)) > c - \varepsilon$. From above $A \supset \bigcap_m \bigcup_{n\geq m} (A - \delta_n)$ and so $\mu_t(A) > c - \varepsilon$. Since $\varepsilon > 0$ was arbitrary, $\mu_t(A) \geqq c$.

LEMMA 4.2. *Let μ be a stationary non-lattice renewal measure on S^∞ with interarrival* cdf *F and finite expected interarrival time, m_1. Let*

$$R_{s,n} = \{x \in S^\infty : s - n^{-1} \leqq \pi_i(x) \leqq s + n^{-1}, \text{ for some } i\}, \qquad \textit{then}$$

$$\mu(R_{s,n}) = \frac{1}{m_1} \textstyle\int_0^{2/n} (1 - F(s) \, ds \geqq \frac{2}{m_1 n} (1 - F(2/n)) .$$

PROOF. If $\{B(t), -\infty < t < \infty\}$ is the stationary backwards recurrence time process associated with μ, then $R_{s,n} = \{B(s + n^{-1}) \leqq 2n^{-1}\}$. We know ([10], [20]) that $B(s + n^{-1})$ has density $(1/m_1)(1 \dot{-} F)$, therefore

$$\mu(R_n) = \frac{1}{m_1} \textstyle\int_0^{2/n} (1 - F(s)) \, ds .$$

LEMMA 4.3. *Again let μ be a stationary non-lattice renewal measure on S^∞ and*

$$R_{s,m} = \left\{x \in S^\infty : s - \frac{1}{m} \leqq \pi_i(x) \leqq s + \frac{1}{m}, \text{ for some } i\right\}, \qquad \textit{then}$$

$$\lim_{t\to\infty} |\mu(R_{s,m} \cap R_{s+t,m}) - \mu(R_{s,m})\mu(R_{s+t,m})| = 0 .$$

PROOF. Consider

$$\begin{aligned}\mu(R_{s,m} \cap R_{s+t,m}) &= P\left\{B\left(s+\frac{1}{m}\right) \leqq 2/m \quad \text{and} \quad B\left(s+t+\frac{1}{m}\right) \leqq 2/m\right\} \\ &= \int_0^{2/m} P\left\{B\left(s+t+\frac{1}{m}\right) \leqq 2/m \,\middle|\, B\left(s+\frac{1}{m}\right) = u\right\} dF_{B(s+m^{-1})}(u) \\ &= \int_0^{2/m} P\left\{B_{s-u+m^{-1}}\left(s+t+\frac{1}{m}\right) \leqq 2/m\right\} dF_{B(s+m^{-1})}(u)\,,\end{aligned}$$

where B_t is the backwards recurrence time process associated with μ_t and B the process associated with μ. From [20]

$$\lim_{t\to\infty} P\left\{B_{s-u+m^{-1}}\left(s+t+\frac{1}{m}\right) \leqq 2/m\right\} = P\left\{B\left(s+t'+\frac{1}{m}\right) \leqq 2/m\right\}\,,$$

for any t'.

The lemma follows by dominated convergence.

LEMMA 4.4. *Let $A_1, A_2, \cdots$ be events in a sample space, and suppose that $\sum_{n=1}^{\infty} P(A_n) = \infty$. Suppose also that for some constants N and $c < \infty$, $P(A_n \cap A_m) \leqq cP(A_n)P(A_m)$ for all $n, m > N$, then $P(\bigcap_m \bigcup_{n\geqq m} A_n) > 0$.*

PROOF. See Lamperti (1963).

4.5. UNIFORMITY THEOREM FOR RENEWAL PROCESSES. *Let $A_1 \supset A_2 \supset A_3 \cdots$ be a sequence of $\mathscr{P}$-measurable, $\mathscr{S}$-closed subsets of S^∞. Let $\{\mu_t, t \geqq 0\}$ be a family of translated non-lattice renewal measures on S^∞. If $\mu_t(\bigcap_{n=1}^{\infty} A_n) = 0$ for all $t > 0$, then $\mu_t(A_n) \searrow 0$, uniformly in t as $n \to \infty$.*

PROOF. Suppose the conclusion of the theorem does not hold, then there exists an $\varepsilon > 0$ such that for each n there exists a t_n for which $\mu_{t_n}(A_n) \geqq \varepsilon$. The sequeuce $\{t_n\}$ is unbounded: suppose the contrary, then $\{t_n\}$ has a limit point, $t_* < \infty$, and $t_{n'} \to t_*$. Since $\mu_t(A_n) \searrow 0$ for all t there exists an m such that $\mu_{t_*}(A_m) < \varepsilon/2$, but by definition of $\{t_{n'}\}$, $\mu_{t_{n'}}(A_m) > \varepsilon$ for $n' \geqq m$. From Section 2 we know that $\mu_{t_{n'}} = \mu_{t_*}(T_{t_{n'}-t_*})^{-1}$ where T_t is the shift operator so $\mu_{t_{n'}}(A_m) = \mu_{t_*}(A_m - (t_{n'} - t_*))$. Since $t_{n'} - t_* \to 0$ we can apply Lemma 4.1 to get $\mu_{t_*}(A_m) \geqq \varepsilon$, which is a contradiction. Thus $\{t_n\}$ is unbounded and in particular, we can take a subsequence which goes to infinity at a rapid rate. If $R_n = \{x \in S^\infty : t_n - n^{-1} \leqq \pi_i(x) \leqq t_n + n^{-1}$, for some $i\}$ then we shall require that $\{t_n\}$ approach infinity rapidly enough to satisfy $\mu(R_n \cap R_m) \leqq 2\mu(R_n)\mu(R_m)$, which can be guaranteed by Lemma 4.3.

Now let us reconsider $\mu_{t_n}(A_n) \geqq \varepsilon > 0$. The measure μ_{t_n} is concentrated on $S_{t_n}^\infty$. Let $B_n = A_n \cap S_{t_n}^\infty$; $\mu_{t_n}(B_n) \geqq \varepsilon$. Clearly since A_n is $\mathscr{S}$-closed and $B_n \subset A_n$ we have $\bigcap_m (\bigcup_{n\geqq m} B_n)^{-\mathscr{S}} \subset \bigcup_n A_n$; thus it suffices to show that $\mu_s(\bigcap_m (\bigcup_{n\geqq m} B_n)^{-\mathscr{S}}) \neq 0$, for some s, to get a contradiction. Equivalently, it suffices to show that the set $\{x \in S^\infty :$ there exists $x_n \to_{\mathscr{S}} x$ such that $x_n \in B_n\}$

has positive μ_s-measure for some s. Using Lemma 2.6, it will suffice to show that this set has positive μ-measure, where $\mu_t \rightarrow_w \mu$.

Let $C_k = \bigcup_{|\delta|<k^{-1}} (B_k + \delta)$. This set is $\mathscr{P}$-measurable for reasons stated in the proof of Lemma 2.4. Let $C = \bigcap_n \bigcup_{k \geqq n} C_k$; if $x \in C$, then there exists a sequence $\{x_k\}$, $x_k \in B_k$ which has x as an $\mathscr{S}$-limit point. Thus to obtain our contradiction it suffices to show that $\mu(C) > 0$.

A slight problem arises because C_k as defined above is not a disjoint union. We can think of C_k as the elements of S^∞ with a renewal in $[t_k - k^{-1}, t_k + k^{-1}]$, say, at $t_k + \delta(x)$; if 2 or more renewals occur we just consider the first one. Then for such an x to be an element of C_k we also require that $x - \delta(x) \in B_k$. (Note that there may be other elements of C_k: namely, those with 2 renewals in $[t_k \pm k^{-1}]$, say, at $t_k + \delta_1(x)$ and $t_k + \delta_2(x)$, such that $x - \delta_1(x) \notin B_k$ but $x - \delta_2(x) \in B_k$.)

Let $T_k(x) = \min_i (\pi_i(x) \geqq t_k - k^{-1})$ and $D_k = \{x \in S^\infty : x - (T_k(x) - t_k) \in B_k\}$ and $R_k = \{x \in S^\infty : t_k - k^{-1} \leqq \pi_i(x) \leqq t_k + k^{-1}$, for some $i\}$; then $\mu(C_k) \geqq \mu(R_k \cap D_k)$. Letting the origin be $t_k - k^{-1}$ and using a symmetric version of Lemma 2.5 for forward recurrence times from the origin, we get

$$(R_k \cap D_k) = \frac{1}{m_1} \int_{t_k-k^{-1}}^{\infty} \mu_s(R_k \cap D_k \,|\, \bar{R}_s)\left(1 - F\left(s - \left(t_k - \frac{1}{k}\right)\right)\right) ds\,, \qquad \text{where}$$

$$R_s = \left\{x \in S^\infty : t_k - \frac{1}{k} \leqq \pi_i(x) < s\,, \quad \text{for some} \quad i\right\}.$$

But, by shifting the renewal sequence by $t_k - s$, we see that $\mu_s(R_k \cap D_k \,|\, \bar{R}_s) = \mu_{t_k}(B_k \,|$ no renewal in $[t_k - k^{-1} + (t_k - s), t_k)$ for $t_k - k^{-1} \leqq s \leqq t_k + k^{-1}$, and equals 0 otherwise. This first expression is greater than $\mu_{t_k}(B_k \cap \{$no renewal in $[t_k - 2k^{-1}, t_k)\})$ which for sufficiently large k is greater than $\varepsilon/2$ because $\mu_{t_k}(B_k) > \varepsilon$ and $\lim_{k\to\infty} \mu_{t_k}$ (no renewal in $[t_k - 2k^{-1}, t_k)) = 1$. Thus for sufficiently large k,

$$\begin{aligned} \mu(C_k) &\geqq \frac{1}{m_1} \int_{t_k-k^{-1}}^{t_k+k^{-1}} \left(\frac{\varepsilon}{2}\right)\left(1 - F\left(s - \left(t_k - \frac{1}{k}\right)\right)\right) ds \\ &\geqq \left(\frac{1}{m_1}\right)\left(\frac{\varepsilon}{2}\right)\left(\frac{2}{k}\right)\left(1 - F\left(\frac{2}{k}\right)\right). \end{aligned}$$

Since F has no atom at 0, we get $\mu(C_k) \geqq \varepsilon/2km_1$ for large k. Therefore $\sum_{k=1}^{\infty} \mu(C_k) = \infty$, satisfying the first hypothesis of Lemma 4.4. From the above we also get $\mu(C_k) > (\varepsilon/2)\mu(R_k)$ for large k.

From the above analysis and the inclusion $R_k \supset C_k$ we get $\mu(R_k) \geqq \mu(C_k) \geqq \varepsilon/2\mu(R_k)$ for k sufficiently large. This and the preceding implies that

$$\begin{aligned} \mu(C_j \cap C_k) &\leqq \mu(R_j \cap R_k) \leqq 2\mu(R_j)\mu(R_k) \\ &\leqq 2\left(\frac{2}{\varepsilon}\,\mu(C_j)\right)\left(\frac{2}{\varepsilon}\,\mu(C_k)\right) = \frac{8}{\varepsilon^2}\,\mu(C_j)\mu(C_k) \end{aligned}$$

for sufficiently large j and k. This fulfills the second hypothesis of Lemma 4.4. Thus $\mu(C) > 0$, which is a contradiction, completing the proof.

For the purpose of a discussion (Section 7) of the general theory of weak convergence we note the following corollary.

COROLLARY 4.6. (Uniform σ-additivity of translated renewal measures). *Let $A_1 \supset A_2 \supset A_3 \cdots$ be a sequence of $\mathscr{P}$-measurable, $\mathscr{S}$-closed subsets of S^∞, such that $\bigcap_{n=1}^\infty A_n = \varnothing$. Let $\{\mu_t, t \geqq 0\}$ be a family of translated non-lattice renewal measures on S^∞; then $\mu_t(A_n) \searrow 0$ uniformly in t as $n \to \infty$.*

4.7. UNIFORMITY THEOREM FOR INPUT PROCESSES. *Let μ_s be a family of translated input-measures on (S^∞, R_+^∞), with ν non-lattice. Suppose $A_1 \subset A_2 \subset \cdots$ are $\mathscr{P}$-measurable, $\mathscr{S}$-closed subsets of (S^∞, R_+^∞) such that $\mu_s(\bigcap_n A_n) = 0$ for all $s \geqq s_0$. Then $\mu_s(A_n) \searrow 0$ uniformly in s $(s \geqq s_0)$ as $n \to \infty$.*

PROOF. The proof of Theorem 4.5 extends to this case virtually unchanged.

5. A continuity theorem.

THEOREM 5.1. *Let μ_t, $t > 0$, be a family of atomless translated renewal measures on S^∞ and let μ be the limiting stationary renewal measure. Suppose g, g_t: $S^\infty \to R$, $t \geqq 0$; g is bounded; $g_t(x) \nearrow g(x)$, as $t \to \infty$; g_t is $\mathscr{P}$-continuous* a.s. *(μ_s) for all $s \neq t$, $t > 0$; and g is $\mathscr{S}$-continuous* a.s. *(μ_s) for all s. Then $\mu_t g_t^{-1} \to_w \mu g^{-1}$, as $t \to \infty$.*

PROOF. By Theorem 2.1 of Billingsley (1968) it suffices to show that $\lim_{t\to\infty} \int f\, d(\mu_t g_t^{-1}) = \int f\, d(\mu g^{-1})$, for all bounded, uniformly continuous real f. Without loss of generality we can assume $0 \leqq f \leqq 1$. Pick any such f; then given $\varepsilon > 0$, there exists $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$.

Let $A_t = \{g - g_t \geqq \delta\}$. Because $g_t \nearrow g$ and g is bounded it follows that $A_t \searrow \varnothing$. Let $D_t = \{\mathscr{S}\text{-discontinuities of } g - g_t\} \subset \{\mathscr{S}\text{-discontinuities of } g\} \cup \{\mathscr{S}\text{-discontinuities of } g_t\}$. By hypothesis then, for $s \neq t$, $\mu_s(D_t) = 0$. Consider the $\mathscr{S}$-closure of A_t: $A_t^{-\mathscr{S}} \subset A_t \cup D_t$; $A_t^{-\mathscr{S}}$ is $\mathscr{P}$-measurable by Lemma 2.4 and by above $\mu_s(A_t^{-\mathscr{S}}) = \mu_s(A_t)$ if $s \neq t$. As $t \to \infty$, $\mu_s(A_t) \searrow 0$ for each s, and consequently so does $\mu_s(A_t^{-\mathscr{S}}) \searrow 0$. Thus $\mu_s(\bigcap_t A_t^{-\mathscr{S}}) = 0$, for each $s \geqq 0$. By Theorem 4.5, $\mu_s(A_t^{-\mathscr{S}}) \searrow 0$ uniformly in s as $t \to \infty$. Since $\mu_s(A_t^{-\mathscr{S}}) \geqq \mu_s(A_t)$ we get $\mu_s(A_t) \searrow 0$ uniformly in s as $t \to \infty$, even though A_t may not be $\mathscr{S}$-closed.

Consider

$$
\begin{aligned}
&\left|\int f\, d(\mu_t g_t^{-1}) - \int f\, d(\mu g^{-1})\right| \\
&\quad = \left|\int (f \circ g_t)\, d\mu_t - \int (f \circ g)\, d\mu\right| \\
&\quad \leqq \int_{A_t} |(f \circ g_t) - (f \circ g)|\, d\mu_t + \int_{\bar{A}_t} |(f \circ g_t) - (f \circ g)|\, d\mu_t \\
&\qquad + \int_{A_s} |(f \circ g) - (f \circ g_s)|\, d|\mu_t - \mu| \\
&\qquad + \int_{\bar{A}_s} |(f \circ g) - (f \circ g_s)|\, d|\mu_t - \mu| + \left|\int_{S^\infty} (f \circ g_s)\, d(\mu_t - \mu)\right| .
\end{aligned}
$$

From the above uniformity, there exists t_0 such that $\mu_t(A_t) < \varepsilon$ and $\mu(A_t) < \varepsilon$ for all $t \geqq t_0$. Hence $\int_{A_t} |f \circ g_t - f \circ g|\, d\mu_t \leqq \int_{A_t} d\mu_t < \varepsilon$ because $0 \leqq f \leqq 1$, for $t \geqq t_0$; $\int_{\bar{A}_t} |f \circ g_t - f \circ g|\, d\mu_t \leqq \int_{\bar{A}_t} \varepsilon\, d\mu_t < \varepsilon$, by uniform continuity of f.

Similar reasoning implies that the third and fourth terms are both less than ε if $s, t \geqq t_0$. Finally, we know (Lemma 2.2) that $\mu_t \to_w \mu$ in the $\mathscr{P}$-topology. By Theorem 5.1 of Billingsley (1968), $\mu_t g_s^{-1} \to_w \mu g_s^{-1}$ since g_s is $\mathscr{P}$-continuous a.s. (μ), which follows from the hypothesis and Lemma 2.6. Thus, the fifth summand goes to 0 as $t \to \infty$.

COROLLARY 5.2. *Theorem* 5.1 *holds also for the imput space* (S^∞, R_+^∞).

PROOF. Same as Theorem 5.1.

6. A limit theorem for queue-length.

THEOREM 6.1. *Let* $\{Q(t), 0 \leqq t < \infty\}$ *be the queue-length process of a* GI/G/s *queueing system with atomless interarrival and service distributions which starts at epoch* 0. *As* $t \to \infty$, $Q(t) \to_w Q_*$, *where* Q_* *is the queue-length of a stationary queueing process.*

PROOF. Using the notation of Section 3, $Q(t) = Q_0(\cdot, t)$. Starting at epoch 0 implies the input has measure μ_0. Thus the distribution of $Q(t)$ is $\mu_0 Q_0(\cdot, t)^{-1}$ which equals $\mu_t Q_t(\cdot, 0)^{-1} = \mu_t h_t^{-1}$. All the hypotheses of Corollary 5.2 are satisfied except h is not bounded. Pick any $N > 0$ and let $h_t' = \min(h_t, N)$ and $h' = \min(h, N)$, then $\mu_t(h_t')^{-1} \to_w \mu(h')^{-1}$ as $t \to \infty$ by Corollary 5.2. If $n < N$, then $\mu_t(h_t')^{-1}(\{n\}) = \mu_t h_t^{-1}(\{n\})$ and $\mu(h')^{-1}(\{n\}) = \mu h^{-1}(\{n\})$. Thus $\lim_{t\to\infty} \mu_t h_t^{-1}(\{n\}) = \mu h^{-1}(\{n\})$ for $n < N$. Since N is arbitrary we get convergence for all integers, n. Define Q_* as a random variable on the nonnegative integers with distribution μh^{-1}, which was established in Section 3 as the distribution of queue-length is a stationary process.

REMARKS 6.2. (i) In all the preceding, h could assume the value ∞, so we have actually dealt with light and heavy traffic.

(ii) There are more general queueing systems to which we cannot apply the theory developed here. For example, the h_t's associated with bulk-service queues will not necessarily be monotone increasing. The simplest bulk service situation is the single server case where 2 customers are serviced simultaneously; if a customer arrives at an empty queue he must wait for the next arrival before he can begin service.

7. Relations to the general theory of weak* convergence. There is a well-established theory of weak* convergence (weak convergence to the probabilist) of probability measures on topological spaces, particularly complete separable metric spaces X; [28] is perhaps the best reference, see also [11] and [25] (for a treatment in the context of duality and weak* convergence).

We have two brief remarks to make relating our work to the general theory. Let X be a completely regular Hausdorff space, $C_b(X)$ the bounded continuous real-valued functions on X. A set of the form $\{x \mid f(x) \geqq s\}$, $f \in C_b(X)$, is called a zero set in X. The reader is no doubt familiar with how the concept of tightness is used to obtain the existence of a weak* limit for a sequence of measures whose finite dimensional distributions converge ([3]). For a complete separable metric space X, tightness is equivalent to the formally *weaker* condition of uniform σ-additivity: $\mu_n(Z_k) \rightarrow_k 0$ uniformly in n where $Z_k \supset Z_{k+1}$ is a sequence of zero sets with void intersection. Recall Corollary 4.6: this corollary does not exactly fit the general theory of weak* convergence because of the incompatibility of the topology and σ-field. Both tightness and uniform σ-additivity imply that the weak* closure of $\{\mu_n\}$ is weak* compact ([25], [28]). Indeed, uniform σ-additivity is equivalent to weak* compact closure, in the space of bounded Baire measures on any completely regular X and available as a tool even when all compact sets in X have measure zero (e.g., $X = (S^\infty, \mathscr{S})$). This is our first point.

Our second remark is that we have not used uniform σ-additivity as tightness is ordinarily used: recall that a limiting measure μ already exists. We were instead motivated by its equivalence with weak* compactness, and this in turn viewed from the functional analytical point of view in conjunction with Dini's classical theorem for uniform convergence of a monotone sequence of functions on a compact set, ([1], page 425). Specifically, if M denotes the weak* closure of a set $\{\mu_s\}$ and M is compact, consider a monotone net $f_t \in C_b(X) \nearrow f \in C_b(X)$ and think of f, f_t as functions on M by $f_t(\mu) = \int_X f_t \, d\mu$, $\mu \in M$. Then each f_t and f is continuous on M and $f_t(\mu) \nearrow f(\mu)$ for each μ. Since M is compact the convergence is uniform. Thus, $f_t(\mu_s) = \int_X f_t \, d\mu_s \rightarrow \int_X f \, d\mu = f(\mu)$ *independently of the rates at which t and s converge*. This is the basis of the ideas resulting in 5.1, whose proof cannot be put in this simple form because of the incompatibility of continuity and measurability.

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