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THE MULTIPLE RANGE OF TWO-DIMENSIONAL RECURRENT WALK

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For each positive integer p, let $R_n{}^p$ be the number of points visited exactly p times by a random walk during the course of its first n steps. We call the random variables $R_n{}^p$ the multiple range of order p for the given walk. We prove that for two-dimensional simple walk, $R_n{}^p$ obeys the strong law of large numbers $\lim_{n\to\infty} R_n{}^p/(\pi^2 n/\log^2 n) = 1$ a.s. The method of proof generalizes to yield a similar result for all genuine two-dimensional walks with 0 mean and finite $2+\varepsilon$ moments ($\varepsilon > 0$).

1. Introduction. Let G be an infinite countable group and $\{X_n\}$, $1 \leqq n < \infty$, a sequence of independent identically distributed G valued random variables. The sequence of partial sums $\{S_n\}$, $S_n = X_1 + \cdots + X_n$, is called a random walk on G. The range of the random walk is defined to be the sequence $\{R_n\}$, R_n being the number of distinct values among $S_0 = 0, S_1, \cdots, S_n$: i.e., R_n is the number of distinct points visited by a walk starting from the origin during the course of its first n steps.

The study of the range of random walks was initiated by Dvoretzky and Erdös [1]. They have shown that for simple random walk on the d-dimensional lattice $Z^d (d \geqq z)$, R_n obeys the strong law of large numbers

$$\lim_{n\to\infty} \frac{R_n}{ER_n} = 1 \quad \text{a.s.,} \tag{1.1}$$

ER_n denoting the expectation of R_n (it is known that (1.1) fails to hold for the simple 1-dimensional walk; see [6]).

Subsequently, Kesten, Spitzer and Whitman ([8], pages 38–40) have employed the ergodic theorem to show that on any countable group G

$$\lim_{n\to\infty} \frac{R_n}{n} = e \quad \text{a.s.,} \tag{1.2}$$

where e is the escape probability $P[S_n \neq 0, 1 \leqq n < \infty]$. A random walk is called transient or recurrent according as to $e > 0$ or $e = 0$. For example, it is well known that the simple walk on Z^d is transient for $d > 2$ and recurrent for $d = 1, 2$ ([3]). Using the dominated convergence theorem, (1.2) implies $\lim_{n\to\infty} (ER_n/n) = e$. It follows that (1.2) implies (1.1) in the transient case, but that (1.1) is a stronger statement than (1.2) for the simple walk on Z^2, which is recurrent.

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It is natural to try to generalize (1.1), (1.2) to the following sequences of random variables. For each positive integer p, let R_n^p be the number of points visited exactly p times by the random walk in the first n steps. We call $\{R_n^p\}$ the multiple range of order p. Elaborating upon the method of Spitzer, Kesten and Whitman, Pitt [7] has recently shown that

$$\lim_{n\to\infty} \frac{R_n^p}{n} = e^2(1-e)^{p-1} \quad \text{a.s.}, \qquad 1 \leqq p < \infty \tag{1.3}$$

for any random walk on a countable group G.

It follows from the dominated convergence theorem that for transient random walks, (1.3) implies the strong law of large numbers

$$\lim_{n\to\infty} \frac{R_n^p}{ER_n^p} = 1 \quad \text{a.s.}, \qquad 1 \leqq p < \infty\,. \tag{1.4}$$

The question arises whether there are recurrent walks for which (1.4) remains true. This problem has been considered by Erdös and Taylor [2] for the case of simple walk on Z^2. They sketch an erroneous proof (the mistake will be pointed out in Section 2) and we provide in this paper a correct proof. For technical reasons which will become apparent later (see the remark following (3.23)), we work with the random variables $T_n^p = R_n^1 + \cdots + R_n^p$ instead of the R_n^p's. Thus T_n^p is the number of points visited at least once and at most p times in the first n steps of the walk. We show in Section 3 that for $1 \leqq p < \infty$, $ER_n^p \sim (\pi^2 n/\log^2 n)$ as $n \to \infty$. (This result should be compared with $ER_n \sim (\pi n/\log n)$, proven in [1]). This fact readily implies that (1.4) is equivalent to

$$\lim_{n\to\infty} \frac{T_n^p}{ET_n^p} = 1 \quad \text{a.s.}, \qquad 1 \leqq p < \infty\,. \tag{1.5}$$

As shown in Section 2, (1.5) can readily be deduced from the following inequality (1.6).

THEOREM 1.1. *Let $\varepsilon, \delta > 0$. For each p, $1 \leqq p < \infty$,*

$$P\left(\left|T_n^p - \frac{\pi^2 pn}{\log^2 n}\right| \geqq \frac{\varepsilon\pi^2 pn}{\log^2 n}\right) = O\left(\frac{1}{\log^\delta n}\right), \tag{1.6}$$

$$P\left(\left|R_n - \frac{\pi n}{\log n}\right| \geqq \frac{\varepsilon\pi n}{\log n}\right) = O\left(\frac{1}{\log^\delta n}\right). \tag{1.7}$$

We remark that (1.7) is derived in [1] for $0 < \delta < 2$. (A mistake occurs in the argument of that paper which is corrected in [5]). As shown in [1], knowledge of (1.7) for some $\delta > 1$ suffices to establish the strong law (1.1). We shall reproduce the argument of [1] in Section 2. It is assumed in [2] that the same argument shows that the validity of (1.6) for some $\delta > 1$ establishes the strong law (1.5) for the multiple range of two-dimensional simple random walk. This is not the case, but we prove in Section 2 that knowledge of (1.6) for some

$\delta > 3$ suffices to establish (1.5). We shall see that the need for a stronger estimate for the multiple range problem stems from the fact that, whereas R_n is monotonically increasing with n, $R_n{}^p$ and $T_n{}^p$ are not. We shall prove only (1.6), as we require it to derive (1.5), but it will be clear from the proof that similar reasoning yields (1.7).

The plan of the paper is as follows. In order that the reader get a better appreciation of the estimates of Theorem 1.1, we first show in Section 2 how they can be used to derive the strong law (1.5). In Section 3, we establish the estimates for the expectation and variance of $T_n{}^p$ which suffice to give (1.6) for $0 < \delta < 1$. In passing over to $\delta \geqq 1$, we establish separately in Sections 4 and 6 the estimates

$$(1.8)\qquad P\left(T_n{}^p \geqq \frac{\pi^2 p(1+\varepsilon)n}{\log^2 n}\right) = O\left(\frac{1}{\log^\delta n}\right),$$

$$(1.9)\qquad P\left(T_n{}^p \leqq \frac{\pi^2 p(1-\varepsilon)n}{\log^2 n}\right) = O\left(\frac{1}{\log^\delta n}\right).$$

The reason for splitting (1.6) into (1.8), (1.9) is that the two parts are proven in different manners, (1.9) being much more difficult to derive. Section 5 is devoted to some moment estimates needed for the proof of (1.9).

We restrict our discussion to simple walk in Z^2, but it will be clear from our proof that (1.4) and (1.5) hold for any genuine two-dimensional walk with mean 0 and finite $2+\varepsilon$ moments, $\varepsilon > 0$ (see the remark following the proof of Theorem 3.1 in Section 3). It seems reasonable to conjecture that (1.4) and (1.5) hold for any aperiodic recurrent walk on a countable Abelian group of rank 2, although we have not been able to prove this. We remark that the analogous statement is known to hold for R_n. (A proof in the Z^2-case is given in [6]. As pointed out in [4], the same proof works in the rank 2 case.)

2. Derivation of the strong law for $T_n{}^p$. We first reproduce the derivation of the strong law $\lim_{n\to\infty} R_n/(\pi n/\log n) = 1$ a.s., found in [1]. It is shown there that $ER_n \sim \pi n/\log n$, so that $\lim_{n\to\infty} R_n/ER_n = 1$.

Suppose that (1.7) holds for some $\delta > 1$. Let $n_k = [e^{k^\theta}]$ where θ is a number satisfying $1/\delta < \theta < 1$. For $n = n_k$, (1.7) becomes

$$(2.1)\qquad P\left(\left|R_{n_k} - \frac{\pi n_k}{\log n_k}\right| \geqq \varepsilon \frac{\pi n_k}{\log n_k}\right) = O\left(\frac{1}{k^{\theta\delta}}\right).$$

Since $\sum_{k=1}^\infty 1/k^{\theta\delta} < \infty$, we conclude from (2.1) that

$$(2.2)\qquad \sum_{k=1}^\infty P\left(\left|\frac{R_{n_k}}{\pi n_k/\log n_k} - 1\right| \geqq \varepsilon\right) < \infty .$$

We conclude from the Borel–Cantelli lemma that

$$(2.3)\qquad \limsup_{k\to\infty}\left|\frac{R_{n_k}}{\pi n_k/\log n_k} - 1\right| \leqq \varepsilon \quad \text{a.s.}$$

Since ε is arbitrary, we have

$$\lim_{k\to\infty} \frac{R_{n_k}}{\pi n_k/\log n_k} = 1 \quad \text{a.s.} \tag{2.4}$$

For given n, choose k so that $n_k \leqq n < n_{k+1}$. Thus

$$R_{n_k} \leqq R_n \leqq R_{n_{k+1}} . \tag{2.5}$$

Since $n/\log n$ increases with n, we conclude from (2.5) that

$$\frac{R_{n_k}}{\pi n_{k+1}/\log n_{k+1}} \leqq \frac{R_n}{\pi n/\log n} \leqq \frac{R_{n_{k+1}}}{\pi n_k/\log n_k} . \tag{2.6}$$

Now $n_{k+1}/\log n_{k+1} \sim n_k/\log n_k$, so that (2.6) yields the strong law

$$\lim_{n\to\infty} \frac{R_n}{\pi n/\log n} = 1 \quad \text{a.s.}$$

The above argument fails to work for $T_n{}^p$ as (2.5) does not hold for $T_n{}^p$: i.e., $T_n{}^p$ is not monotonically increasing with n. The following modification of the above argument gives the strong law for $T_n{}^p$, provided we know (1.6) to hold for some $\delta > 3$. For $\delta > 1$ and $1/\delta < \theta < 1$, we prove as above that

$$\lim_{k\to\infty} \frac{T^p_{n_k}}{\pi^2 p n_k/\log^2 n_k} = 1 \quad \text{a.s.} \tag{2.7}$$

Let $n_k \leqq n < n_{k+1}$. Then

$$|T_n{}^p - T^p_{n_k}| \leqq n_{k+1} - n_k = O[e^{(k+1)^\theta} - e^{k^\theta}] = O(e^{k^\theta} \cdot k^{\theta-1}) . \tag{2.8}$$

Now

$$e^{k^\theta} = O(n_k) , \qquad k = O((\log n_k)^{1/\theta}) . \tag{2.9}$$

Using (2.8) and (2.9), we get

$$\frac{T_n{}^p}{\pi^2 p n_k/\log^2 n_k} = \frac{T^p_{n_k}}{\pi^2 p n_k/\log^2 n_k} + O((\log n_k)^{3-1/\theta}) . \tag{2.10}$$

Choose θ so that $1/\delta < \theta < \frac{1}{3}$, which is possible when $\delta > 3$. Then $3 - 1/\theta < 0$, and we conclude from (2.7), (2.10) that

$$\lim_{k\to\infty} \frac{T_n{}^p}{\pi^2 p n_k/\log n_k} = 1 \quad \text{a.s.} \tag{2.11}$$

Since $n_k/\log^2 n_k \sim n/\log^2 n$, (2.11) yields the strong law $\lim_{n\to\infty} T_n{}^p/(\pi^2 pn/\log n) = 1$ a.s.

3. The estimates for the expectation and variance of $T_n{}^p$. We obtain estimates for $ET_n{}^p$, $\sigma^2 T_n{}^p$, which are respectively the expectation and variance of $T_n{}^p$. These estimates enable us to prove a weak law for $T_n{}^p$ (Theorem 3.4). For $p = 1$ ($p = 0$ for Theorem 3.1) the results of the present section can be found in [2]. We show here that these results hold for all p.

We first estimate r_n^p, $0 \leqq p, n < \infty$, where r_n^p $(1 \leqq n < \infty)$ is the probability that the walk return to the origin 0 exactly p times in the first n steps. For $n = 0$, it is convenient to define $r_0^0 = 1$ and $r_0^p = 0$ $(p \geqq 1)$. We denote r_n^0 as r_n.

THEOREM 3.1. *For given p, $0 \leqq p < \infty$,*

$$r_n^p = \frac{\pi}{\log n} + O\left(\frac{1}{\log^2 n}\right) \qquad \textit{as } n \to \infty . \tag{3.1}$$

PROOF. For $p = 0$, (3.1) is derived in [2], formula (2.5). We prove the result for all p by induction. Suppose (2.1) holds for p. Let f_j = probability that walk returns first to 0 upon the jth step $(1 \leqq j < \infty)$ (f_0 is defined to be 0).

We have

$$r_n^{p+1} = \sum_{j=0}^{n} f_j r_{n-j}^p = \sum_{j=0}^{[n/2]} f_j r_{n-j}^p + \sum_{j=[n/2]+1}^{n} f_j r_{n-j}^p . \tag{3.2}$$

(3.1) readily yields

$$\max_{0 \leq j \leq [n/2]} \left| r_{n-j}^p - \frac{\pi}{\log n} \right| = O\left(\frac{1}{\log^2 n}\right) . \tag{3.3}$$

Hence

$$\begin{aligned} \sum_{j=0}^{[n/2]} f_j r_{n-j}^p &= \frac{\pi}{\log n} \sum_{j=0}^{[n/2]} f_j + \sum_{j=0}^{[n/2]} f_j \left(r_{n-j}^p - \frac{\pi}{\log n}\right) \\ &= \left(\frac{\pi}{\log n} + O\left(\frac{1}{\log^2 n}\right)\right)(1 - r_{[n/2]}) . \end{aligned} \tag{3.4}$$

Using (3.1) once more, $r_{[n/2]} = O(1/\log n)$ so that (3.4) becomes

$$\sum_{j=0}^{[n/2]} f_j r_{n-j}^p = \frac{\pi}{\log n} + O\left(\frac{1}{\log^2 n}\right) . \tag{3.5}$$

(3.1) also implies $r_{[n/2]} - r_n = O(1/\log^2 n)$ so that

$$\sum_{j=[n/2]+1}^{n} f_j r_{n-j}^p \leqq \sum_{j=[n/2]+1}^{n} f_j = r_{[n/2]} - r_n = O\left(\frac{1}{\log^2 n}\right) . \tag{3.6}$$

We conclude from (3.5), (3.6) that (3.1) holds for $p + 1$.

REMARK. It can be shown that for two-dimensional walk with 0-mean and finite $(2 + \varepsilon)$ moments $(\varepsilon > 0)$, we have the estimate $r_n^p = A/\log n + O(1/\log^2 n)$, A being a positive constant depending on the walk. The results and proofs of the present paper go through verbatim for this class of random walks, the number π being replaced by A.

We require the following:

LEMMA 3.1. *The probability that the point S_n is visited at time n for the pth time $(0 \leqq n < \infty, 1 \leqq p < \infty)$ equals r_n^{p-1}.*

PROOF. We have

$P[S_n$ is visited for the pth time$]$
$= P[S_j = S_n$ for $p-1$ of the indices $0 \leqq j \leqq n-1]$
$= P[p-1$ of the sums $X_n, X_n + X_{n-1}, \cdots, X_n + \cdots + X_1$ equal $0]$
$= P[p-1$ of the sums $X_1, X_1 + X_2, \cdots, X_1 + \cdots + X_n$ equal $0]$
$= r_n^{p-1}$.

THEOREM 3.2. *For each* p, $1 \leqq p < \infty$,

$$ET_n^p \sim \frac{\pi^2 pn}{\log^2 n} \qquad \text{as } n \to \infty . \tag{3.7}$$

PROOF. Let $0 \leqq n < \infty$, $0 \leqq i \leqq n$. We define

$$Z_{ni}^p = 1, \quad \text{if the point } S_i \text{ is visited for the } p\text{th time at time } i \text{ and not visited again in the next } n-i \text{ steps}$$
$$= 0, \quad \text{otherwise.} \tag{3.8}$$

Thus

$$R_n^p = Z_{n0}^p + \cdots + Z_{nn}^p . \tag{3.9}$$

Now $EZ_{ni}^p = P$ (S_i is visited for the pth time at time i) $\cdot r_{n-i}$. We conclude from Lemma 3.1 that

$$EZ_{ni}^p = r_i^{p-1} \cdot r_{n-i} , \tag{3.10}$$

and

$$ER_n^p = \sum_{i=0}^n EZ_{ni}^p = \sum_{i=0}^n r_i^{p-1} r_{n-i} . \tag{3.11}$$

Let $A = [n/\log^3 n]$ $(n \geqq 2)$ and $\varepsilon > 0$. We conclude from (3.1) that $\exists N_\varepsilon$ such that for $n > N_\varepsilon$,

$$(1-\varepsilon)\frac{\pi}{\log n} \leqq r_i^{p-1} , \qquad r_{n-i} \leqq (1+\varepsilon)\frac{\pi}{\log n} , \qquad A \leqq i \leqq n - A . \tag{3.12}$$

Since $0 \leqq r_i^{p-1}$, $r_{n-i} \leqq 1$, (3.11) and (3.12) yield

$$(1-\varepsilon)^2 \frac{\pi^2}{\log^2 n}(n-2A) \leqq ER_n^p \leqq \frac{(1+\varepsilon)^2\pi^2(n+1)}{\log^2 n} + 2A , \qquad n > N_\varepsilon . \tag{3.13}$$

Dividing (3.13) by $\pi^2 n/\log^2 n$ and letting $n \to \infty$, we obtain

$$(1-\varepsilon)^2 \leqq \liminf_{n\to\infty} \frac{ER_n^p}{\pi^2 n/\log^2 n} \leqq \limsup_{n\to\infty} \frac{ER_n^p}{\pi^2 n/\log^2 n} \leqq (1+\varepsilon)^2 . \tag{3.14}$$

As $\varepsilon > 0$ is arbitrary, we conclude

$$ER_n^p \sim \frac{\pi^2 n}{\log^2 n} , \qquad 1 \leqq p < \infty . \tag{3.15}$$

Since $ET_n^p = ER_n^1 + \cdots + ER_n^p$, (3.7) is an immediate consequence of (3.15).

REMARK. The above derivation is based on the asymptotic formula $r_n^p \sim \pi/\log n$, which is weaker than (3.1). (3.1) may be used to prove the stronger result $ET_n^p = \pi^2 pn/\log^2 n + O(n/\log^3 n)$. We do not prove this here as it is not required in the sequel. The complete information contained in (3.1) will be required for the variance estimate given in Theorem 3.3.

THEOREM 3.3. *For each* p, $1 \leqq p \leqq \infty$,

$$\sigma^2 T_n^p = O\left(\frac{n^2 \log\log n}{\log^5 n}\right). \tag{3.16}$$

PROOF. Let $0 \leqq n < \infty$, $0 \leqq i \leqq n$. We define

$$\begin{aligned} W_{ni}^p &= 1\,, && \text{if } S_i \text{ has been visited at most } p-1 \text{ times before time } i \text{ and not visited again in the remaining } n-i \text{ steps} \\ &= 0\,, && \text{otherwise.} \end{aligned} \tag{3.17}$$

Thus

$$W_{ni}^p = Z_{ni}^1 + \cdots + Z_{ni}^p\,, \tag{3.18}$$

and

$$T_n^p = W_{n0}^p + \cdots + W_{nn}^p\,. \tag{3.19}$$

(3.10), (3.18) yield

$$EW_{ni}^p = \rho_i^{p-1} r_{n-i}\,, \tag{3.20}$$

where

$$\rho_n^{p-1} = r_n^0 + \cdots + r_n^{p-1} = \text{probability of returning at most } p-1 \text{ times to the origin in the first } n \text{ steps.} \tag{3.21}$$

It follows from (3.19), (3.20) that

$$\sigma^2 T_n^p = E((T_n^p)^2) - E^2(T_n^p) \leqq 2 \textstyle\sum_{0 \leqq i \leqq j \leqq n} (p_{ij} - \rho_i^{p-1} r_{n-i} \rho_j^{p-1} r_{n-j})\,, \tag{3.22}$$

where

$p_{ij} = E(W_{ni}^p W_{nj}^p) = P[S_i$ has been visited at most $p-1$ times before time i and is not visited again in remaining $n-i$ steps, and S_j has been visited at most $p-1$ times before time j and is not visited again in remaining $n-j$ steps$]$.

Call the latter event A_{ij}. We have

$$A_{ij} \subseteqq B_{ij} \tag{3.23}$$

where B_{ij} denotes the event: S_i has been visited at most $p-1$ times before time i and is not visited again in the $[(j-i)/2]$ steps succeeding the ith one, and the point S_j has been visited at most $p-1$ times in the $[(j-i)/2]$ steps preceding the jth one and is not visited again in the $n-j$ steps succeeding the jth one. (The inclusion $A_{ij} \subseteq B_{ij}$ becomes false if we replace the phrase "at most $p-1$ times" by "exactly $p-1$ times." This is the reason which prompts us to work with the T_n^p's instead of the R_n^p's.)

We conclude from (3.23) and Lemma 3.1 that

$$p_{ij} \leqq p(B_{ij}) = \rho_i^{p-1} r_{[(j-i)/2]} \rho_{[(j-i)/2]}^{p-1} r_{n-j} . \tag{3.24}$$

Hence

$$\sigma^2 T_n^p \leqq 2 \sum_{0 \leqq i \leqq j \leqq n} \rho_i^{p-1} r_{n-j} \{ r_{[(j-i)/2]} \rho_{[(j-i)/2]}^{p-1} - r_{n-i} \rho_j^{p-1} \} . \tag{3.25}$$

We estimate the sum $\sum$ of (3.25). Let $x_{ij} = \rho_i^{p-1} r_{n-j} | r_{[(j-i)/2]} \rho_{[(j-i)/2]}^{p-1} - r_{n-i} \rho_j^{p-1} |$; we have $x_{ij} \leqq 1$ for all i, j. Let $A = [n/(\log n)^5]$, $n \geqq 2$. Then

$$\begin{aligned} |\textstyle\sum| \leqq \sum_{0 \leqq i \leqq j \leqq n} x_{ij} &\leqq \textstyle\sum_{0 \leqq i \leqq A; i \leqq j \leqq n} 1 + \sum_{n-A \leqq j \leqq n; 0 \leqq i \leqq j} 1 \\ &+ \textstyle\sum_{0 \leqq i \leqq n; i \leqq j \leqq i+A} 1 + \sum_{A \leqq i \leqq n-A; i+A \leqq j \leqq n-A} x_{ij} . \end{aligned} \tag{3.26}$$

The first three sums on the right side of (3.26) are $O(n^2/(\log n)^5)$. The fourth sum, call it $\sum_4$, is estimated as follows. We conclude from (3.1) that

$$\max_{A/4 \leqq k \leqq n} \left| r_k - \frac{\pi}{\log k} \right| = O\left(\frac{1}{\log^2 n} \right) . \tag{3.27}$$

Write

$$r_k - \frac{\pi}{\log n} = r_k - \frac{\pi}{\log k} + \pi \left(\frac{\log n/k}{\log n \cdot \log k} \right) .$$

We have $n/k \leqq 4n/A = O(\log^5 n)$ for $k \geqq A/4$. It follows from (3.27) that

$$\max_{A/4 \leqq k \leqq n} \left| r_k - \frac{\pi}{\log n} \right| = O\left(\frac{\log \log n}{\log^2 n} \right) . \tag{3.28}$$

A similar estimate may be derived for ρ_n^{p-1}.

Thus for $A \leqq i \leqq n-A$, $i+A \leqq j \leqq n-A$,

$$\max | r_{[(j-i)/2]} \rho_{[(j-i)/2]}^{p-1} - r_{n-i} \rho_j^{p-1} | = O\left(\frac{\log \log n}{\log^3 n} \right) . \tag{3.29}$$

Since $\max_{A \leqq i \leqq n; 0 \leqq j \leqq n-A} \rho_i^{p-1} r_{n-j} = O(1/\log^2 n)$, we conclude that

$$\max_{A \leqq i \leqq n-A; i+A \leqq j \leqq n-A} x_{ij} = O\left(\frac{\log \log n}{\log^5 n} \right) .$$

Hence $\sum$, $\sum_4 = O(n^2 \log \log n / \log^5 n)$ and Theorem 3.3 then follows from (3.25).

Using Chebychev's inequality, we obtain the weak law of large numbers for T_n^p.

THEOREM 3.4. *Let $1 \leqq p < \infty$. $\exists\, C > 0$ depending only on p such that*

$$P(|T_n^p - ET_n^p| \geqq \varepsilon ET_n^p) \leqq \frac{C \log \log n}{\varepsilon^2 \log n} \tag{3.30}$$

holds for $\varepsilon > 0$ and $3 \leqq n < \infty$.

4. Derivation of inequality (1.8). We introduce the following

DEFINITION 4.1. (i) Let p be a fixed positive integer. The real number $\delta > 0$ is said to have the property $A(\delta)$ iff $\forall\, \varepsilon_0 > 0\ \exists\, C(\varepsilon_0, \delta) > 0$ such that

$$P\left(T_n^p \geqq (1+\varepsilon)\frac{\pi^2 pn}{\log^2 n}\right) \leqq \frac{C(\varepsilon_0, \delta)}{\varepsilon^2 \log^\delta n} \quad \text{for } \varepsilon \geqq \varepsilon_0 \text{ and } n \geqq 2\,. \tag{4.1}$$

(ii) δ is said to have the property $B(\delta)$ iff $\forall\, \varepsilon > 0$, $\exists\, D(\varepsilon, \delta) > 0$ such that

$$P\left(T_n^p \geqq (1+\varepsilon)\frac{\pi^2 pn}{\log^2 n}\right) \leqq \frac{D(\varepsilon, \delta)}{\log^\delta n} \quad \text{for } n \geqq 2\,. \tag{4.2}$$

We observe that $A(\delta) \Rightarrow B(\delta)$ as we may then choose $D(\varepsilon, \delta) = C(\varepsilon, \delta)/\varepsilon^2$. The following remark will be useful later on.

REMARK. δ has the property $A(\delta)$ iff $\forall\, \varepsilon_0 > 0\ \exists\, C'(\varepsilon_0, \delta), n(\varepsilon_0, \delta) > 0$ such that

$$P\left(T_n^p \geqq (1+\varepsilon)\frac{\pi^2 pn}{\log^2 n}\right) \leqq \frac{C'(\varepsilon_0, \delta)}{\varepsilon^2 \log^\delta n} \quad \text{for } \varepsilon \geqq \varepsilon_0 \text{ and } n \geqq n(\varepsilon_0, \delta)\,. \tag{4.3}$$

Clearly (4.1) $\Rightarrow$ (4.3). Suppose that (4.3) holds. For $2 \leqq n \leqq n(\varepsilon_0, \delta)$, let $C_n = \sup_{0<\varepsilon<\infty} \varepsilon^2 \log^\delta n \cdot P(T_n^p \geqq (1+\varepsilon)\pi^2 pn/\log^2 n)$. C_n is finite as $P(T_n^p \geqq (1+\varepsilon)\pi^2 pn/\log^2 n) = 0$ for ε sufficiently large. We conclude that (4.1) holds with $C(\varepsilon_0, \delta) = \max(C_2, \cdots, C_N, C'(\varepsilon_0, \delta))$ where $N = [n(\varepsilon_0, \delta)]$.

We first prove

THEOREM 4.1. *$A(\delta)$ holds for $0 < \delta < 1$.*

PROOF. We have $ET_n^p \sim \pi^2 pn/\log^2 n$ as $n \to \infty$ (Theorem 3.2) and $(1 + \varepsilon/2)/(1+\varepsilon) \leqq (1+\varepsilon_0/2)/(1+\varepsilon_0)$ for $\varepsilon \geqq \varepsilon_0$. It follows that $\exists\, n(\varepsilon_0)$ such that

$$\left(1+\frac{\varepsilon}{2}\right) ET_n^p \leqq (1+\varepsilon)\frac{\pi^2 pn}{\log^2 n} \quad \text{for } \varepsilon \geqq \varepsilon_0\,, \qquad n \geqq n(\varepsilon_0)\,. \tag{4.4}$$

We conclude from (4.4) and Theorem 3.4 that $\exists\, C > 0$ such that

$$P\left(T_n^p \geqq (1+\varepsilon)\frac{\pi^2 pn}{\log^2 n}\right) \leqq P\left(T_n^p \geqq \left(1+\frac{\varepsilon}{2}\right) ET_n^p\right) \leqq C\frac{\log\log n}{\varepsilon^2 \log n} \tag{4.5}$$

whenever $\varepsilon \geqq \varepsilon_0$, $n \geqq n(\varepsilon_0)$. The result follows from the remark following Definition 4.1.

THEOREM 4.2. *$A(\delta) \Rightarrow A(\frac{4}{3}\delta)$. It follows from Theorem 4.1 that $A(\delta)$ holds for all $\delta > 0$.*

REMARK. Inequality (1.8) states that $B(\delta)$ holds for all $\delta > 0$. Since $A(\delta) \Rightarrow B(\delta)$, Theorem 4.2 implies (1.8). The reason for proving the stronger assertion $A(\delta)$ is that we have the implication $A(\delta) \Rightarrow A(\frac{4}{3}\delta)$. The following proof applied to $B(\delta)$ would have yielded $B(\delta) \Rightarrow B(\delta')$ for $\delta \leqq \delta' < \frac{2}{3}\delta + \frac{2}{3}$ (see the remark following (4.12)) which, in conjunction with Theorem 4.1, would only have yielded $B(\delta)$ for $0 < \delta < 2$.

PROOF OF THEOREM 4.2. It is shown in [1] that $\forall \varepsilon > 0$, we have

$$P\left(R_n \geqq (1+\varepsilon)\frac{\pi n}{\log n}\right) = O\left(\frac{1}{\log^\delta n}\right) \tag{4.6}$$

whenever $0 < \delta < 2$: i.e., $B(\delta)$ holds for $0 < \delta < 2$ ($T_n{}^p$ being replaced by R_n). The following proof is just a careful analysis of the argument presented in [1], showing that it may be applied to prove the implication $A(\delta) \Rightarrow A(\frac{4}{3}\delta)$.

Let $N = [\log^{\delta/3} n]$ $(n \geqq 3)$ and $n_i = [ni/N]$, $1 \leqq i \leqq N$. We divide the points $\{S_0, \cdots, S_n\}$ into the N blocks $\{S_0, \cdots, S_{n_1}\}, \cdots, \{S_{n_{N-1}+1}, \cdots, S_n\}$ and let T^p_{ni} = number of points in ith block visited at most p times in that block, $i \leqq i \leqq N$. Thus T^p_{ni} has the same distribution as $T^p_{m_i}$ where $m_i = n_i - n_{i-1} - 1$, $1 \leqq i \leqq N$ (n_0 is defined to be -1).

We clearly have

$$T_n{}^p \leqq T^p_{n1} + \cdots + T^p_{nN} . \tag{4.7}$$

Let

$$A_i = \left[T^p_{ni} \geqq \left(1 + \frac{\varepsilon}{2}\right)\frac{\pi^2 pn/N}{\log^2 n}\right], \qquad B_i = \left[T^p_{ni} \geqq \left(1 + \frac{\varepsilon N}{2}\right)\frac{\pi^2 pn/N}{\log^2 n}\right], \qquad 1 \leqq i \leqq N .$$

Then

$$\left[T_n{}^p \geqq (1+\varepsilon)\frac{\pi^2 pn}{\log^2 n}\right] \subseteqq \bigcup_{1 \leqq i < j \leqq N} A_i A_j \cup \bigcup_{1 \leqq i \leqq N} B_i . \tag{4.8}$$

To prove (4.8), we observe that it is equivalent to saying that the occurrence of at most one A_i and no B_i implies $T_n{}^p < (1+\varepsilon)\pi^2 pn/\log^2 n$. The latter follows from (4.7), as we then have

$$\begin{aligned} T_n{}^p &\leqq \left(\frac{N-1}{N}\right)\left(1 + \frac{\varepsilon}{2}\right)\frac{\pi^2 pn}{\log^2 n} + \left(\frac{1}{N} + \frac{\varepsilon}{2}\right)\frac{\pi^2 pn}{\log^2 n} \\ &< (1+\varepsilon)\frac{\pi^2 pn}{\log^2 n} . \end{aligned} \tag{4.9}$$

We estimate $P(A_i)$ and $P(B_i)$. We have $((n/N)/(\log^2 n))/(m_i/\log^2 m_i) \to 1$ uniformly in i as $n \to \infty$, and $(1 + \varepsilon/4)/(1 + \varepsilon/2) \leqq (1 + \varepsilon_0/4)/(1 + \varepsilon_0/2) < 1$ for $\varepsilon \geqq \varepsilon_0$. Hence $\exists\, n_1(\varepsilon_0, \delta)$ such that

$$\left(1 + \frac{\varepsilon}{4}\right)\frac{m_i}{\log^2 m_i} \leqq \left(1 + \frac{\varepsilon}{2}\right)\frac{n/N}{\log^2 n}, \qquad 1 \leqq i \leqq N, \tag{4.10}$$

holds for $\varepsilon \geqq \varepsilon_0$ and $n \geqq n_1(\varepsilon_0, \delta)$.

Choose $n_2(\delta)$ so that $\log^\delta m_i \geqq \frac{1}{2}\log^\delta n$ holds for $i \leqq i \leqq N$ and $n \geqq n_2(\delta)$. It follows from (4.1), (4.10) that

$$P(A_i) \leqq P\left(T^p_{ni} \geqq \left(1 + \frac{\varepsilon}{4}\right)\frac{\pi^2 pm_i}{\log^2 m_i}\right) \leqq \frac{32C(\varepsilon_0/4, \delta)}{\varepsilon^2 \log^\delta n}, \qquad 1 \leqq i \leqq N, \tag{4.11}$$

whenever $\varepsilon \geqq \varepsilon_0$, $n \geqq n_3(\varepsilon_0, \delta) = \max(n_1(\varepsilon_0, \delta), n_2(\delta))$.

Replacing ε by εN, we conclude that

$$(4.12)\qquad P(B_i) \leqq P\left(T^p_{ni} \geqq \left(1+\frac{\varepsilon N}{4}\right)\frac{\pi^2 p m_i}{\log^2 m_i}\right) \leqq \frac{32C(\varepsilon_0/4,\delta)}{\varepsilon^2 N^2 \log^\delta n}, \qquad 1 \leqq i \leqq N,$$

whenever $\varepsilon \geqq \varepsilon_0$, $n \geqq n_3(\varepsilon_0, \delta)$.

At this point of the argument, it becomes necessary to work with the $A(\delta)$ property instead of the $B(\delta)$ property. Suppose we tried to derive $B(\delta)$. We can still employ Theorem 4.1 to estimate $P(B_i)$, but we would find that the term $\log^\delta n$ occurring in the right side of (4.12) must be replaced by $\log^{\delta_0} n$, where δ_0 is any number in (0, 1). This would only yield the implication $B(\delta) \Rightarrow B(\delta')$ for any $\delta' \in [\delta, \frac{2}{3}\delta + \frac{2}{3})$.

Since the A_i's are independent events, we conclude from (4.11), (4.12) that

$$(4.13)\qquad P[\bigcup_{1\leqq i<j\leqq N} A_iA_j \cup \bigcup_{1\leqq i\leqq N} B_i] \leqq \left[\frac{32C(\varepsilon_0/4,\delta)}{\varepsilon_0}\right]^2 \frac{N^2}{\varepsilon^2\log^{2\delta} n} + \frac{32C(\varepsilon_0/4,\delta)}{\varepsilon^2 N\log^\delta n}$$

for $\varepsilon \geqq \varepsilon_0$, $n \geqq n_3(\varepsilon_0, \delta)$.

Choose $n_4(\delta)$ so that $N \geqq \frac{1}{2}\log^{\delta/3} n$ for $n \geqq n_4(\delta)$, and let $n_5(\varepsilon, \delta) = \max(n_3(\varepsilon, \delta), n_4(\delta))$. We conclude from (4.8), (4.13) that

$$(4.14)\qquad P\left(T_n{}^p \geqq (1+\varepsilon)\frac{\pi^2 pn}{\log^2 n}\right) \leqq \frac{C(\varepsilon_0, \frac{4}{3}\delta)}{\varepsilon^2 \log^{\frac{4}{3}\delta} n}$$

for $\varepsilon \geqq \varepsilon_0$, $n \geqq n_5(\varepsilon_0, \delta)$, where

$$(4.15)\qquad C(\varepsilon_0, \tfrac{4}{3}\delta) = \left[\frac{32C(\varepsilon_0/4,\delta)}{\varepsilon_0}\right]^2 + 64C\left(\frac{\varepsilon_0}{4}, \delta\right).$$

Using the remark following Definition 4.1, we conclude from (4.14) that $A(\delta) \Rightarrow A(\frac{4}{3}\delta)$.

5. Some moment estimates. In order to prove inequality (1.9), we require the moment estimates provided by Theorems 5.1 and 5.2. The proofs of these theorems are rather long and complicated. The reader is advised to accept them on faith upon a first reading and to go on to Section 6, where inequality (1.9) is established.

We first prove several lemmas. In the sequel the symbol $\rho_n{}^\infty$, $0 \leqq n < \infty$, is used to designate 1.

LEMMA 5.1. *Let $x, y \in Z^2$, $0 \leqq p$, $q \leqq \infty$, $n = i + j + k$ where $i, j \geqq 0$ and $k > 0$. Let $P^n(x, y; i, j; p, q)$ = probability that the simple random walk on Z^2, starting from x, reaches y after n steps, the walk returning to x at most p times in the first i steps and visiting y at most q times in the j steps preceding the nth one. Then*

$$(5.1)\qquad P^n(x, y; i, j; p, q) \leqq \frac{C}{k}\rho_i{}^p\rho_j{}^q$$

where ρ_i^p is the quantity defined in (3.21) *and* C *is a positive constant independent of* x, y, i, j, k, p, q. *Furthermore*

(5.2) $$\sum_{y \in Z^2} P^n(x, y: i, j; p, q) = \rho_i^p \rho_i^q$$

for all x, i, j, k, p, q.

REMARK. In the applications of (5.1), (5.2), we only require the case $q = 0$. We then write $P^n(x, y; i, j; p)$ instead of $P^n(x, y; i, j; p, 0)$.

PROOF. Let

$P^n(x, y) = P$[walk starting at x, reaches y after n steps]

$P_p^n(x, y) = P$[walk starting at x, reaches y after n steps, the walk returning to x at most p times in those n steps]

$Q_p^n(x, y) = P$[walk starting at x, reaches y after n steps, there being at most p visits to y prior to time n].

We have

(5.3) $$P^n(x, y; i, j; p, q) = \sum_{z, w \in Z^2} P_p^i(x, z) P^k(z, w) Q_q^j(w, y) .$$

Imitating the proof of Lemma 3.1, one readily shows that

(5.4) $$Q_q^j(w, y) = P_q^j(w, y) .$$

In view of (5.4), (5.3) becomes

(5.5) $$P^n(x, y; i, j; p, q) = \sum_{z, w \in Z^2} P_p^i(x, z) P^k(z, w) P_q^j(w, y) .$$

Now

(5,6) $$P^k(z, w) \leqq \frac{C}{k}$$

where $C > 0$ is independent of k, z, w ([8], page 72).

We conclude from (5.5), (5.6) that

(5.7) $$\begin{aligned} P^n(x, y; i, j; p, q) &\leqq \frac{C}{k} \sum_{z, w \in Z^2} P_p^i(x, z) P_q^j(w, y) \\ &= \frac{C}{k} \sum_{z \in Z^2} P_p^i(x, z) \cdot \sum_{w \in Z^2} P_q^j(w, y) \\ &= \frac{C}{k} \sum_{z \in Z^2} P_p^i(0, z - x) \cdot \sum_{w \in Z^2} P_q^j(o, y - w) \\ &= \frac{C}{k} \sum_{z \in Z^2} P_p^i(0, z) \cdot \sum_{w \in Z^2} P_q^j(0, w) = \frac{C}{k} \rho_i^p \rho_j^q , \end{aligned}$$

thus proving (5.1).

Using (5.5), we get

(5.8) $$\begin{aligned} \sum_{y \in Z^2} P^n(x, y; i, j; p, q) &= \sum_{z, w \in Z^2} P_p^i(x, z) P^k(z, w) \sum_{y \in Z^2} P_q^j(w, y) \\ &= \sum_{z \in Z^2} P_p^i(x, z) \cdot \sum_{w \in Z^2} P^k(z, w) \cdot \rho_j^q \\ &= \sum_{z \in Z^2} P_p^i(x, z) \cdot \rho_j^q = \rho_i^p \rho_j^q , \end{aligned}$$

thus proving (5.2).

In the following two lemmas log 0, log 1 are interpreted to be 1, log n retaining its usual meaning for $n > 1$.

LEMMA 5.2. *Let* $\gamma_1, \cdots, \gamma_j \geqq 0$ $(j \geqq 2)$. *Then*

$$\sum_{i_1+\cdots+i_j=n;i_1,\cdots,i_j\geqq 0} \frac{1}{\log^{\gamma_1} i_1 \cdots \log^{\gamma_j} i_j} = O\left(\frac{n^{j-1}}{\log^{\gamma} n}\right) \tag{5.9}$$

where $\gamma = \gamma_1 + \cdots + \gamma_j$.

PROOF. Let $\sum$ denote the sum of (5.9). Let $A = [n^{\frac{1}{2}}]$. The sum obtained from $\sum$ by retaining only those terms for which $i_k \leqq A$ is denoted by $\sum_{i_k \leqq A}$. A similar meaning is attached to $\sum_{i_1,\cdots,i_j \geqq A}$. We have

$$\sum \leqq \sum_{i_1 \leqq A} + \cdots + \sum_{i_j \leqq A} + \sum_{i_1,\cdots,i_j \geqq A} . \tag{5.10}$$

The terms occurring in $\sum$ are uniformly bounded for $0 \leqq n < \infty$. It is readily seen that the number of terms in $\sum$ is $O(n^{j-1})$, while the number of terms in each $\sum_{i_k \leqq A}$ is $O(n^{j-\frac{3}{2}})$. Hence

$$\sum_{i_k \leqq A} = O(n^{j-\frac{3}{2}}) , \quad 1 \leqq k \leqq j ; \qquad \sum_{i_1,\cdots,i_j \geqq A} = O\left(\frac{n^{j-1}}{\log^{\gamma} n}\right) \tag{5.11}$$

(5.10) and (5.11) yield (5.9).

LEMMA 5.3. *Let* $\alpha_1, \cdots, \alpha_j, \beta_1, \cdots, \beta_j \geqq 0$, $0 \leqq \gamma_1, \cdots, \gamma_{j-1} \leqq 1$, *where* $j \geqq 2$. *Then*

$$\begin{aligned}\sum_{1\leqq \nu_1,\cdots,\nu_j \leqq n} 1/(&\log^{\alpha_1} \nu_1 \cdots \log^{\alpha_j} \nu_j \log^{\beta_1}(n - \nu_1) \cdots \\ &\log^{\beta_j}(n - \nu_j)(n - \nu_1 + \nu_2)^{\gamma_1} \cdots (n - \nu_{j-1} + \nu_j)^{\gamma_{j-1}}) \\ &= O\left(\frac{n^{j-\gamma}}{\log^{\alpha+\beta} n}\right)\end{aligned} \tag{5.12}$$

where $\alpha = \sum_{i=1}^{j} \alpha_i$, $\beta = \sum_{i=1}^{j} \beta_i$, $\gamma = \sum_{i=1}^{j-1} \gamma_i$.

PROOF. We designate the sum of (5.12) by $\sum = \sum (n; \alpha_1, \cdots, \alpha_j; \beta_1, \cdots, \beta_j; \gamma_1, \cdots, \gamma_{j-1})$ and the sum $\sum_{1 \leqq \nu \leqq n} 1/\log^{\alpha} \nu \log^{\beta}(n - \nu)$ by $\sum (n; \alpha; \beta)$. Consider first the case $j = 2$. Let $k = n - \nu_1 + \nu_2$ so that $1 \leqq k \leqq 2n - 1$.
Let $\sum = \sum_{k \leqq n} + \sum_{k > n}$. We have

$$\begin{aligned}\sum_{k\leqq n} &= \sum_{k=1}^{n} \frac{1}{k^{\gamma}} \sum_{\nu_2=1}^{k} \frac{1}{\log^{\alpha_1}(n-k+\nu_2) \log^{\alpha_2} \nu_2 \log^{\beta_1}(k-\nu_2) \log^{\beta_2}(n-\nu_2)} \\ &\leqq \sum_{k=1}^{n} \frac{1}{k^{\gamma}} \cdot \sum (k; \alpha; \beta) .\end{aligned} \tag{5.13}$$

Lemma 5.2 yields

$$\sum (k; \alpha; \beta) = O\left(\frac{k}{\log^{\alpha+\beta} k}\right) \tag{5.14}$$

so that

$$\begin{aligned}\sum_{k \leqq n} &= \sum_{k=1}^{n} O\left(\frac{k^{1-\gamma}}{\log^{\alpha+\beta} k}\right) = O(n^{1-\gamma}) \cdot \sum_{k=1}^{n} \frac{1}{\log^{\alpha+\beta} k} \\ &= O\left(\frac{n^{2-\gamma}}{\log^{\alpha+\beta} n}\right).\end{aligned} \tag{5.15}$$

Using Lemma 5.2 again, we have

$$(5.16)\qquad \sum_{k>n} \leqq \frac{1}{n^{\gamma}} \sum (n; \alpha_1; \beta_1) \cdot \sum (n; \alpha_2; \beta_2)$$
$$= \frac{1}{n^{\gamma}} O\left(\frac{n}{\log^{\alpha_1+\beta_1} n}\right) O\left(\frac{n}{\log^{\alpha_2+\beta_2} n}\right) = O\left(\frac{n^{2-\gamma}}{\log^{\alpha+\beta} n}\right).$$

We have thus proved the lemma for $j = 2$. Suppose that the result holds for $j-1$; we show that it holds for j. Decompose $\sum = \sum (n; \alpha_1, \cdots, \alpha_j, \beta_1, \cdots, \beta_j; \gamma_1, \cdots, \gamma_{j-1})$ as $\sum = \sum_1 + \sum_2$, $\sum_1$ extending over those indices for which $\nu_2 \leqq n/2$, $\sum_2$ over those indices for which $\nu_2 > n/2$. If $\nu_2 \leqq n/2$, then $n - \nu_2 + \nu_3 \geqq n/2$. Hence

$$(5.17)\qquad \sum_1 \leqq \frac{2^{\gamma_2}}{n^{\gamma_2}} \sum (n; \alpha_1, \alpha_2; \beta_1, \beta_2; \gamma_1)$$
$$\times \sum (n; \alpha_3, \cdots, \alpha_j; \beta_3, \cdots, \beta_j; \gamma_3, \cdots, \gamma_{j-1}).$$

We conclude from (5.17) that

$$(5.18)\qquad \sum_1 = O\left(\frac{1}{n^{\gamma_2}}\right) \cdot O\left(\frac{n^{2-\gamma_1}}{(\log n)^{\alpha_1+\alpha_2+\beta_1+\beta_2}}\right) \cdot O\left(\frac{n^{j-2-(\gamma_3+\cdots+\gamma_{j-1})}}{(\log n)^{\alpha_3+\cdots+\alpha_j+\beta_3+\cdots+\beta_j}}\right)$$
$$= O\left(\frac{n^{j-\gamma}}{\log^{\alpha+\beta} n}\right).$$

If $\nu_2 > n/2$, then $n - \nu_1 + \nu_2 > n/2$. Hence

$$(5.19)\qquad \sum_2 \leqq \frac{2^{\gamma_1}}{n^{\gamma_1}} \sum (n; \alpha_1; \beta_1) \cdot \sum (n; \alpha_2, \cdots, \alpha_j; \beta_2, \cdots, \beta_j; \gamma_2, \cdots, \gamma_{j-1}).$$

We conclude as before that $\sum_2 = O(n^{j-\gamma}/\log^{\alpha+\beta} n)$.

Let L be an integer $\geqq 2$. The set of points $\{S_{(i-1)L}, \cdots, S_{iL-1}\}$, $1 \leqq i < \infty$, is referred to as the ith block of the random walk and $S_{(i-1)L+j-1}$, $1 \leqq j \leqq L$, as the jth point of the ith block. We prove

THEOREM 5.1. *Let p be a given positive integer. For distinct positive integers i, j, let T_{ij} be the number of points in the* ith *block visited at most p times in that block which also appear in the* jth *block. Then for any $2m$ distinct positive integers $i_1, i_2, \cdots, i_{2m}$,*

$$(5.20)\qquad E(T_{i_1 i_2} \cdot T_{i_3 i_4} \cdots T_{i_{2m-1} i_{2m}}) \leqq \frac{CL^m}{\log^{3m} L},$$

C being a positive constant depending only on m and p (i.e. *C is independent of L and the choice of $2m$-tuples $i_1, i_2, \cdots, i_{2m}$*).

REMARK. For $m = 1$, (5.20) becomes $E(T_{i_1, i_2}) \leqq CL/\log^3 L$. If $i_1, i_2 < i_3, i_4 < \cdots < i_{2m-1}, i_{2m}$ then the random variables $T_{i_1 i_2}, \cdots, T_{i_{2m-1} i_{2m}}$ are independent and (5.20) would be a trivial consequence of the special case $m = 1$. In general, though, the random variables $T_{i_1 i_2}, \cdots, T_{i_{2m-1} i_{2m}}$ are not independent. Nevertheless, we obtain the same estimate as in the independent case.

PROOF OF THEOREM 5.1. We have

$$T_{ij} = \sum_{1 \leqq \mu, \mu' \leqq L; x \in Z^2} Z(i, j; \mu, \mu'; x) \tag{5.21}$$

where $Z(i, j; \mu, \mu'; x)$ is the indicator function of the event

$A(i, j; \mu, \mu'; x) =$ [μth point of ith block is visited for first time in that block and revisited at most $p-1$ times in that block; μ'th point of jth block is visited for first time in that block; μth point of ith block $=$ μ'th point of jth block $= x$] .

It follows that

$$\begin{aligned} &E(T_{i_1 i_2} \cdots T_{i_{2m-1} i_{2m}}) \\ &\quad = \sum_{1 \leqq \mu_1, \cdots, \mu_{2m} \leqq L; x_1, \cdots, x_m \in Z^2} P(A(i_1, i_2; \mu_1, \mu_2; x_1) \cap \cdots \\ &\qquad \cap A(i_{2m-1}, i_{2m}; \mu_{2m-1}, \mu_{2m}; x_m)) . \end{aligned} \tag{5.22}$$

We express the event on the right side of (5.22) as intersections of independent events. Arrange $i_1, \cdots, i_{2m}$ in increasing order as $j_1 < \cdots < j_{2m}$. Thus $j_k = i_{\sigma(k)}$, $1 \leqq k \leqq 2m$, where $\sigma(k)$ is some permutation of $1, \cdots, 2m$. Let $\nu_k = \mu_{\sigma(k)}$, $y_k = x_{[(\sigma(k)+1)/2]}$,

$$\begin{aligned} \varepsilon_k &= p - 1 && \text{if } \sigma(k) \text{ is odd} \\ &= +\infty && \text{if } \sigma(k) \text{ is even,} \qquad 1 \leqq k \leqq 2m . \end{aligned}$$

Then

$$\begin{aligned} &A(i_1, i_2; \mu_1, \mu_2; x_1) \cap \cdots \cap A(i_{2m-1}, i_{2m}; \mu_{2m-1}, \mu_{2m}; x_m) \\ &\quad \subseteqq B(\nu_1; y_1) \cap B(1; \nu_1, \nu_2; y_1, y_2) \cap \cdots \\ &\qquad \cap B(2m-1; \nu_{2m-1}, \nu_{2m}; y_{2m-1}, y_{2m}) \cap B'(\nu_{2m}; y_{2m}) \end{aligned} \tag{5.23}$$

where

$B(\nu, y) =$ [νth point of j_1th block is visited for first time in that block and equals y]

$B(k; \nu, \nu'; y, y') =$ [νth point of j_kth block $= y$ and is revisited at most ε_k times in the next $[(L-\nu)/2]$ steps; ν'th point of j_{k+1}th block $= y'$ and is not visited on prior $[(\nu'-1)/2]$ steps]

$B'(\nu, y) =$ [νth point of j_{2m}th block $= y$ and is revisited at most ε_{2m} times in that block] .

We refer to the above events as the B-events. Since they are independent, we conclude from (5.23) that

$$\begin{aligned} &P(A(i_2, i_2; \mu_1, \mu_2; x_1) \cap \cdots \cap A(i_{2m-1}, i_{2m}; \mu_{2m-1}, \mu_{2m}; x_m)) \\ &\quad \leqq P(B(\nu_1, y_1)) \cdot P(B(1; \nu_1, \nu_2; y_1, y_2)) \cdots \\ &\qquad P(B(2m-1; \nu_{2m-1}, \nu_{2m}; y_{2m-1}, y_{2m})) \cdot P(B'(\nu_{2m}, y_{2m})) . \end{aligned} \tag{5.24}$$

Using the notation of Lemma 5.1 (see the remark following the lemma), we have

$$P(B(k;\nu,\nu';y,y')=P^{s}\left(y,y';\left[\frac{L-\nu}{2}\right],\left[\frac{\nu'-1}{2}\right];\varepsilon_k\right) \tag{5.25}$$

where

$$s=(j_{k+1}-j_k)L+\nu'-\nu\,.$$

We estimate the probabilities of the B events. We use the letter C in the sequel to denote various positive constants which do not depent on any of the appearing indices. Let $d=s-[(L-\nu)/2]-[(\nu'-1)/2]$. It follows from (3.1) and (5.1) that

$$P(B(k;\nu,\nu';y,y'))\leqq\frac{C}{d}\frac{1}{\log\nu'\cdot\log^{\beta_k}(L-\nu)}\,,\qquad 1\leqq k\leqq 2m-1\,, \tag{5.26}$$

where

$$\begin{aligned}\beta_k&=1\,,\qquad \text{if}\quad \varepsilon_k=p-1\\ &=0\,,\qquad \text{if}\quad \varepsilon_k=+\infty\,.\end{aligned}$$

We have

$$d\geqq L+\nu'-\nu-\left(\frac{L-\nu}{2}\right)-\left(\frac{\nu'-1}{2}\right)\geqq\frac{L+\nu'-\nu}{2}\,. \tag{5.27}$$

We conclude from (5.26), (5.27) that

$$P(B(k;\nu,\nu';y,y'))\leqq\frac{C}{\log\nu'\cdot\log^{\beta_k}(L-\nu)\cdot(L+\nu'-\nu)}\,,\qquad 1\leqq k\leqq 2m-1\,. \tag{5.28}$$

We also obtain from (3.1),

$$P(B'(\nu,y))=O\left(\frac{1}{\log^{\beta_{2m}}(L-\nu)}\right)\,. \tag{5.29}$$

We use the estimates (5.28), (5.29) to derive (5.20). Choose from each of the m pairs $(\sigma^{-1}(1),\sigma^{-1}(2)),\cdots,(\sigma^{-1}(2m-1),\sigma^{-1}(2m))$ the smaller number, and write these in increasing order as $1=k_1<\cdots<k_m$. Similarly, let $l_1<\cdots<l_m=2m$ be the bigger number of these pairs. $y_{k_1}\cdots y_{k_m}$ are a rearrangement of $x_1,\cdots,x_m$, and so vary independently over Z^2. We use (5.22), (5.23), applying (5.28) to the terms $P(B(l_r-1;\nu_{l_r-1},\nu_{l_r};y_{l_r-1},y_{l_r}))$, $1\leqq r\leqq m$, and (5.29) to $P(B'(\nu_{2m},y_{2m}))$. We obtain

$$\begin{aligned}&E(T_{i_1i_2}\cdots T_{i_{2m-1}i_{2m}})\\ &\quad\leqq C\cdot\textstyle\sum_{1\leq\nu_1,\cdots,\nu_{2m}\leq L;\,y_{k_1},\cdots,y_{k_m}\in Z^2}\\ &\quad\quad\times\prod_{t=1}^{m}\frac{1}{\log\nu_{l_t}\log^{\beta_{l_t-1}}(L-\nu_{l_t-1})(L-\nu_{l_t-1}+\nu_{l_t})}\cdot\frac{1}{\log^{\beta_{2m}}(L-\nu_{2m})}\\ &\quad\quad\times P(B(\nu_1,y_1))\prod_{t=2}^{m}P^{s_{k_t-1}}\Big(y_{k_t-1},y_{k_t};\\ &\quad\quad\quad\left[\frac{L-\nu_{k_t-1}}{2}\right],\left[\frac{\nu_{k_t-1}}{2}\right],\varepsilon_{k_t-1}\Big)\end{aligned} \tag{5.30}$$

where $s_k=(j_{k+1}-j_k)L+\nu_{k+1}-\nu_k$.

To estimate the sum in the right side of (5.30), we sum first over $y_{k_m}, \cdots, y_{k_1}$, in that respective order. Using formula (5.2), we see that summation over y_{k_m} has the effect of replacing the term

$$P^{s_{k_m-1}}\left(y_{k_m-1}, y_{k_m}; \left[\frac{L-\nu_{k_m-1}}{2}\right], \left[\frac{\nu_{k_m-1}}{2}\right], \varepsilon_{k_m-1}\right)$$

by

$$\rho^{\varepsilon_{k_m-1}}_{[(L-\nu_{k_m-1})/2]}\rho_{[(\nu_{k_m-1})/2]}$$

which, by Theorem 3.1, is $\leqq C/\log \nu_{k_m} \log^{\beta_{k_m-1}}(L-\nu_{k_m-1})$. Repeating this procedure for $y_{k_{m-1}}, \cdots, y_{k_1}$, (5.30) yields

$$\begin{aligned} E(T_{i_1i_2} \cdots T_{i_{2m-1}i_{2m}}) & \\ \leqq C \textstyle\sum_{1\leqq\nu_1,\cdots,\nu_{2m}\leqq L} \prod_{t=1}^m & \frac{1}{\log \nu_{l_t} \log^{\beta_{l_t-1}}(L-\nu_{l_t-1})(L-\nu_{l_t-1}+\nu_{l_t})} \\ \times & \frac{1}{\log^{\beta_{2m}}(L-\nu_{2m})} \cdot \frac{1}{\log \nu_1} \prod_{r=2}^m \frac{1}{\log \nu_{k_r} \log^{\beta_{k_r-1}}(L-\nu_{k_r-1})} \\ = C \textstyle\sum_{1\leqq\nu_1,\cdots,\nu_{2m}\leqq L} & 1/(\log \nu_1 \cdots \log \nu_{2m} \cdot \log^{\beta_1}(L-\nu_1) \cdots \\ & \log^{\beta_{2m}}(L-\nu_{2m})(L-\nu_{l_1-1}+\nu_{l_1}) \cdots (L-\nu_{l_m-1}+\nu_{l_m})) . \end{aligned} \tag{5.31}$$

We apply Lemma 5.3 to the latter sum. We have $\alpha = 2m$, $\beta = m$ (as m of the β_i's $= 1$ and m of the β_i's $= 0$), $\gamma = m$. It follows that for given $l_1, \cdots, l_m$,

$$E(T_{i_1i_2} \cdots T_{i_{2m-1}i_{2m}}) \leqq C \frac{L^m}{\log^{3m} L} . \tag{5.32}$$

(5.32) is equivalent to (5.20) as there are only a finite number of ways of choosing $l_1, \cdots, l_m$ from $1, \cdots, 2m$.

THEOREM 5.2. *Let p be a given positive integer. Let $i_0 < i_1 < \cdots < i_m$ or $i_m < \cdots < i_1 < i_0$, the i_j's being nonnegative integers. Then*

$$E(T_{i_1i_0} \cdots T_{i_mi_0}) \leqq C \frac{L^m}{\log^{3m} L} , \tag{5.32}$$

C being a positive constant depending only on m and p.

PROOF. Suppose that $i_m < \cdots < i_1 < i_0$. A standard reversal argument shows that

$$E(T_{i_1i_0} \cdots T_{i_mi_0}) = E(T_{1+i_0-i_1,1} \cdots T_{1+i_0-i_m,1}) .$$

It suffices therefore to consider the case $i_0 < i_1 < \cdots < i_m$. Furthermore, we assume without loss of generality that $i_0 = 1$.

We have

$$\begin{aligned} E(T_{i_1i_0} \cdots T_{i_mi_0}) & \\ = \textstyle\sum_{1\leqq\mu_1,\cdots,\mu_m\leqq L; 1\leqq\nu_1,\cdots,\nu_m\leqq L; x_1,\cdots,x_m\in Z^2} & P(A(i_1, i_0; \mu_1, \nu_1; x_1) \cap \cdots \\ & \cap A(i_m, i_0; \mu_m, \nu_m; x_m)) . \end{aligned} \tag{5.33}$$

Let $\lambda_1 \leqq \lambda_2 \leqq \cdots \leqq \lambda_m$ be the ν_i's written in increasing order and let $\lambda_0 = 0$.

We have

$$(5.34)\qquad A(i_1, i_0; \mu_1, \nu_1; x_1) \cap \cdots \cap A(i_m, i_0; \mu_m, \nu_m, x_m) \subseteqq B(\lambda_1, \cdots \lambda_m; \mu_1, \cdots, \mu_m; x_1, \cdots, x_m)$$

where

$B(\lambda_1, \cdots, \lambda_m; \mu_1, \cdots, \mu_m; x_1, \cdots, x_m)$
$= [\lambda_j$th point in first block has not been visited in prior $\lambda_j - \lambda_{j-1}$ steps, $1 \leqq j \leqq m$. μ_jth point of the i_jth block is visited for the first time in that block and revisited $\leqq p - 1$ times in that block, $1 \leqq j \leqq m$; μ_1th point of i_1th block, $\cdots$, μ_mth point of i_mth block form some permutation of λ_1th, $\cdots$, λ_mth points of first block, and μ_jth point of i_jth block $= x_j$, $1 \leqq j \leqq m]$.

Mimicking the proof of Theorem 5.1, we obtain

$$(5.35)\qquad \begin{aligned} &P(B(\lambda_1, \cdots, \lambda_m; \mu_1, \cdots, \mu_m; x_1, \cdots, x_m)) \\ &\leqq C \sum_\sigma P_0^{\lambda}(0, X_{\sigma(1)}) P_{x_{\sigma(1)}}^{\lambda_2-\lambda_1}(x_{\sigma(1)}, x_{\sigma(2)}) \cdots P_{x_{\sigma(m)}}^{\lambda_m-\lambda_{m-1}}(x_{\sigma(m-1)}, x_{\sigma(m)}) \\ &\quad \times \frac{1}{\log \mu_1 \cdots \log \mu_m} \cdot \frac{1}{\log (L - \mu_1) \cdots \log (L - \mu_m)} \\ &\quad \times \frac{1}{(L - \lambda_m + \mu_1) \cdots (L - \mu_{m-1} + \mu_m)}, \end{aligned}$$

the above summation extending over all permutations of $1, \cdots, m$.

Sum up both sides of (5.35), first over $x_1, \cdots, x_m$, then over $\lambda_1, \cdots, \lambda_{m-1}$, fixing $\lambda_m, \mu_1, \cdots, \mu_m$. It follows from Theorem 3.1 and Lemma 5.2 that

$$(5.36)\qquad \begin{aligned} &\sum_{0 \leqq \lambda_1 \leqq \cdots \leqq \lambda_{m-1} \leqq \lambda_m; x_1, \cdots, x_m \in Z^2} P(B(\lambda_1, \cdots, \lambda_m; \mu_1, \cdots, \mu_m; x_1, \cdots, x_m)) \\ &\leqq C \frac{L^{m-1}}{\log^m \lambda_m} \frac{1}{\log \mu_1 \cdots \log \mu_m} \cdot \frac{1}{\log (L - \mu_1) \cdots \log (L - \mu_m)} \\ &\quad \times \frac{1}{(L - \lambda_m + \mu_1)(L - \mu_1 + \mu_2) \cdots (L - \mu_{m-1} + \mu_m)}. \end{aligned}$$

Sum both sides of (5.36) over the remaining indices $\lambda_m, \mu_1, \cdots, \mu_m$. We conclude from (5.33), (5.34), and Lemma 5.3, that

$$(5.37)\qquad E(T_{i_1 i_0} \cdots T_{i_{2m-1} i_0}) \leqq C L^{m-1} \frac{L^{m+1-m}}{\log^{3m} L} = C \frac{L^m}{\log^{3m} L}.$$

6. Derivation of inequality (1.9). Let $N = [\log \log n]$ $(n \geqq 3)$ and $L = [n/N]$. We divide the points of the random walk into blocks of size L as $\{S_0, \cdots, S_{L-1}\}$, $\{S_L, \cdots, S_{2L-1}\}$, etc.

Fix L and p and let T_i = number of points in ith block visited at most p times in that block, T_{ij} = number of points in ith block visited at most p times in that block which also appear in jth block, $i \neq j$. A direct counting argument gives

$$T_n^p \geqq \sum_{i \in \mathscr{T}} T_i - \sum_{i \in \mathscr{T}, 1 \leqq j \leqq N+1; i \neq j} T_{ij} \tag{6.1}$$

where $\mathscr{T}$ is any subset of the indices $1, \cdots, N$.

Let $0 < \varepsilon, \eta$ and let

$$A_i = \left[T_i \leqq \frac{\pi^2 p(1 - \varepsilon/2)(n/N)}{\log^2 n} \right], \qquad 1 \leqq i \leqq N,$$

$$B_{ij} = \left[T_{ij} \geqq \frac{n}{\log^{2+\eta} n} \right], \qquad 1 \leqq i, j \leqq N+1 \text{ and } i \neq j.$$

Let m be any positive integer. We define the events $\mathscr{E}_1, \mathscr{E}_2, \mathscr{E}_3$ as follows:

$\mathscr{E}_1$: A_i occurs for m i's which are $\leqq N+1$;

$\mathscr{E}_2$: some $B_{i_1 j_1} \cap \cdots \cap B_{i_m j_m}$ occurs, with the indices $i_1, j_1, \cdots, i_m, j_m$ all distinct and $\leqq N+1$;

$\mathscr{E}_3$: some $B_{i_1 i_0} \cap \cdots \cap B_{i_m i_0}$ occurs, where either $i_0 < i_1 < \cdots < i_m$ or $i_1 < \cdots < i_m < i_0$, all indices $\leqq N+1$.

We show that

$$\left[T_n^p \leqq \frac{\pi^2 p(1 - \varepsilon)u}{\log^2 n} \right] \subseteqq \mathscr{E}_1 \cup \mathscr{E}_2 \cup \mathscr{E}_3 \tag{6.2}$$

for n sufficiently large.

(6.2) is equivalent to $\mathscr{E}_1' \cap \mathscr{E}_2' \cap \mathscr{E}_3' \subseteqq [T_n^p > \pi^2 p(1-\varepsilon)n/\log^2 n]$ ($\mathscr{E}_i'$ denotes the complement of $\mathscr{E}_i$). Let $\omega \in \mathscr{E}_1' \cap \mathscr{E}_2' \cap \mathscr{E}_3'$. Choose $\{B_{i_1 j_1}, \cdots, B_{i_r j_r}\}$ with biggest possible r, so that $i_1, j_1, \cdots, i_r, j_r$ are distinct indices $\leqq N+1$ for which $\omega \in B_{i_1 j_1} \cap \cdots \cap B_{i_r j_r}$. Since $\omega \in \mathscr{E}_2'$, we have $r < m$. Let I_1 be the set of indices $i_1, j_1, \cdots, i_r, j_r$, so that its cardinality $|I_1| < 2m$. Let I_2 be the set of indices $i \leqq N+1$ such that $\omega \in B_{ij}$ for some $j \in I_1$. Since $\omega \in \mathscr{E}_3'$, the number of i's for which $\omega \in B_{ij}$, with fixed j, is $< 2m$. Hence $|I_2| < 4m^2$. Let I_3 be the set of indices $i \leqq N+1$ for which $\omega \in A_i$. Since $\omega \in \mathscr{E}_1'$, $|I_3| < m$. Let $I = I_1 \cup I_2 \cup I_3$ and $I' = \{1, \cdots, N\} - I$. Thus $|I'| \geqq N - M$ where $M = 4m^2 + 3m$. Let $i \in I'$. Since $i \notin I_3$, $\omega \notin A_i$, so that $T_i > \pi^2 p(1 - \varepsilon/2)(n/N)/\log^2 n$. Furthermore $\omega \notin B_{ij}$ for $j \neq i$ and $j \leqq N+1$, so that $T_{ij} < n/\log^{2+\eta} n$. For suppose $\omega \in B_{ij}$ for some $j \leqq N+1$, $j \neq i$. If $j \in I_1$, then $i \in I_2$, contradicting that $i \in I'$. If $j \notin I_1$, then $i, j \notin I_1$. Thus $\omega \in B_{ij} \cap B_{i_1 j_1} \cap \cdots \cap B_{i_r j_r}$ where $i, j, i_1, j_1, \cdots, i_r, j_r$ are all distinct, contradicting the maximality of r. We conclude from (6.1), with $\mathscr{T}$ set equal to I', that for given $\varepsilon > 0$

$$T_n^p \geqq \left(\frac{N-M}{N} \right) \frac{\pi^2 p(1 - \varepsilon/2)n}{\log^2 n} - \frac{(N+1)^2 n}{\log^{2+\eta} n} > \frac{\pi^2 p(1-\varepsilon)n}{\log^2 n} \tag{6.3}$$

holds for $n > n_0(\varepsilon)$, thus proving (6.2).

We now estimate $P(\mathscr{E}_i)$, $1 \leqq i \leqq 3$. It follows from Theorem 3.4 that

$$P(A_i) = O\left(\frac{1}{\log^{\frac{1}{2}} n} \right), \qquad \text{uniformly for } 1 \leqq i \leqq N+1. \tag{6.4}$$

Since the A_i's are independent, (6.4) implies

$$P(\mathscr{E}_1) = O\left(\frac{N^m}{\log^{m/2} n}\right). \tag{6.5}$$

Using Theorem 5.1 and setting $\eta = \frac{1}{2}$, we have

$$P(B_{i_1 i_2} \cap \cdots \cap B_{i_{2m-1} i_{2m}}) = O\left(\frac{L^m}{\log^{3m} L}\right)\left(\frac{\log^{\frac{5}{2}} n}{n}\right)^m = O\left(\frac{1}{\log^{m/2} n}\right), \tag{6.6}$$

uniformly in all $2m$-tuples $i_1, \cdots, i_{2m}$.

Hence

$$P(\mathscr{E}_2) = O\left(\frac{N^{2m}}{\log^{m/2} n}\right). \tag{6.7}$$

Using Theorem 5.2 and setting $\eta = \frac{1}{2}$, we have

$$P(B_{i_1 i_0} \cap \cdots \cap B_{i_m i_0}) = O\left(\frac{1}{\log^{m/2} n}\right) \tag{6.8}$$

uniformly in all $(m+1)$-tuples $i_0, i_1, \cdots, i_m$.

Hence

$$P(\mathscr{E}_3) = O\left(\frac{N^{m+1}}{\log^{m/2} n}\right). \tag{6.9}$$

We conclude from (6.2), (6.5), (6.7), (6.9) that

$$P\left(T_n \leqq \frac{\pi^2 p(1-\varepsilon)n}{\log^2 n}\right) = O\left(\frac{N^{2m}}{\log^{m/2} n}\right) = O\left(\frac{1}{\log^{m/4} n}\right). \tag{6.10}$$

Since m can be chosen arbitrarily large, (6.10) is identical with the desired inequality (1.9).

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