The Annals of Probability
1976, Vol. 4, No. 2, 175–194

ON A CLASS OF SET-VALUED MARKOV PROCESSES[1]

By T. E. Harris

The University of Southern California

Let Z be a finite or countable set, Ξ the set of subsets of Z, $\{\xi_t\}$ a Markov process with state space Ξ. A process $\{\xi_t^*\}$ with the same state space is called associate to $\{\xi_t\}$ if $P_\xi\{\xi_t \cap \eta \neq \varnothing\} = P_\eta^*\{\xi_t^* \cap \xi \neq \varnothing\}$ whenever ξ and η are subsets of Z, at least one of which is finite. Criteria are found for the existence of a process associate to a given one. Examples and applications are given.

1. Introduction. We treat Markov processes $\{\xi_t\}$ whose states are subsets of a finite or countable set Z. A much-studied example is the symmetric *simple exclusion process*, shown by Spitzer (1970) to have a property that can be expressed thus (see the remark at the end of this section):

$$P_\xi\{\xi_t \cap \eta \neq \varnothing\} = P_\eta\{\xi_t \cap \xi \neq \varnothing\}, \qquad \xi, \eta \subset Z, \tag{1.1}$$

provided ξ or η is finite. If ξ is infinite and η finite, then P_η is much easier to study than P_ξ, and hence (1.1) is of great use in treating the process.

For convenience we will let $\xi \# \eta$ denote $\xi \cap \eta \neq \varnothing$; thus the left side of (1.1) is $P_\xi\{\xi_t \# \eta\}$.

Very few processes have the property (1.1), so we ask whether there is a *different* process $\{\xi_t^*\}$, which we will call *associate* to $\{\xi_t\}$, satisfying, if ξ or η is finite,

$$P_\xi\{\xi_t \# \eta\} = P_\eta^*\{\xi_t^* \# \xi\}, \qquad \xi, \eta \subset Z,\ t \geqq 0. \tag{1.2}$$

If $\{\xi_t^*\}$ has the same distribution as $\{\xi_t\}$ we are back to (1.1), in which case we say $\{\xi_t\}$ is *self-associate*. Quite a number of processes have associates, although the property of having one is quite special. One important property of any process having an associate is that "creation from nothing" cannot occur; that is, $P_\varnothing\{\xi_t = \varnothing\} = 1$.

In case Z is finite, we will find necessary and sufficient conditions for an associate process to exist and show how to construct one if it exists. In case Z is countable we give sufficient conditions applying to many processes; in some cases these conditions are necessary.

As an example consider the case where $Z = Z_d$, the d-dimensional integers. Let N be a finite nonempty subset of Z_d not containing the origin O, let $n = |N|$, the cardinality of N, and let $\mu, \lambda_0, \lambda_1, \cdots, \lambda_n$ be nonnegative numbers with $\lambda_0 = 0$. Let $\{\xi_t\}$ have transitions described informally as follows, letting $\xi_t(x)$, the x-coordinate of ξ_t, be 1 if $x \in \xi_t$ and 0 if not. If at time t we have $|\xi_t \cap N| = k$

Received December 26, 1974.

[1] Research supported in part by NSF Grant GP-24239.

AMS 1970 *subject classification*. Primary 60K35.

Key words and phrases. Interaction, association, set-valued process.

and if $\xi_t(O) = 0$ there is an intensity λ_k for a change of $\xi_t(O)$ from 0 to 1. If $\xi_t(O) = 1$, there is an intensity μ not depending on $\xi_t \backslash O$ for a change to 0. The situation is similar for each coordinate $\xi_t(x)$ except that the translated set $N + x$ replaces N. In such a process two coordinates do not change simultaneously. We will see that an associate process exists if and only if the following inequalities hold:

$$(1.3) \qquad \sum_{r=0}^{k} (-1)^{1+r} \binom{k}{r} \lambda_{n-k+r} \geqq 0 \,, \qquad 1 \leqq k \leqq n \,.$$

In particular, if $n = 2$ the condition is $\lambda_1 \leqq \lambda_2 \leqq 2\lambda_1$. If $n \geqq 2$, we will see that the associate process ξ_t^*, if it exists, has simultaneous changes of several coordinates.

The existence of an associate $\{\xi_t^*\}$ can be useful in studying $\{\xi_t\}$. First, we observe that if S is any random subset of Z and if we put $f(\xi) = \text{Prob}\,\{S \,\#\, \xi\}$, then the incidence function f is clearly a monotone function of ξ and moreover satisfies a further series of inequalities of which the most useful is submodularity (strong subadditivity):

$$f(\xi \cup \eta) + f(\xi \cap \eta) \leqq f(\xi) + f(\eta) \,.$$

In fact

$$\begin{aligned} f(\xi \cap \eta) &= \text{Prob}\,(S \,\#\, (\xi \cap \eta)) \\ &\leqq \text{Prob}\,[(S \,\#\, \xi) \cap (S \,\#\, \eta)] = f(\xi) + f(\eta) - f(\xi \cup \eta) \,. \end{aligned}$$

Referring to (1.2) we see that the right side, and hence the left side, is monotone and submodular in ξ. The usefulness of monotonicity is shown in Section 9, while submodularity was used in [4]. (Some processes not having associates have the submodularity property.)

If $\{\xi_t\}$ has an associate, if the transition functions are respectively P and P^*, if μ is an invariant probability measure for ξ_t and if $h_\mu(\xi) = \mu\{\eta : \eta \,\#\, \xi\}$ then we have the relation

$$(1.4) \qquad h_\mu(\xi) = \int P^*(t, \xi, d\eta) h_\mu(\eta) \,.$$

This form of the relation of association was used by Holley and Liggett (1975). The harmonicity of h for P^* proved very useful in their study of the extremal invariant distributions for the voter model. We will use (1.4) and similar relations in Section 9.

When the present paper was partly completed the author received a preprint of [7] by Holley and Liggett, which uses the notion of association for certain processes, and a copy of the preprint (1973) by Vasil'ev, Mityushin, Pyatetskiĭ-Shapiro, and Toom (1973). The notion of association, for special processes, appears in (2.4) of that paper. A similar notion was used in a special form in the proof of Theorem 1 of Vasil'ev (1969), and related ideas have been useful for percolation processes (Broadbent and Hammersley (1957), page 635). The present paper has benefited from the work cited above. However, the results here are mainly different; overlaps will be pointed out where they occur.[2]

REMARK. It was shown in [11] that if $\{\xi_t\}$ is a symmetric simple exclusion process then $P_\xi\{\xi_t \subset \eta\} = P_\eta\{\xi_t \supset \xi\}$ if $|\xi| < \infty$. At first sight this appears different from (1.1) but the two statements are equivalent because of the fact (which can be seen from Section 7a) that the complementary process ξ_t^c has the same law as ξ_t.

2. Preliminary relations.

Notation. Let Z be a finite set, Ξ the set of subsets of Z, C the set of functions $\Xi \to R_1$, C_0 the set of functions $f \in C$ such that $f(\varnothing) = 0$. Let M be the set of signed measures on the family $\mathscr{P}(\Xi)$ of all subsets of Ξ; M_0 denotes the set of $\mu \in M$ such that $\mu(\Xi) = 0$. If $\xi \subset Z$, $\eta \subset Z$, then $\xi \# \eta$ means $\xi \cap \eta \neq \varnothing$, i.e., ξ touches η. Note the distinction between $\mu(\{\varnothing\})$, in general not 0, $\mu(\varnothing) = 0$. Subsets of Ξ are denoted by ξ, η, ζ, and sometimes other letters. If $x \in Z$, we will sometimes write $\xi \cup x$ instead of $\xi \cup \{x\}$, etc.

We introduce a linear mapping $\varphi : M \to C$ defined as follows:

$$\varphi\mu(\xi) = \textstyle\sum_{\eta:\eta\#\xi} \mu(\{\eta\}), \quad \xi \in \Xi. \tag{2.1}$$

If μ is the distribution of a random subset S of Z, then $q = \varphi\mu$ is the "incidence function" of S, $q(\xi)$ being the probability of $S \# \xi$. The function $\xi \to 1 - q(\xi) = \text{Prob}\{S^c \supset \xi\}$ is the *correlation function* of the random set S^c.

We see from (2.1) that φ maps M ($2^{|Z|}$ dimensions) into C_0 ($2^{|Z|} - 1$ dimensions) and is hence not invertible. However, assuming $\mu(\Xi)$ known, it follows from a modification of the Moebius inversion formula [10] that if $f(\xi)$ denotes the left side of (2.1) where $\mu \in M$, then μ is obtained from f by

$$\mu(\{\eta\}) = \mu(\Xi) \cdot \delta_\varnothing(\{\eta\}) + \psi f(\{\eta\}), \tag{2.2}$$

where $\delta_\xi(\{\eta\}) = 1$ if $\eta = \xi$ and 0 otherwise, and $\psi : C \to M$ is given by

$$\psi f(\{\eta\}) = \textstyle\sum_{A\subset\eta} (-1)^{|A|+1} f(\eta^c \cup A), \qquad \eta \in \Xi, f \in C. \tag{2.3}$$

Note that $\psi 1(\{\eta\}) = 0$ if $\eta \neq \varnothing$, $\psi 1(\{\varnothing\}) = -1$.

The set I of incidence functions, which is the image under φ of the set M_1 of probability measures in M, is a subset of C_0. Since M_1 is $(2^{|Z|} - 1)$-dimensional and can be recovered from I by (2.2) with $\mu(\Xi) = 1$, we see that I is $(2^{|Z|} - 1)$-dimensional. From (2.2), ψ maps I onto $M_1 - \delta_\varnothing$, a $(2^{|Z|} - 1)$-dimensional subset of M_0; since $\varphi\delta_\varnothing = 0$, it follows from (2.1) that φ maps $M_1 - \delta_\varnothing$ onto I. Hence, from linearity:

(2.4) LEMMA. *φ maps M_0 1 : 1 onto C_0, ψ maps C_0 1 : 1 onto M_0, $\varphi\psi f = f$ for $f \in C_0$, $\psi\varphi\mu = \mu$ for $\mu \in M_0$.*

---

[2] In the abstract [5], the author used the word "reciprocal" rather than "associate," but the former word has been used for other kinds of processes.

(2.5) DEFINITIONS. Let $\theta_\xi(\eta) = 1$ if $\xi \# \eta$ and 0 otherwise. Then $\theta_\xi \in C_0$. From the definition of φ and the inverse relationship of φ and ψ we have

(2.6) $$\varphi(\delta_\xi - \delta_\varnothing) = \theta_\xi , \qquad \psi\theta_\xi = \delta_\xi - \delta_\varnothing .$$

Since the measures $\{\delta_\xi - \delta_\varnothing, \xi \neq \varnothing\}$ are a basis for M_0, the functions $\{\theta_\xi, \xi \neq \varnothing\}$ are a basis for C_0, and M_0 and C_0 are duals.

(2.7) DEFINITIONS. Let $(f, \mu) = \sum_\xi f(\xi)\mu(\{\xi\})$, $f \in C$, $\mu \in M$. This is also written as (μ, f), since it will be clear which element is from C and which from M. It is easy to verify that $(\theta_\xi, \psi\theta_\eta) = (\psi\theta_\xi, \theta_\eta)$, $\xi, \eta \in \Xi$, whence

(2.8) $$(f, \psi g) = (\psi f, g) , \quad f, g \in C_0 .$$

Similarly, since $(\varphi(\delta_\xi - \delta_\varnothing), \delta_\eta - \delta_\varnothing) = (\delta_\xi - \delta_\varnothing, \varphi(\delta_\eta - \delta_\varnothing))$,

(2.9) $$(\varphi\mu, \nu) = (\mu, \varphi\nu) , \quad \mu, \nu \in M_0 .$$

If $f \in C_0$, we have $f = \sum_{\eta\neq\varnothing} \psi f(\{\eta\})\theta_\eta$; this can be checked from (2.6). It follows that

(2.10) $$f = f(\varnothing) + \textstyle\sum_{\eta\neq\varnothing} \psi f(\{\eta\})\theta_\eta , \quad f \in C .$$

3. Associate processes. With the notation of Section 2, still assuming Z finite, let $\{T_t\}$ denote a Markov semigroup on C. Our Markov semigroups will always be continuous in t and satisfy $T_t 1 = 1$, even in later sections when Z is infinite; $P(t, \xi, \Gamma)$, P_ξ, and $\mathscr{A}$ will denote the corresponding transition function, probability measures and generator; $\{\xi_t\}$ will be a corresponding standard process. A Markov semigroup $\{T_t^*\}$ on C is called *associate* to $\{T_t\}$ if

(3.1) $$T_t\theta_\xi(\eta) = T_t^*\theta_\eta(\xi) , \qquad \xi, \eta \in \Xi ,$$

corresponding to (1.2). If $\{T_t^*\}$ exists it is uniquely determined by (3.1) and the relation $T_t^*1 = 1$. The corresponding transition function, etc. will be denoted by $P^*(t, \xi, \Gamma)$, etc.

It follows from (3.1) that $\{T_t\}$ is associate to $\{T_t^*\}$. Putting $\eta = \varnothing$ in (3.1), we have $T_t^*\theta_\varnothing(\xi) = 0 = T_t\theta_\xi(\varnothing)$, which shows that $T_t C_0 \subset C_0$. Hence $\xi_0 = \varnothing$ implies $\xi_t = \varnothing$ for all $t > 0$, a.s. $(P_\varnothing)$. Note that also $T_t^*C_0 \subset C_0$.

If $\{T_t^*\}$ exists, let $\{U_t^*\}$ be its conjugate semigroup, acting on M:

(3.2) $$U_t^*\mu(A) = \int \mu(d\xi)P^*(t, \xi, A) , \qquad \mu \in M, A \subset \Xi ,$$

(3.3) $$(U_t^*\mu, f) = (\mu, T_t^*f) , \qquad f \in C, \mu \in M .$$

Putting $f = 1$, we see that U_t^* maps M_0 into itself. Moreover from (3.2) and the relation $P^*(t, \varnothing, \{\varnothing\}) = 1$ we have

(3.4) $$U_t^*\delta_\varnothing = \delta_\varnothing , \qquad t \geqq 0 .$$

From (3.3), (3.1), (2.6) and (2.8), putting $\delta_\xi' = \delta_\xi - \delta_\varnothing$, we get

$$\begin{aligned}(\theta_\xi, U_t^*\delta_\eta') &= (T_t^*\theta_\xi, \delta_\eta') = (T_t\theta_\eta, \delta_\xi') = (T_t\varphi\delta_\eta', \psi\theta_\xi) \\ &= (\theta_\xi, \psi T_t\varphi\delta_\eta') , \qquad \xi, \eta \in \Xi .\end{aligned}$$

Hence

(3.5) $$U_t{}^*\mu = \psi T_t \varphi \mu\,, \qquad \mu \in M_0\,.$$

Combining (3.4) and (3.5), we get

(3.6) $$U_t{}^*\mu = \psi T_t \varphi(\mu - \mu(\Xi)\delta_\varnothing) + \mu(\Xi)\delta_\varnothing = \psi T_t\varphi\mu + \mu(\Xi)\delta_\varnothing\,, \qquad \mu \in M\,.$$

4. Existence condition (finite case).

(4.1) DEFINITIONS. Let $\mathscr{S}$ be the class of continuous semigroups on C (*not* necessarily Markov) satisfying (a) $T_t 1 = 1$, $t \geqq 0$ and (b) $T_t C_0 \subset C_0$. If $\{T_t\} \in \mathscr{S}$, call $\{T_t{}^*\}$ *associate* to $\{T_t\}$ if $\{T_t{}^*\} \in \mathscr{S}$ and (3.1) is satisfied. If $\mathscr{A}$ is the generator of $\{T_t\} \in \mathscr{S}$, we say that $\mathscr{A} \in \mathscr{S}$.

(4.2) LEMMA. *If* $\{T_t\} \in \mathscr{S}$, *there is a unique associate* $\{T_t{}^*\}$, *whose conjugate semigroup* $\{U_t{}^*\}$ *is defined by* (3.6). *Furthermore*:

(a) $U_t{}^*\delta_\varnothing = \delta_\varnothing$; $U_t{}^*M_0 \subset M_0$; $U_t{}^*\mu(\Xi) = \mu(\Xi)$ *for* $\mu \in M$.
(b) $T_t{}^*C_0 \subset C_0$, $T_t{}^*1 = 1$.
(c) $(T_t{}^*)^* = T_t$, $t \geqq 0$.

PROOF. The definition of $\{U_t{}^*\}$ shows that it is a continuous semigroup satisfying (a). Define $\{T_t{}^*\}$ by (3.3). Since $(T_t{}^*f, \delta_\varnothing) = (f, U_t{}^*\delta_\varnothing) = f(\varnothing)$, we have $T_t{}^*C_0 \subset C_0$. Since $(T_t{}^*1, \mu) = (1, U_t{}^*\mu) = 0$ if $\mu \in M_0$, $T_t{}^*1$ is constant as a function of ξ, if t is fixed. Since $T_t{}^*1(\varnothing) = (1, U_t{}^*\delta_\varnothing) = (1, \delta_\varnothing) = 1$, we have $T_t{}^*1 = 1$, proving (b). Moreover, $T_t{}^*\theta_\xi(\eta) = (T_t{}^*\theta_\xi, \delta_\eta - \delta_\varnothing) = (\theta_\xi, U_t{}^*(\delta_\eta - \delta_\varnothing)) = (\theta_\xi, \psi T_t \varphi(\delta_\eta - \delta_\varnothing)) = (\psi\theta_\xi, T_t\theta_\eta) = (\delta_\xi - \delta_\varnothing, T_t\theta_\eta) = T_t\theta_\eta(\xi)$. Hence $\{T_t{}^*\}$ is associate to $\{T_t\}$, and we have uniqueness because $\{T_t{}^*\}$ is determined by (3.1) and $T_t{}^*1 = 1$. It is easily seen that $(T_t{}^*)^* = T_t$. □

The following lemma is little more than a rephrasing of (4.2) in terms of generators.

(4.3) LEMMA. *If* $\{T_t\} \in \mathscr{S}$, *the generator* $\mathscr{A}^*$ *of* $\{T_t{}^*\}$ *and* $\mathscr{B}^*$ *of* $\{U_t{}^*\}$ *have the following properties.*

(a) $\mathscr{A}^*\theta_\eta(\xi) = \mathscr{A}\theta_\xi(\eta)$, $\xi, \eta \in \Xi$;
(b) $\mathscr{A}^*1 = 0$, $\mathscr{B}^*\delta_\varnothing = 0$, $\mathscr{B}^*\delta_\xi = \psi\mathscr{A}\theta_\xi$, $\xi \in \Xi$;
(c) $\mathscr{A}^*C_0 \subset C_0$, $\mathscr{B}^*M_0 \subset M_0$;
(d) $\mathscr{B}^*\mu = \psi\mathscr{A}\varphi\mu = \psi\mathscr{A}\varphi(\mu - \mu(\Xi)\delta_\varnothing)$, $\mu \in M$;
(e) $\mathscr{A}^*$ *is determined uniquely by* $\mathscr{A}^*1 = 0$ *and* (a).

(4.4) THEOREM. *Let* $\{T_t\}$ *be a Markov semigroup on* C. *Then* $\{T_t\}$ *has a Markov associate if and only if* (a) $\mathscr{A}C_0 \subset C_0$ *and* (b) $\psi\mathscr{A}\theta_\xi(\{\eta\}) \geqq 0$ *for each* $\xi, \eta \in \Xi$, $\xi \neq \eta$. *Moreover*, $\psi\mathscr{A}\theta_\xi(\{\eta\})$ *is the intensity for the transition* $\xi \to \eta$, $\xi \neq \eta$ *in the associate process.*

PROOF. Suppose (a) and (b) hold. From Lemma 4.2, there is an associate semigroup $\{T_t{}^*\}$ whose conjugate generator $\mathscr{B}^*$ satisfies $\mathscr{B}^*1 = 0$, and $\mathscr{B}^*\delta_\xi(\{\eta\}) = \psi\mathscr{A}\theta_\xi(\{\eta\})$ from (4.3) (b). Consider the Markov semigroup $\{T_t'\}$

whose conjugate generator $\mathscr{B}'$ satisfies $\mathscr{B}'\delta_\xi(\{\eta\}) = \psi\mathscr{A}\theta_\xi(\{\eta\})$, $\xi \neq \eta$, this being the intensity for the transition from ξ to η in $\{T_t'\}$; since also $\mathscr{B}'1 = 0$, we have $\mathscr{B}^* = \mathscr{B}'$, whence $\mathscr{B}^*$ and $\{T_t^*\}$ are Markovian. Conversely if a Markovian reciprocal $\{T_t^*\}$ exists, then $T_t^*C_0 \subset C_0$, as pointed out in Section 3. Since $\mathscr{B}^*\delta_\xi(\{\eta\}) = \psi\mathscr{A}\theta_\xi(\{\eta\})$ is the intensity for the transition $\xi \to \eta$ if $\xi \neq \eta$, (b) is true. □

(4.5) DEFINITION. Let $\mathscr{S}_m$ be the class of generators in $\mathscr{S}$ that are Markov and have associates.

(4.6) REMARK. It follows from Theorem (4.4) the $\mathscr{S}_m$ is closed under formation of positive linear combinations, and that $(c_1\mathscr{A}_1 + c_2\mathscr{A}_2)^* = c_1\mathscr{A}_1^* + c_2\mathscr{A}_2^*$. Later we will see that there are cases of probabilistic interest where $\mathscr{A}_1$ and $\mathscr{A}_2$ are Markovian and $\mathscr{A}_1$ or $\mathscr{A}_2$ is not in $\mathscr{S}_m$ but $\mathscr{A}_1 + \mathscr{A}_2$ is in $\mathscr{S}_m$.

4.1. Discrete time. Still supposing Z finite, let P be a transition matrix (i.e., Markov) with elements $p(\xi, \eta)$ and let $Pf(\xi) = \sum_\eta p(\xi, \eta)f(\eta)$. A transition matrix P^* is called associate to P if $P^*\theta_\xi(\eta) = P\theta_\eta(\xi)$. As in the continuous case existence of an associate implies $PC_0 \subset C_0$. By arguments like those above, if an associate transition matrix exists its elements are

$$p^*(\xi, \eta) = \delta_\varnothing(\{\eta\}) + \psi P\theta_\xi(\{\eta\}) . \tag{4.7}$$

It can be shown that if P is any transition matrix such that $PC_0 \subset C_0$ and if $\psi P\theta_\xi(\{\eta\}) \geqq 0$ for each $\xi, \eta \neq \varnothing$, then (4.7) defines an associate transition matrix. (If $\eta = \varnothing$, the right side of (4.7) is $1 - P\theta_\xi(Z)$, which is $\geqq 0$ for every transition matrix P.)

If U^* is conjugate to P^*, (4.7) implies

$$U^*(\delta_\xi - \delta_\varnothing) = \psi P\theta_\xi = \psi P\varphi(\delta_\xi - \delta_\varnothing) ,$$

implying $U^*\mu = \psi P\varphi\mu$, $\mu \in M_0$. It can be shown from this that if Q is another transition matrix with an associate transition matrix Q^*, then PQ has the associate transition matrix $(PQ)^* = Q^*P^*$. By induction, $(P^n)^* = (P^*)^n$, so that existence of an associate transition matrix implies existence of an associate process.

5. The case of countable Z. Let Z be a countably infinite set, Ξ the set of all subsets ξ of Z. If $A \subset Z$, then ξ_A means $\xi \cap A$, Ξ_A is the set of all subsets of A, $\xi(A) = |\xi \cap A|$; if $x \in Z$, $\xi(x)$ means $\xi(\{x\})$ and is a *coordinate* of ξ. Ξ is a compact matrizable space with convergence $\xi_n \to \xi$ meaning $\xi_n(x) \to \xi(x)$ for each x, and the measurable subsets of Ξ are the Borel sets $B(\Xi)$. C is the space of continuous functions $\Xi \to R_1$, supremum norm. If $A \subset Z$, $A \neq \varnothing$, then C_A is the set of f in C depending only on coordinates $\xi(x)$, $x \in A$. Let $C^0 = \bigcup_{1 \leqq |A| < \infty} C_A$ be the class of cylinder functions. The functions θ_ξ, $\xi \in \Xi$, are defined as in (2.5), allowing ξ and η to be infinite.

A *Feller semigroup* (FSG) will mean a continuous Markov semigroup $\{T_t\}$ on

C; according to our convention this implies $T_t 1 = 1$. An operator $\mathscr{A}$ with domain in C will be called a *generator* if it is the generator of a FSG. A FSG $\{T_t^*\}$ is called *associate* to $\{T_t\}$ if

(5.1) $$T_t\theta_\xi(\eta) = T_t^*\theta_\eta(\xi)\,, \qquad t \geqq 0,\ |\xi| \text{ or } |\eta| < \infty\,.$$

The corresponding generators $\mathscr{A}$ and $\mathscr{A}^*$ are also called associates.

NOTE. *In Section 5, the words "generator," "semigroup," and "associate" will refer only to the Markov case, unless the contrary is stated.*

If (5.1) holds for finite ξ and η, it also holds if only one is finite. For example if $|\xi| < \infty$, take $\eta_m \uparrow \eta$, $|\eta_m| < \infty$, and use the Feller property (continuity in η) on the left side of (5.1) and monotone convergence on the right side.

$\{T_t^*\}$ is uniquely determined by (5.1), since if $f \in C_V$, $|V| < \infty$, f is a finite linear combination of 1 and the incidence functions θ_ξ, $\xi \subset V$.

As in the case of finite Z, we see that if $\{\xi_t\}$ has an associate, then $P_\varnothing\{\xi_t = \varnothing\} = 1$, $t \geqq 0$.

(5.2) DEFINITIONS. If $V \subset Z$, $1 \leqq |V| < \infty$, let $\mathscr{G}_V$ be the set of generators $\mathscr{A}$ satisfying (a) $f \in C^0 \cap C_{Z\backslash V}$ implies $\mathscr{A}f = 0$; (b) $\mathscr{A}C_V \subset C_V$. Let $\mathscr{G} = \bigcup_{1\leqq|V|<\infty} \mathscr{G}_V$.

Intuitively if $\mathscr{A} \in \mathscr{G}_V$, coordinates $\xi_t(x)$, $x \notin V$, do not change with time, while the distribution of $V \cap \xi_t$ does not depend on $(Z\backslash V) \cap \xi_0$. We have the following elementary facts about $\mathscr{G}_V$.

(5.3) LEMMA. *Let $\mathscr{A} \in \mathscr{G}_V$, corresponding to the* FSG $\{T_t\}$. *Then* (a) $f \in C_{Z\backslash V}$ *implies $T_t f = f$; $f \in C_V$ implies $T_t f \in C_V$.* (b) *$\mathscr{A}$ is bounded and if $f = f_1 f_2$ where $f_1 \in C_V$ and $f_2 \in C_{Z\backslash V}$ then $\mathscr{A}f = f_2\mathscr{A}f_1$ and $T_t f = f_2 T_t f_1 = f_2 \sum_{n=0}^\infty (t^n/n!)\mathscr{A}^n f_1$.*

PROOF. Since $\mathscr{A}C_V \subset C_V$ and C_V is finite-dimensional, $T_t f = \exp(t\mathscr{A})f \subset C_V$ if $f \in C_V$. Let $A_\lambda = \lambda\mathscr{A}R_\lambda$, $\lambda > 0$, where R_λ is the resolvent of $\mathscr{A}$. If $f \in C^0 \cap C_{Z\backslash V}$, then $T_t f = \lim_{\lambda\to\infty} \exp(tA_\lambda)f$, and $A_\lambda f = \lambda R_\lambda \mathscr{A}f = 0$, so $T_t f = f$. This extends to $f \in C_{Z\backslash V}$ by continuity. It follows that $P_\xi\{\xi_t(x) = \xi(x), x \notin V\} = 1$, $t \geqq 0$, $\xi \in \Xi$. Hence $T_t f = f_2 T_t f_1$ if f is as in (b), and the other assertions in (b) follow. □

Note that if $\mathscr{A} \in \mathscr{G}_V$, C^0 is in the domain of $\mathscr{A}$, since any $f \in C^0$ is a finite sum of products $f_1 f_2$, $f_1 \in C_V$, $f_2 \in C_{Z\backslash V}$.

(5.4) LEMMA. (a) *If $V \subset V'$, then $\mathscr{G}_V \subset \mathscr{G}_{V'}$.* (b) *If $\mathscr{A} \in \mathscr{G}_V$ and $\mathscr{A}^*$ is associate to $\mathscr{A}$, then $\mathscr{A}^* \in \mathscr{G}_V$.* (c) *Let $C_{(V)}$ be the set of functions $\Xi_V \to R_1$. Let $\mathscr{A}_1, \mathscr{A}_2 \in \mathscr{G}_V$, with projections $\mathscr{A}_1'$, $\mathscr{A}_2'$ defined in a natural manner on $C_{(V)}$. Then $\mathscr{A}_1$ and $\mathscr{A}_2$ are associates* iff *$\mathscr{A}_1'$ and $\mathscr{A}_2'$ are associates.*

PROOF. (a) follows from the definition of $\mathscr{G}_V$ and Lemma (5.3)(b), noting that each $f \in C_{V'}$ is a finite sum of products $f_1 f_2$, where $f_1 \in C_V$ and $f_2 \in C_{V'\backslash V}$. To prove (b), suppose $W \subset Z\backslash V$, $0 < |W| < \infty$. If $\eta \subset W$ then $T_t^*\theta_\eta(\xi) = T_t\theta_\xi(\eta) = P_\eta\{\xi_t \# \xi\} = \theta_\eta(\xi)$, whence $\mathscr{A}^*\theta_\eta = 0$, implying $\mathscr{A}^*f = 0$ for $f \in C^0 \cap C_{Z\backslash V}$. If

$\eta \subset V$, $T_t^*\theta_\eta(\xi) = T_t\theta_\xi(\eta) = T_t\theta_{\xi\cap V}(\eta)$ because $\xi_t(Z\backslash V) = 0$ a.s. (P_η). Hence

$$(5.5)\qquad \frac{T_t^*\theta_\eta(\xi) - \theta_\eta(\xi)}{t} = \frac{T_t\theta_{\xi\cap V}(\eta) - \theta_{\xi\cap V}(\eta)}{t}, \qquad \eta \subset V.$$

The limit of the right side of (5.5) as $t \downarrow 0$ is $\mathscr{A}\theta_{\xi\cap V}(\eta)$ uniformly in $\eta \subset V$ and $\xi \subset Z$. For fixed $\eta \subset V$, the function $\xi \to \mathscr{A}\theta_{\xi\cap V}(\eta) \in C_V$ whence $\mathscr{A}^*\theta_\eta \in C_V$ if $\eta \subset V$, from (5.5). Hence $\mathscr{A}^*C_V \subset C_V$. The proof of (c) comes readily from the definition of association and is omitted. □

(5.6) LEMMA. *Let $\mathscr{A} \in \mathscr{G}_V$ and let $\mathscr{A}^*$ be associate to $\mathscr{A}$. Then $\|\mathscr{A}^*\| \leqq d_k\|\mathscr{A}\|$ where $d_k < \infty$ depends only on the cardinality k of V.*

PROOF. It can be verified that $\|\mathscr{A}\|$ is equal to the norm of the projection of $\mathscr{A}$ on $C_{(V)}$, so we may suppose $Z = V$. From (4.3) (d), $\mathscr{B}^*\mu = \psi\mathscr{A}\varphi\mu$, $\mu \in M$. It is easily verified that $\|\varphi\mu\| \leqq \|\mu\|$ (variation norm for μ) whence $\|\mathscr{A}\varphi\mu\| \leqq \|\mathscr{A}\| \cdot \|\mu\|$. The norm of ψ evidently depends only on $|V|$; call it d_k if $|V| = k$. Then $\|\mathscr{B}^*\mu\| \leqq d_k\|\mathscr{A}\| \cdot \|\mu\|$ whence $\|\mathscr{B}^*\| \leqq d_k\|\mathscr{A}\|$ if $|V| = k$. Finally, as for any finite-state chain, we have

$$\|\mathscr{A}^*\| = \sup_{\|f\|=1} \sup_\xi |\mathscr{A}^*f(\xi)| = \sup_{\|f\|=1} \sup_\xi |(f, \mathscr{B}^*\delta_\xi)| \leqq \|\mathscr{B}^*\|. \quad \square$$

(5.7) THEOREM. *Let $\mathscr{A}_1, \mathscr{A}_2, \cdots$ be generators satisfying the following conditions.*

(a) $\mathscr{A}_i \in \mathscr{G}_{V_i}$, $|V_i| \leqq N < \infty$, *and* $\|\mathscr{A}_i\| \leqq K < \infty$, $i = 1, 2, \cdots$;

(b) *for each x in Z, there are at most R values of i such that $x \in V_i$, where $R < \infty$ does not depend on x;*

(c) *each $\mathscr{A}_i$ has an associate $\mathscr{A}_i^*$.*

Let $\{T_t^n\}$ and $\{T_t^{n*}\}$ be the semigroups corresponding to $\sum_{i=1}^n \mathscr{A}_i$ and $\sum_{i=1}^n \mathscr{A}_i^*$. Then $\{T_t^n\}$ and $\{T_t^{n*}\}$ converge to associate Feller semigroups $\{T_t\}$ and $\{T_t^*\}$; i.e., *if $f \in C$, $\lim_{n\to\infty} \|T_t^nf - T_tf\| = 0$ uniformly on finite t-intervals and similarly for $\{T_t^{n*}\}$. The corresponding generators have the form $\mathscr{A}f = \sum \mathscr{A}_if$ and $\mathscr{A}^*f = \sum \mathscr{A}_i^*f$ for $f \in C^0$.*

PROOF. The convergence for T_t^n follows from the existence theorem of Holley (1972) with minor modifications of the proof. From Lemmas 5.6 and 5.4(b), we see that the $\mathscr{A}_i^*$ satisfy the same conditions as the $\mathscr{A}_i$ with the same V_i but possibly a different K. Hence $\{T_t^{n*}\}$ converges to a FSG $\{T_t^*\}$. Clearly (5.1) holds if ξ and η are finite, and hence if one is infinite. □

REMARK. If $\{T_t^n\} \to \{T_t\}$ and there are associate semigroups $\{T_t^{n*}\}$, then obviously $\lim T_t^{n*}\theta_\xi(\eta) = \lim T_t^n\theta_\eta(\xi)$ exists for each finite η, uniformly in ξ, since $\theta_\eta \in C^0$. However, under the conditions of Theorem 5.7, we have now shown convergence uniformly in η for each finite ξ.

A stronger version of Theorem 5.7 might be constructed using results of [9], where an existence theorem is given with fewer restrictions on the generators.

6. Examples: some elementary processes. We consider, for finite sets Z, some elementary generators which all map C_0 into itself. Even those not having

Markov associates may have probabilistic interest; see Section 7, where we obtain various processes by combining these elementary generators. See Section 5 for the definition of $\xi(x)$ and $\xi(A)$.

a. *Pure death.* Let $Z = z \cup y$, $y \neq z$; given $\mu_0, \mu_1 \geqq 0$, define $\mathscr{A}$ by

$$\mathscr{A}f(\xi) = \xi(z)[\mu_1\xi(y) + \mu_0(1 - \xi(y))](f(\xi\backslash z) - f(\xi)) . \tag{6.1}$$

Thus the intensity for $1 \to 0$ at z is μ_1 if $\xi(y) = 1$ and μ_0 if $\xi(y) = 0$. We calculate $\mathscr{A}\theta_z(z) = -\mu_0$, $\mathscr{A}\theta_z(z \cup y) = -\mu_1$, $\mathscr{A}\theta_{z\cup y}(z) = -\mu_0$, and $\mathscr{A}\theta_\xi(\eta) = 0$ otherwise. From Theorem 4.4, the matrix $\mathscr{B}^*\delta_\xi(\{\eta\})$ has the form

		η			
		$\emptyset$	z	y	$z \cup y$
	$\emptyset$	0	0	0	0
ξ:	z	μ_1	$-\mu_1$	$\mu_0 - \mu_1$	$-(\mu_0 - \mu_1)$
	y	0	0	0	0
	$z \cup y$	0	0	μ_0	$-\mu_0$

This is Markov if and only if $\mu_0 = \mu_1$, in which case $\mathscr{A}$ is self-associate.

b. *Jump.* Again let $Z = z \cup y$. Let $\mathscr{A}$ correspond to intensity 1 for the transition $z \to y$, with no other transitions. That is $\mathscr{A}f(\xi) = \xi(z)(1 - \xi(y))(f(y) - f(z))$. Then $\mathscr{B}^*\delta_z(\{y\}) = 1$, $\mathscr{B}^*\delta_z(\{y \cup z\}) = -1$, $\mathscr{B}^*\delta_y(\{y\}) = -1$, $\mathscr{B}^*\delta_y(\{y \cup z\}) = 1$, and $\mathscr{B}^*\delta_\xi(\eta) = 0$ otherwise. Hence $\mathscr{A}$ does *not* have a Markov associate.

c. *Permutation.* Suppose $2 \leqq |Z| < \infty$. Let π be a permutation of Z and let $\mathscr{A}f(\xi) = f(\pi\xi) - f(\xi)$, corresponding to an application of π at random times with intensity 1. We take $\mathscr{A}f(\emptyset) = 0$. Then $\mathscr{A}$ has a Markov associate, corresponding to the permutation π^{-1}.

d. *Birth processes.* Let $Z = z \cup N$, $z \notin N$, $1 \leqq |N| < \infty$. Let $\lambda(\xi) \geqq 0$ be defined for $\xi \subset N$, with $\lambda(\emptyset) = 0$. Let

$$\begin{aligned} \mathscr{A}f(\xi) &= \lambda(\xi)(f(\xi \cup z) - f(\xi)) , \quad z \notin \xi , \\ \mathscr{A}f(\xi) &= 0 , \quad z \in \xi . \end{aligned} \tag{6.2}$$

The only transition is $0 \to 1$ at z, with intensity $\lambda(\xi)$. A special case of this process is a component of the contact processes of [4], and the process is related to the proximity processes of [7].

After some calculation, the inequalities of Theorem 4.4 (b) become

$$\begin{aligned} 0 &\leqq \mathscr{B}^*\delta_\xi(\{\eta\}) \\ &= \textstyle\sum_{A\subset\eta\backslash\xi} (-1)^{|A|+1}\lambda(\eta^c \cup A) , \qquad z \in \xi \subset \eta,\ \xi \neq \eta , \end{aligned} \tag{6.3}$$

since $\mathscr{B}^*\delta_\xi(\{\eta\}) = 0$ for all other pairs $\xi \neq \eta$. Hence an associate process $\{\xi_t^*\}$ exists iff the inequalities in (6.3) hold.

If λ depends only on $|\xi|$, say $\lambda(\xi) = \lambda_{|\xi|}$, the inequalities in (6.3) become

$$(6.4) \qquad (-1)^{k-r+1}\Delta^{-(k-r)}\lambda_{n-r} \geqq 0\,, \qquad 0 \leqq r < k \leqq n\,,$$

where $n = |N|$, $\Delta^{-1}\lambda_m = \lambda_m - \lambda_{m-1}$, and $\Delta^{-(p+1)}\lambda_m = \Delta^{-p}\lambda_m - \Delta^{-p}\lambda_{m-1}$, $p \geqq 1$.

It can be verified by induction on r that (6.4) holds iff

$$(6.5) \qquad (-1)^{k+1}\Delta^{-k}\lambda_n \geqq 0\,, \qquad 1 \leqq k \leqq n\,.$$

These inequalities are the same as (1.3).

REMARK. The simplest cases where $\lambda_0 = 0$ and (6.5) holds are (a) $\lambda_k = k\lambda$, $k = 0, 1, \cdots, n$; and (b) $\lambda_0 = 0$, $\lambda_k = \lambda$ for $1 \leqq k \leqq n$.

To determine the nature of $\{\xi_t^*\}$ when it exists, note from (6.3) that

$$(6.6) \qquad \mathscr{B}^*\delta_{z\cup\xi\cup y}(\{\eta \cup \xi \cup y\}) = \mathscr{B}^*\delta_{z\cup\xi}(\{\eta \cup \xi\}) + \mathscr{B}^*\delta_{z\cup\xi}(\{\eta \cup \xi \cup y\})$$

for $z \in \eta$, $z \neq \eta$, $\xi \cap \eta = \varnothing$, $y \notin \xi \cup \eta$. We then obtain

$$(6.7) \qquad \mathscr{B}^*\delta_{z\cup\xi}(\{\eta \cup \xi\}) = \textstyle\sum_{\gamma\subset\xi} \mathscr{B}^*\delta_z(\{\eta \cup \gamma\})\,, \qquad z \in \eta,\ z \neq \eta,\ \xi \cap \eta = \varnothing\,,$$

using (6.6) and induction on the size of ξ.

From (6.3), the process $\{\xi_t^*\}$ has the intensity

$$\textstyle\sum_{A\subset\eta\backslash z} (-1)^{|A|+1}\lambda(\eta^c \cup A) = \mathscr{B}^*\delta_z(\{\eta\})$$

for the transition $z \to \eta$ if $z \in \eta$. We see from (6.7) that if $\{\xi_t^*\}$ is in the state ξ' at any time, with $z \in \xi'$, and if $z \in \eta' \neq z$, the intensity is $\mathscr{B}^*\delta_z(\{\eta'\})$ for the "event" that simultaneously all coordinates of η' that are not already 1 become 1. This "event" is not observable if $\eta' \subset \xi'$. Such processes were called "branching processes with interference" in [7].

In the special case where $N = y$, i.e., $|N| = 1$, and $\lambda(y) = \lambda$, we have $\mathscr{B}^*\delta_z(\{z \cup y\}) = \lambda$ from (6.3). That is, if $\{\xi_t\}$ has the intensity $\lambda \cdot \xi_t(y)$ for the transition $0 \to 1$ at z, then $\{\xi_t^*\}$ has the intensity $\lambda \cdot \xi_t^*(z)$ for the transition $0 \to 1$ at y. From the additive nature of the relation of association, it follows that if $1 \leqq |N| < \infty$ and $\lambda(\xi) = \lambda \cdot |\xi|$, then $\mathscr{B}^*\delta_z(\{z \cup y\}) = \lambda$ for each $y \in N$, and $\mathscr{B}^*$ is then determined by (6.7).

In the special case $\lambda(\varnothing) = 0$, $\lambda(\xi) = \lambda$ for $\xi \neq \varnothing$, we have $\mathscr{B}^*\delta_z(\{z \cup N\}) = \lambda$ and $\mathscr{B}^*\delta_z(\{\eta\}) = 0$ for every other $\eta \neq z$.

7. Processes in Z_d. Let Z be the set Z_d of d-dimensional integers, with O denoting the origin.

If $\mathscr{A}_O \in \mathscr{G}_V$ (see Definition 5.2) and has a Markov associate $\mathscr{A}_O^*$, then Theorem 5.7 implies that the generator $\mathscr{A} = \sum_{x\in Z_d} \mathscr{A}_x$ has the Markov associate $\mathscr{A}^* = \sum_x \mathscr{A}_x^*$, where we define

$$(7.1) \qquad \mathscr{A}_x f(\xi) = \mathscr{A}_O f_x(\xi - x)\,, \qquad f_x(\xi) = f(\xi + x)\,,$$

and $\mathscr{A}_x^* = (\mathscr{A}_x)^* = (\mathscr{A}_O^*)_x$. That is, $\mathscr{A}_x$ is $\mathscr{A}$ shifted to x. We now combine some of the elementary generators of Section 6, taking $Z = Z_d$ throughout this section.

a. *Stirring and symmetric simple exclusion.* Let $N_1, \cdots, N_k$ be finite subsets of Z, let $\lambda_1, \cdots, \lambda_k$ be > 0, and let π_j be a permutation of N_j. Let $\mathscr{A}_O = \sum \lambda_j \mathscr{A}_{(j)}$, where $\mathscr{A}_{(j)}$ is constructed from π_j as in Section 6c. (The case $k = 1$ corresponds to the discrete version of the stirring process treated by Lee (1974)). Take $V = \bigcup N_j$. Then $\mathscr{A} = \sum \mathscr{A}_x$ has a Markov associate with π_j^{-1} replacing π_j.

Now suppose $N_j = O \cup x_j$, $1 \leqq j \leqq k$, where the x_j are distinct from each other and from O. Suppose no π_j is the identity. Then $\mathscr{A}$ corresponds to a symmetric simple exclusion process where a particle at x has intensity λ_j for jumping to $x + x_j$ if the latter is empty, and the same intensity for jumping to $x - x_j$. Since $\pi_j = \pi_j^{-1}$, this process is self-associate, as shown essentially by Spitzer (1970). The proof just given is more special because it requires an upper bound on the jump size.

Taking $k = 1$ and $|N| = 3$, we obtain examples which may be interpreted as motion of particles with *speed change*. These will not be described here.

b. *Contact processes* (*extended*). Let $N \subset Z$, $1 \leqq |N| < \infty$, $O \notin N$. Let $\lambda(\xi) \geqq 0$ depend only on $\{\xi(x), x \in N\}$, $\lambda(\xi) = 0$ if $\xi(N) = 0$. Let a constant $\mu \geqq 0$ be given. Take $V = O \cup N$, let

$$\mathscr{A}_O f(\xi) = (1 - \xi(O))\lambda(\xi)(f(\xi \cup O) - f(\xi)) + \mu\xi(O)(f(\xi \backslash O) - f(\xi)) , \tag{7.2}$$

and put $\mathscr{A} = \sum \mathscr{A}_x$. We will call this a *contact process*, extending the terminology of [4]. The birth rate at x depends on the condition of $N + x$; the death rate is constant.

Suppose $\lambda(\xi)$ depends only on $\xi(N)$, say $\lambda(\xi) = \lambda_k$ if $\xi(N) = k$. Let $n = |N|$. Using Theorem 5.7 and a and d of Section 6, we see that $\{\xi_t\}$ has an associate if (6.5) is true (or more generally (6.3) with appropriate changes of notation).

Moreover, (6.5) *is necessary for the existence of an associate process.* For, suppose $\{\xi_t^*\}$ exists. Let $V = O \cup N$, and let $\psi = \psi_V$ be defined as in (2.3), taking $Z = V$. Fix t and define a 1-step transition function $P'(\xi, d\eta)$, for $\xi, \eta \subset V$, by

$$P'(\xi, d\eta) = P_\xi(\xi_t \cap V \in d\eta) .$$

Define P'^* similarly using ξ_t^*. Then

$$P'\theta_\xi(\eta) = T_t\theta_\xi(\eta) = T_t^*\theta_\eta(\xi) = P'^*\theta_\eta(\xi) , \qquad \xi, \eta \subset V ,$$

so that P' and P'^* are associates. From (4.7) we have

$$\psi P'\theta_\xi(\{\zeta\}) \geqq 0 , \qquad \xi, \zeta \subset V, \zeta \neq \varnothing . \tag{7.3}$$

Letting $\xi = O$ in (7.3), replacing $P'\theta_O$ by $\theta_O + t\mathscr{A}\theta_O + o(t)$, and noting that

$\psi\theta_O(\{\eta\}) = 0$ if $\eta \neq \emptyset$ or O, since $\psi\theta_O = \delta_O - \delta_\emptyset$, we find

$$\psi\mathscr{A}\theta_O(\{\eta\}) \geqq 0\,, \qquad \eta \subset V,\ \eta \neq \emptyset \text{ or } O\,. \tag{7.4}$$

It can be verified that $\mathscr{A}\theta_O = \mathscr{A}_O\theta_O$, where $\mathscr{A}_O$ is given by (7.2). Moreover, writing the right side of (7.2) as $\mathscr{A}_O' + \mathscr{A}_O''$ where $\mathscr{A}_O''$ corresponds to death, we find that $\psi\mathscr{A}''\theta_O(\{\eta\}) = 0$ if $O \in \eta$, $O \neq \eta$ because this is the intensity for the transition $O \to \eta$ of the associate to a death process and hence of the process itself. Hence we have

$$\psi\mathscr{A}_O'\theta_O(\{\eta\}) \geqq 0\,, \qquad O \in \eta \subset V,\ O \neq \eta\,. \tag{7.5}$$

But the inequalities in (7.5) are just (6.5) or (1.3), and necessity is proved.

If $|N| = 1$, say $N = \{y\}$, it follows from Section 6 (2 paragraphs below (6.7)) that $\{\xi_t^*\}$ has the same law as $\{\xi_t\}$ with $\{y\}$ replaced by $\{-y\}$. Hence if $\lambda_k = k \cdot \lambda$, $0 \leqq k \leqq |N|$, then $\{\xi_t^*\}$ has the same law as $\{\xi_t\}$ with N replaced by $-N$. In particular *if N is symmetric with respect* to O, *then $\{\xi_t\}$ is self-associate.*

If $\lambda_k = \lambda$, $1 \leqq k \leqq |N|$, we have the case mentioned at the end of Section 6, and $\{\xi_t^*\}$ is an example of a branching process with interference (see [7]).

Since a pure death process is self-associate, the associate of a contact process has the same death intensities as the original process.

c. *Variable death rates; the voter model.* We are led to a wider class of processes, where μ depends on ξ, if we combine the death process of 6a with the process of 6d. Again take $N \subset Z$, $1 \leqq |N| < \infty$, $O \notin N$, $V = O \cup N$ and let $\mu_0 \geqq \mu_1 \geqq 0$ be given. For each $y \in N$ define $\mathscr{A}_y'$ and $\mathscr{A}_y''$ by

$$\mathscr{A}_y'f(\xi) = \xi(O)[\mu_1\xi(y) + \mu_0(1 - \xi(y))](f(\xi\backslash O) - f(\xi))\,, \qquad f \in C^0\,,$$
$$\mathscr{A}_y''f(\xi) = \lambda\xi(y)(1 - \xi(O))(f(\xi \cup O) - f(\xi))\,, \qquad f \in C^0\,,$$

where $\lambda \geqq \mu_0 - \mu_1$. Then $\mathscr{A}_y' + \mathscr{A}_y''$ corresponds to death at O with rate μ_1 if y is occupied and μ_0 if not, and birth at O with rate λ if y is occupied. From Section 6a and the next-to-last paragraph of Section 6, we find that the associate of $\mathscr{A}_y' + \mathscr{A}_y''$ has the following matrix $\mathscr{B}^*\delta_\xi(\eta)$, restricting ξ and η to $O \cup y$.

	η: $\emptyset$	O	y	$O \cup y$
ξ: $\emptyset$	0	0	0	0
O	μ_1	$-\mu_1 - \lambda$	$\mu_0 - \mu_1$	$\lambda - (\mu_0 - \mu_1)$
y	0	0	0	0
$O \cup y$	0	0	μ_0	$-\mu_0$

This matrix is Markov if $\lambda \geqq \mu_0 - \mu_1 \geqq 0$, which we henceforth assume.

Putting $\mathscr{A}_O = \sum_{y \in N}(\mathscr{A}_y' + \mathscr{A}_y'')$, $\mathscr{A} = \sum \mathscr{A}_x$, we get a process $\{\xi_t\}$ with an associate process $\{\xi_t^*\}$. The intensities are as follows, given $\xi_t = \xi$ or $\xi_t^* = \xi$ respectively, putting $N' = -N$. In the table, $y \in N$.

Before	After	Intensity
$\xi_t(x) = 0$	$\xi_t(x) = 1$	$(1 - \xi(x))\lambda \cdot \xi(x + N)$
$\xi_t(x) = 1$	$\xi_t(x) = 0$	$\xi(x)[\mu_0\|N\| - (\mu_0 - \mu_1)\xi(x + N)]$
$\begin{cases}\xi_t^*(x) = 1\\ \xi_t^*(x + y) = 0\end{cases}$	$\begin{cases}\xi_t^*(x) = 0\\ \xi_t^*(x + y) = 1\end{cases}$	$\xi(x)(1 - \xi(x + y))(\mu_0 - \mu_1)$
$\xi_t^*(x) = 0$	$\xi_t^*(x) = 1$	$(1 - \xi(x))(\lambda - (\mu_0 - \mu_1))\xi(N' + x)$
$\xi_t^*(x) = 1$	$\xi_t^*(x) = 0$	$\xi(x)(\mu_1\|N\| + (\mu_0 - \mu_1)\xi(N + x))$

Note that Z is an absorbing state for $\{\xi_t\}$ iff $\mu_1 = 0$. The following cases are of interest.

(i) If $\mu_0 = \mu_1$ we have a contact process. If also N is symmetric ($N = N'$), then $\{\xi_t\}$ is *self-associate*. If $\mu_0 < \mu_1$, we have transport as well as birth and death for ξ_t^*.

(ii) If $\lambda = \mu_0 - \mu_1$, we see that $|\xi_t^*|$ can never increase with t, and hence has a relatively simple behavior. This can be of great help in studying $\{\xi_t\}$ by means of (5.1). If in addition $\mu_1 = 0$, $\{\xi_t\}$ is a special case of the "voter model", which Holley and Liggett studied by means of $\{\xi_t^*\}$.

(iii) If $Z = Z_1$, $\mu_1 = 0$, and $\lambda = \mu_0$, then $\{\xi_t\}$ is a degenerate case of the time-dependent Ising model of Glauber (1963), if we identify the coordinates 1 and 0 with the spin states 1 and -1 and take $N = \{-1, 1\}$.

(iv) Some of the processes $\{\xi_t\}$ of the present Section 6 are examples of the proximity processes of [7], although the author has not verified that all of them are.

8. Processes of constant size; double stochasticity. The process $\{\xi_t\}$ is said to have *constant size* if $P_\xi\{|\xi_t| = |\xi|\} = 1$, $t \geqq 0$ whenever $|\xi| < \infty$.

(8.1) THEOREM. *Suppose* $|Z| < \infty$. *If* $\{\xi_t\}$ *has constant size and has an associate* $\{\xi_t^*\}$, *then* (a) $\{\xi_t^*\}$ *has constant size*; (b) *both processes are doubly stochastic*, i.e.,

$$\sum_{\xi:|\xi|=|\eta|} P(t, \xi, \{\eta\}) = 1 \,, \tag{8.2}$$

and similarly for P^*.

PROOF. Let $|Z| = n$; let m be an integer, $0 \leqq m \leqq n$. If S is any random subset of Z such that Prob $(|S| = m) = 1$, and if $q(A) =$ Prob $(S = A)$, then

$$\sum_{|A|=n-m} q(A) = \tbinom{n}{m} - 1 \,, \tag{8.3}$$

$$\sum_{A\subset Z} q(A) = 2^n(1 - (\tfrac{1}{2})^m) \,. \tag{8.4}$$

To get (8.3), write the left side as

$$\sum_{|C|=m} \text{Prob}\,(S = C) \sum_{|A|=n-m} \theta_C(A) \,. \tag{8.5}$$

Only one of the $\binom{n}{m}$ terms in the inner sum is 0 rather than 1, whence we get (8.3). A similar argument leads to (8.4).

If $|\xi| + |\eta| > n$, then $1 = P_\eta\{\xi_t \# \xi\} = P_\xi^* \{\xi_t^* \# \eta\}$, showing that $P_\xi^*\{|\xi_t^*| \geqq |\xi|\} = 1$. From this and association,

$$P_A(\xi_t \# B) = \sum_{C:|C|\geqq|B|} P^*(t, B, \{C\})\theta(C, A) \qquad B \subset Z, \quad A \subset Z. \tag{8.6}$$

Summing over all B such that $|B| = n - m$, where $m = |A|$, and using (8.3), we get

$$\begin{aligned}\tbinom{n}{m} - 1 &= \textstyle\sum_{|B|=n-m} (1 - P^*(t, B, \{A^c\})) \\ &= \tbinom{n}{m} - \textstyle\sum_{|B|=n-m} P^*(t, B, \{A^c\}) .\end{aligned}\tag{8.7}$$

Hence for each $A' \subset Z$ we have

$$\sum_{B:|B|=|A'|} P^*(t, B, A') = 1 . \tag{8.8}$$

Summing over all $A' \subset Z$ and changing the order of summation, we get

$$\sum_{B\subset Z} \sum_{A':|A'|=B} P^*(t, B, A') = 2^n . \tag{8.9}$$

Each inner sum in (8.9) is $\leqq 1$, whence each must equal 1. Hence $\{\xi_t^*\}$ is of constant size and, from (8.8), is doubly stochastic. Then $\{\xi_t\}$ is doubly stochastic since it is associate to $\{\xi_t^*\}$. □

REMARK. Double stochasticity was used by Lee (1974) in proving a limit theorem for the "stirring processes" mentioned in Section 7, and might be of use in some other cases with infinite Z. Theorem 7.1 might be extended to the case of infinite Z, with suitable technical assumptions.

The author does not have an example of associate processes of constant size other than stirring processes like those of Section 7a.

9. Application: convergence and invariant measures. In this section $Z = Z_d$. A *measure* will always mean a probability measure in Ξ. Convergence means weak convergence. A measure μ will be called *time invariant* (for ξ_t) if $\mu(A) = \int \mu(d\xi)P(t, \xi, A)$, *translation invariant* if $\mu(A + x) = \mu(A)$, and *invariant* if it is *both* time and translation invariant. The processes ξ_t considered in this section will automatically have translation-invariant transition functions: $P(t, \xi + x, A + x) = P(t, \xi, A)$. In this section μ will always be a measure and $U_t\mu(A) = \int \mu(d\xi)P(t, \xi, A)$.

(9.1) DEFINITION. Let $\mathscr{K}$ be the class of Feller semigroups with generator $\mathscr{A} = \sum_{x\in Z} \mathscr{A}_x$, where $\mathscr{A}_x$ is as in (7.1), $\mathscr{A}_0 \in \mathscr{G}_{V_0}$ for some V_0 (Definition 5.2), and $\mathscr{A}_0$ has a Markov associate.

(9.2) THEOREM. *Let $\{T_t\} \in \mathscr{K}$ and satisfy the following conditions.*

(a) *For each $t > 0$ and $n = 1, 2, \cdots$*

$$\sup_{\xi:|\xi|=n} P_Z\{\xi_t \# \xi\} < 1 .$$

(b) *For each $t > 0$, $x \in Z$, $\xi \neq \varnothing$, we have $P_\xi\{\xi_t \# x\} > 0$.*

Then: (i) $U_t\delta_Z$ *converges to a measure* ν. (ii) *If* $\nu = \delta_\varnothing$ *then for each* μ *we have* $\lim U_t\mu = \delta_\varnothing$. (iii) *If* $\nu \neq \delta_\varnothing$, *then* $\nu(\{\varnothing\}) = 0$, *and if* μ *is any translation invariant measure then* $\lim U_t\mu = \mu(\{\varnothing\})\delta_\varnothing + (1 - \mu(\{\varnothing\}))\nu$. *Hence every invariant* μ *has the form* $c\delta_\varnothing + (1 - c)\nu$.

For criteria whether $\nu = \delta_\varnothing$ or not, see [3], [4] or [7].

Assumption (a) of the theorem does not hold for processes having $\xi = Z$ as an absorbing state; for such processes, the situation can be very different. See for example the discussion of the voter model in [7].

Assumption (b) cannot be dropped entirely. However, at least if $d = 1$, it can be weakened; see Theorem 9.20 below.

I am indebted to David Griffeath for the present form of Theorem 9.2; the original statement gave only the form of the invariant measures, not convergence.

Theorem 9.2 is very similar to Theorem 4.11 of [14], although not contained in it. The results of [14], applying to certain discrete time processes named for O. N. Stavskaya, extend results of Vasil'ev (1969), Vasershtein and Leontovich (1970), and others. The present proof borrows several ideas of [14], but makes extensive use of (9.5), which has no counterpart in [14]. Some ideas of [6] and [7] are also used.

See Section 11 for a brief discussion of hypermonotonicity, a notion used in [14], which is closely related to association.

We need several lemmas for the proof. The conditions of the theorem will be assumed, and $\{\xi_t\}$ and $\{\xi_t^*\}$ will be standard processes governed by $\{T_t\}$ and $\{T_t^*\}$.

(9.3) LEMMA. $\lim_{t\to\infty} P_\xi^*\{0 < |\xi_t^*| \leqq k\} = 0$, $k > 0$, $\xi \in \Xi$.

PROOF. We have from 9.2(a)

$$\sup_{|\xi|\leqq k} P_\xi^*\{\xi_t^* \# Z\} = \sup_{|\xi|\leqq k} P_Z\{\xi_t \# \xi\} < 1 , \qquad t > 0 .$$

That is, $\inf_{|\xi|\leqq k} P_\xi^*\{\xi_t^* = \varnothing\} > 0$ for fixed $t > 0$. Since $\varnothing$ is an absorbing state, the lemma follows from familiar results about Markov processes. ∎

(9.4) DEFINITION. If μ is a probability measure in Ξ let $h_\mu(\xi) = \mu\{\eta : \eta \# \xi\}$, and when dealing with a fixed μ, write h_t instead of $h_{U_t\mu}$, putting $h = h_0$. Note that $h_\mu(\varnothing) = 0$.

Writing

$$\begin{aligned} h_{t+s}(\eta) &= \textstyle\int\int \mu(d\xi)P(t+s, \xi, d\gamma)\theta_\eta(\gamma) \\ &= \textstyle\int \mu(d\xi) \int P(t, \xi, d\zeta) \int P(s, \zeta, d\gamma)\theta_\eta(\gamma) , \end{aligned}$$

replacing the inmost integral by $\int P^*(s, \eta, d\gamma)\theta_\zeta(\gamma)$, and changing the order of integration, we obtain

$$h_{t+s}(\eta) = \textstyle\int P^*(s, \eta, d\gamma)h_t(\gamma) , \qquad s, t \geqq 0, |\eta| < \infty , \tag{9.5}$$

and if μ is time-invariant,

$$h(\eta) = \textstyle\int P^*(s, \eta, d\gamma)h(\gamma) . \tag{9.6}$$

These useful forms of the relation of association were utilized in [7].

(9.7) LEMMA. *For any μ, and any k, $t > 0$, $|\eta| < \infty$, we have*

$$\begin{aligned}\liminf_{s\to\infty} \textstyle\int_{|\gamma|>k} P^*(s, \eta, d\gamma)h_t(\gamma) &= \liminf_{s\to\infty} h_{t+s}(\eta) \leqq \limsup_{s\to\infty} h_{t+s}(\eta) \\ &\leqq \limsup_{s\to\infty} \textstyle\int_{|\gamma|>k} P^*(s, \eta, d\gamma)h_t(\gamma) .\end{aligned}$$

PROOF. This is an immediate consequence of (9.5) and Lemma 9.3. □

The main step remaining is to show that if μ is any translation invariant measure and $\mu(\{\emptyset\}) = 0$, then $\inf_{|\gamma|>k} h_{t_1}(\gamma) \to 1$ as $k \to \infty$ for an appropriate $t_1 > 0$. For this we use some ideas of [14], but in order to avoid certain independence assumptions of [14] we use some results of [6].

(9.8) DEFINITIONS. Recall the definition of V_0 in (9.1). If $V \subset Z$, let $S(V) = V \cup \bigcup_x' (V_0 + x)$, where $\bigcup'$ is taken over all x such that $(V_0 + x) \# V$. Put $S_0(V) = V$, $S_1(V) = S(V)$ and $S_{n+1}(V) = S(S_n(V))$, $n = 1, 2, \cdots$. Call V' and V'' *strongly disjoint* if $S(V') \cap V'' = V' \cap S(V'') = \emptyset$. Below, V' and V'' are finite.

(9.9) LEMMA. *Suppose $f' \in C_{V'}$, $f'' \in C_{V''}$; $||f'||$, $||f''|| \leqq 1$; and $S_m(V')$ and $S_n(V'')$ are strongly disjoint for $m + n \leqq k - 1$. Then for some $t_1 > 0$ depending on $||\mathscr{A}_0||$ and $|V_0|$ (not V' or V'')*

$$||T_t(f'f'') - (T_tf')(T_tf'')|| \leqq 5e^{|V'|+|V''|} \cdot k(\tfrac{1}{2})^k , \quad 0 \leqq t \leqq t_1 , \qquad k = 1, 2, \cdots .$$

The proof is omitted, since it follows closely the methods and results, especially (2.8) and Lemma 3.5, of [6]. However, note the following results of [6] which are used.

(a) If $f \in C_V$ for some finite V, then $T_tf = \sum_{k=0}^\infty (t^k/k!)\mathscr{A}^kf$, $0 \leqq t \leqq t_1$.
(b) If $S_m(V')$ and $S_n(V'')$ are strongly disjoint for all $m + n \leqq k - 1$, and f' and f'' are as above, then $\mathscr{A}^k(f'f'') = \sum_{r=0}^k \binom{k}{r}\mathscr{A}^rf'\mathscr{A}^{k-r}f''$, and $\mathscr{A}^rf' \in C_{S_r(V')}$, $r = 0, 1, \cdots$.

(9.10) DEFINITION. For $|\xi| \geqq 2$, let $\Delta(\xi) = \min_{x,y\in\xi,x\neq y} |x - y|$.

(9.11) LEMMA. *Given $\varepsilon > 0$ and $m = 1, 2, \cdots$, there exists $L = L(\varepsilon, m)$ such that $|\eta| = m$, $\Delta(\eta) \geqq L$ implies*

(9.12) $$|\mathscr{E}_\xi \textstyle\prod_{x\in\eta} (1 - \xi_t(x)) - \prod_{x\in\eta} \mathscr{E}_\xi(1 - \xi_t(x))| < \varepsilon , \qquad 0 \leqq t \leqq t_1, \xi \in \Xi$$

where t_1 is as in (9.9).

The proof is readily carried out by induction on m, using (9.9).
For convenience we give a result used in [14].

(9.13) LEMMA. *Let $X_1, \cdots, X_k$ be random variables such that $0 \leqq X_i \leqq 1$ and* $\text{Prob}\,(X_i > \rho) \leqq \varepsilon$, $1 \leqq i \leqq k$, *for some $\varepsilon > 0$ and some $0 \leqq \rho \leqq 1$. Then* $\mathscr{E}(X_1 \cdots X_k) \leqq \varepsilon + \rho^k$.

For $k = 2$, this result follows from number 378 of Hardy–Littlewood–Pólya (1964), and the method used there also can be used for general k.

(9.14) LEMMA. *Let μ be translation invariant with $\mu(\{\varnothing\}) = 0$. Given $\varepsilon > 0$, there exists an integer $k_1 = k_1(\varepsilon)$ such that $|\eta| \geqq k_1$ implies $h_{t_1}(\eta) \geqq 1 - 3\varepsilon$.*

PROOF. Since both μ and the transition function are translation invariant, we have

$$\mu\{\xi : \mathscr{E}_\xi(1 - \xi_{t_1}(x)) = 1\} = c\,,$$

where c does not depend on x. From 9.2(b), $c = 0$ and hence there exists $\rho_\varepsilon < 1$ such that

$$\mu\{\xi : \mathscr{E}_\xi(1 - \xi_{t_1}(x)) > \rho_\varepsilon\} < \varepsilon\,. \tag{9.15}$$

Now write

$$1 - h_{t_1}(\eta) = \int \mu(d\xi)\mathscr{E}_\xi \textstyle\prod_{x\in\eta}(1 - \xi_{t_1}(x))\,. \tag{9.16}$$

From (9.12), (9.13), (9.15) and (9.16) we have

$$1 - h_{t_1}(\eta) \leqq \int \mu(d\xi) \textstyle\prod_{x\in\eta}\mathscr{E}_\xi(1 - \xi_{t_1}(x)) + \varepsilon \leqq 2\varepsilon + \rho_\varepsilon^{|\eta|}\,,$$
$$2 \leqq |\eta| < \infty,\ \Delta(\eta) \geqq L(\varepsilon, |\eta|)\,.$$

Letting N_ε be the smallest integer such that $(\rho_\varepsilon)^{N_\varepsilon} \leqq \varepsilon$, we have

$$1 - h_{t_1}(\eta) \leqq 3\varepsilon \quad \text{if} \quad |\eta| = N_\varepsilon \quad \text{and} \quad \Delta(\eta) \geqq L(\varepsilon, N_\varepsilon)\,.$$

Let $k_1(\varepsilon)$ be an integer such that $|\eta| \geqq k_1$ implies the existence of $\eta' \subset \eta$ with $|\eta'| = N_\varepsilon$ and $\Delta(\eta') \geqq L(\varepsilon, N_\varepsilon)$. Then $|\eta| \geqq k_1$ implies $1 - h_{t_1}(\eta) \leqq 1 - h_{t_1}(\eta') \leqq 3\varepsilon$, and the lemma is proved. □

PROOF OF THEOREM 9.2. If $|\eta| < \infty$, $P_Z\{\xi_t \# \eta\} = P_\eta^*\{\xi_t^* \neq \varnothing\}$ decreases as t increases and hence has a limit. The corresponding weak limit ν is invariant because of the Feller property. This proves (i). If the measure ν of (i) is $\delta_\varnothing$, and if μ is any measure, then

$$\int \nu(d\xi)P_\xi\{\xi_t \# \eta\} \leqq \int \mu(d\xi)P_Z\{\xi_t \# \eta\} \to 0$$

if $|\eta| < \infty$, proving (ii). Here we used the fact, a consequence of the existence of ξ_t^*, that $P_\xi\{\xi_t \# \eta\}$ is monotone in ξ. Now suppose $\nu \neq \delta_\varnothing$. Putting $\nu(\{\varnothing\}) = c$, we show $c = 0$. Let $W_n = \{\xi : \xi(x) = 0 \text{ if } |x| \leqq n\}$, $n = 1, 2, \cdots$. Since $P(t, \varnothing, W_n) = 1$ we have

$$\begin{aligned}\nu(W_n) &= c + \textstyle\int_{\xi\neq\varnothing} \nu(d\xi)P(t, \xi, W_n)\\ &\geqq c + \textstyle\int_{\xi\neq\varnothing} \nu(d\xi)P(t, Z, W_n)\\ &= c + (1 - c)P(t, Z, W_n)\,.\end{aligned}$$

Letting $t \to \infty$, we have $\nu(W_n) \geqq c + (1 - c)\nu(W_n)$, or $c\nu(W_n) \geqq c$, $n = 1, 2, \cdots$. Since $\nu(W_n) < 1$ for some n, we must have $c = 0$.

Let $p_\infty^*(\eta) = P_\eta^*\{\xi_t^* \neq \varnothing\ \forall\, t\}$. This is $\lim_{t\to\infty} P_\eta^*\{|\xi_t^*| > k\}$ for each $k > 0$, in view of Lemma 9.3. Let μ be a translation invariant measure such that $\mu(\{\varnothing\}) = 0$. From Lemma 9.7 and Lemma 9.14 we have, given $\varepsilon > 0$,

$$\begin{aligned}(1 - 3\varepsilon)p_\infty^*(\eta) &\leqq \liminf_{s\to\infty} h_{t_1+s}(\eta) \leqq \limsup_{s\to\infty} h_{t_1+s}(\eta)\\ &\leqq p_\infty^*(\eta)\,, \qquad\qquad |\eta| \geqq k_1(\varepsilon)\,,\end{aligned}$$

showing that $\lim_{t\to\infty} h_t(\eta) = p_\infty^*(\eta)$. This limit is the same for all permissible μ and hence $\lim U_t\mu = \lim U_t\nu = \nu$.

Finally if μ is any translation invariant measure with $\mu(\{\emptyset\}) = c < 1$, put $\mu = c\delta_\emptyset + (1-c)\mu'$, where $\mu' = (1-c)^{-1}(\mu - c\delta_\emptyset)$ and $\mu'(\{\emptyset\}) = 0$. Since $U_t\delta_\emptyset = \delta_\emptyset$ and $U_t\mu' \to \nu$, the proof of Theorem 9.2 is complete. □

EXAMPLE. The process $\{\xi_t\}$ given by the table in Section 7 b satisfies the conditions of Theorem 9.2 if $\mu_1 > 0$, $\lambda \geqq \mu_0 - \mu_1 \geqq 0$, $\lambda > 0$, and the additive semigroup generated by N includes all of Z.

A one-sided example. The following example in Z_1, for which 9.2(b) does not hold, was suggested by David Griffeath, and the strengthened form given below for Theorem 9.2 in Z_1 arose in correspondence with him.

Consider a contact process $\{\xi_t\}$ in Z_1 where $N = \{-1\}$, $\lambda_1 = \lambda > 0$, $\mu > 0$. Thus $\{\xi_t\}$ can spread only to the right. If λ/μ is sufficiently large, there is a nontrivial invariant measure and $P_\xi\{\xi_t \neq \emptyset \text{ for all } t\} > 0$ if $\xi \neq \emptyset$. This can be shown by arguments somewhat like those of Section 9 of [4], although changes are required because of the one-sided nature of the present process. The following result covers $\{\xi_t\}$.

(9.17) THEOREM. *Let $d = 1$ and suppose* 9.2(a) *is true. Suppose for each $\xi \neq \emptyset$, $t > 0$, $x \in Z_1$ there exists $y \in Z_1$ such that $z \leqq y$ implies $P_{\xi+z}\{\xi_t(O) = 1\} > 0$. Then the conclusions of Theorem* 9.2 *hold.*

There is a similar result with $z \leqq y$ replaced by $z \geqq y$.

PROOF. We need only show (9.15). Thus it suffices to show that if μ is translation invariant and $\mu(\{\emptyset\}) = 0$, then

$$\mu\{\xi : \mathscr{E}_\xi(1 - \xi_t(x)) = 1\} = 0\,, \qquad x \in Z_1,\ t > 0\,. \tag{9.18}$$

Let $Y_x(\xi) = 0$ if $P_\xi(\xi_t(x) = 1) > 0$ and 1 if not. Since $P_\xi(\xi_t(x) = 1) = P_{\xi-x}(\xi_t(O) = 1)$, the sequence $\cdots Y_{-1}, Y_0, Y_1, \cdots$ is stationary (μ), and by assumption $Y_x(\xi) = 0$ for all sufficiently large x, for a.e. $(\mu)\xi$. Hence

$$\frac{Y_1 + Y_2 + \cdots + Y_n}{n} \to 0 \qquad \text{a.s.} \quad (\mu)\,,$$

implying $\int Y_x\, d\mu = 0$, which is (9.18). □

10. Extinction. There are several variants of the idea that a process ξ_t "becomes extinct," including (1) $P(t, \xi, \cdot)$ converges weakly to $\delta_\emptyset$ if $|\xi| < \infty$ or (1′) if $|\xi| \leqq \infty$, and (2) $P_\xi\{\xi_t = \emptyset \text{ eventually}\} = 1$ if $|\xi| < \infty$. Clearly (2) implies (1), but the relations among the three notions have not been clarified in general. Griffeath has pointed out that the 1-dimensional process discussed at the end of Section 9 is a case where (1) holds but (1′) and (2) do not, if λ/μ is sufficiently large.

If $\{\xi_t\}$ is self-associate, then (1′) and (2) are equivalent, as we find by removing the asterisks from the following statement.

(10.1) THEOREM. *Let* $\{\xi_t\}$ *be a Feller process* (*countable* Z) *having a Feller associate* $\{\xi_t^*\}$. *Then either* (a), (b) *and* (c) *all hold or none holds.*

(a) $\lim_{t\to\infty} P_\xi\{\xi_t = \emptyset\} = 1$, *all* $|\xi| < \infty$.
(b) $\{\xi_t^*\}$ *has no invariant measure different from* $\delta_\emptyset$.
(c) $P^*(t, \xi, \cdot)$ *converges weakly to* $\delta_\emptyset$ *for each* ξ, $|\xi| \leqq \infty$.

PROOF. The equivalence of (a) and (c) follows from $P_\xi^*\{\xi_t^* \# \eta\} \leqq P_Z^*\{\xi_t^* \# \eta\} = P_\eta\{\xi_t \neq \emptyset\}$, $|\eta| < \infty$. Suppose (b) is true. Arguing as below (9.19), $P^*(t, Z, \cdot)$ converges to an invariant ν which must be $\delta_\emptyset$, implying (c). If (c) is true, the inequality $P_\xi^*\{\xi_t^* \# \eta\} \leqq P_Z^*\{\xi_t^* \# \eta\}$ shows there is no time-invariant measure except $\delta_\emptyset$. □

11. Association and hypermonotonicity. In this section Z is finite and time is discrete. Let P be a transition matrix corresponding to a process $\xi_0, \xi_1, \cdots$. Let $\xi_n'(x) = 1 - \xi_n(x)$. The process $\{\xi_n'\}$ has the transition matrix P', where

$$P'f(\xi) = Pf'(1 - \xi)\,, \qquad f'(\xi) = f(1 - \xi)\,. \tag{11.1}$$

Let $l_\eta(\xi) = \prod_{x\in\eta} \xi(x)$, $l_\emptyset(\xi) = 1$. Then $l_\xi'(\eta) = 1 - \theta_\xi(\eta)$. From (11.1) and (2.10),

$$P'l_\xi = 1 - P\theta_\xi(Z) + \textstyle\sum_{\gamma\neq\emptyset} \psi P\theta_\xi(\{\gamma\})l_\gamma\,. \tag{11.2}$$

From this and Section 4.1 we find that $\{\xi_n\}$ has an associate iff $\{\xi_n'\}$ has the following properties: $\xi_0' = Z$ implies $\xi_n' = Z$, $n = 1, 2, \cdots$, and the expansion of each function $P'l_\xi$, $\xi \subset Z$, in terms of functions l_γ has nonnegative coefficients. If, as in [14], we have the independence property

$$P'l_\xi = \textstyle\prod_{x\in\xi} P'l_x\,, \tag{11.3}$$

we need only require that each expansion $P'l_x$, $x \in Z$, has nonnegative coefficients. But this is precisely the property of hypermonotonicity for $\{\xi_n'\}$. See [14].

Acknowledgments. I wish to acknowledge some very helpful suggestions by David Griffeath, particularly those mentioned in Sections 9 and 10. I am also indebted to Thomas Liggett for a helpful comment.

REFERENCES

[1] BROADBENT, S. R. and HAMMERSLEY, J. M. (1957). Percolation processes. *Proc. Combridge Philos. Soc.* **53** 629–641.
[2] GLAUBER, R. J. (1963). Time-dependent statistics of the Ising model. *J. Mathematical Phys.* **4** 294–307.
[3] GRIFFEATH, DAVID (1975). Ergodic theorems for graph interactions. *Advances in Appl. Probability* **7** 179–194.
[3a] HARDY, G. H., LITTLEWOOD, J. E. and POLYA, G. (1964). *Inequalities.* Cambridge Univ. Press.
[4] HARRIS, T. E. (1974). Contact interactions on a lattice. *Ann. Probability* **2** 969–988.
[5] HARRIS, T. E. (1974). Reciprocal processes. I: finite case. *Notices Amer. Math. Soc.* **21** A-502.

[6] HOLLEY, R. (1972). Markovian interaction processes with finite range interactions. *Ann. Math. Statist.* **43** 1961–1967.
[7] HOLLEY, R. and LIGGETT, T. (1975). Ergodic theorems for weakly interacting systems and the voter model. *Ann. Probability* **3** 643–663.
[8] LEE, W. C. (1974). Random stirring of the real line. *Ann. Probability* **2** 580–592.
[9] LIGGETT, T. (1972). Existence theorems for infinite particle systems. *Trans. Amer. Math. Soc.* **165** 471–481.
[10] ROTA, G.-C. (1964). On the foundations of combinatorial theory I, theory of Möbius functions. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **2** 340–368.
[11] SPITZER, F. (1970). Interaction of Markov processes. *Advances in Math.* **5** 246–290.
[12] VASERSHTEIN, L. N. and LEONTOVICH, A. M. (1970). On invariant measures of some Markov operators describing a homogeneous random medium. *Problems of the Transmission of Information* (in Russian) **6** No. 1, 71–80.
[13] VASIL'EV, N. B. (1969). On the limiting behavior of a random medium. *Problems of the Transformation of Information* (in Russian) **5** No. 4, 68–74.
[14] VASIL'EV, N. B., MITYUSHIN, L. G., PYATETESKIĬ-SHAPIRO, I. I. and TOOM, A. L. (1973; work dated 1972). Operators of Stavskaya (in Russian). Preprint 12, Inst. of Applied Math., Academy of Sciences, USSR, Moscow.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF SOUTHERN CALIFORNIA
UNIVERSITY PARK
LOS ANGELES, CALIFORNIA 90007