The Annals of Probability
1976, Vol. 4, No. 3, 490–496

THE LOCAL LIMIT THEOREM FOR THE GALTON–WATSON PROCESS

BY S. DUBUC AND E. SENETA

Université de Montréal and Australian National University

The usual form of local limit theorem is extended to an arbitrary supercritical Galton–Watson process with arbitrary initial distribution. The existence of a continuous density on $(0, \infty)$ for the limit random variable W, in the process initiated by a single ancestor, follows from the derivation.

1. Introduction. Let $\{Z_n\}$, $n \geqq 0$, be the Galton–Watson process initiated by a random number Z_0 of initial individuals, the probability generating function (pgf) of the law of reproduction being $f(s) = \sum_{k=0}^{\infty} p_k s^k$; and the pgf of the initial distribution $a(s) = \sum_{k=0}^{\infty} a_k s^k$, where $a_k = P[Z_0 = k]$. To avoid trivialities we suppose that the reproduction law is not degenerate, and also that $a_0 \neq 1$. We assume $m = f'(1-)$ satisfies $1 < m < \infty$.

Before formulating our principal assertion, we need certain known results. Let q be the probability of extinction of a process which begins with $Z_0 = 1$. In this case, Heyde [11] has shown that, for a sequence $\{c_n\}$, $n \geqq 0$, of constants $(c_n > 0)$ used by Seneta [16], Z_n/c_n converges almost surely to a proper nondegenerate random variable W, since $\{\exp(-Z_n/c_n)\}$ is a martingale; we work with these same constants. Dubuc [7], [8] and Athreya [1] have demonstrated that there exists a nonnegative function $w(t)$ on $(0, \infty)$ such that $\int_t^\infty w(u)\,du = P[W > t]$ if $t \in (0, \infty)$; indeed, according to Dubuc, the density $w(t) > 0$ and is continuous in $t \in (0, \infty)$. We shall designate by W_k the sum of k independent copies of W. W_k takes the value 0 with probability q^k, but otherwise $P[W_k > t] = \int_t^\infty w_k(u)\,du$, where w_k is a continuous function defined on $(0, \infty)$; specifically,

$$w_k(t) = \sum_{j=1}^{k} \binom{k}{j} q^{k-j} w^{*j}(t)$$

where w^{*j} is the jth order of convolution of the function w. We need two further characteristics of the process. The period L of a Galton–Watson process is the greatest common divisor of integers of the form $h - s$, where $p_h \neq 0$ and $p_s \neq 0$. We say the process is of type (L, r) if L is the period and r is the residue (mod L) of any k for which $p_k \neq 0$. (The more usual definition of r as the first integer k for which $p_k \neq 0$, though not in general equivalent, can just as easily be used in the sequel also.) The general form of local limit theorem, under the assumptions of the preceding paragraph, is:

THEOREM. *If the supercritical process is of type* (L, r) *and if* $\{y_n\}$ *is a sequence*

Received May 15, 1975.
AMS 1970 *subject classifications.* Primary 60J80; Secondary 60E05.
Key words and phrases. Supercritical branching process, general norming constants, limit density, local limit theorem, subcritical analogue, characteristic functions.

of integers such that $y_n/c_n \to t$, *a positive number, then*

$$\lim_{n\to\infty} (c_n P[Z_n = y_n] - L \sum_{k\in I_n} a_k w_k(t)) = 0$$

where $I_n = \{k : k > 0$ *and* $kr^n \equiv y_n \pmod L\}$.

This local limit theorem has a chequered history. Several authors have sought to demonstrate it under more restrictive hypotheses. In 1957, Čistyakov [4] gave such a theorem for the analogous continuous-time (Markov) branching process, and stated a corresponding discrete-time version, but an incorrect estimate cast the issue into doubt. (He later claimed [5] that the result could be salvaged.) Imai [12] made an effort in the same direction in 1968, but this turned out to need his correction [13] also. Finally, and still under a finite variance condition, Athreya and Ney [2] have proved such a local limit theorem in 1970; but their proof will fail if $L > 1$. The same authors have observed that Dubuc [6], in the course of study of the Green's function of the Galton–Watson process, had demonstrated a local limit theorem. In their book [3], they subsequently establish the local limit theorem under the hypotheses that $\sum_{k=1}^{\infty} p_k k \log k < \infty$, $L = 1$, $a_i = 1$. Athreya [1] mentions the possibility of relaxing the first of these.

Two concluding remarks are in order. At no extra labour, we shall actually derive, in Lemma 9, the existence (by construction) of a continuous density $w(t)$ on $(0, \infty)$ for W, and do not make use of its existence prior to this. Following the proof of our theorem in the final section, we mention a trivial analogue of it for the subcritical case $(0 < m < 1)$.

2. Lemmas and the limit density. As usual, we write $f_n(s)$ for the nth functional iterate of $f(s)$. Put $g(\xi) = E(\exp -\xi W)$ for ξ a complex number with positive real part. We know that $g(\xi) = \lim_{n\to\infty} f_n(\exp -\xi/c_n)$; that $c_{n+1}/c_n < m$, $c_{n+1}/c_n \uparrow m$ and $c_n \uparrow \infty$ as $n \to \infty$.

LEMMA 1. *If* $\operatorname{Re} \xi \geqq 0$ *and* $\xi \neq 0$, *then* $|g(\xi)| < 1$. *As* $\theta \to \pm\infty$, $g(-i\theta) \to q$.

PROOF. Clearly $|g(\xi)| \leqq g(\operatorname{Re} \xi)$; < 1 if $\operatorname{Re} \xi > 0$. If $\operatorname{Re} \xi = 0$ but $\xi \neq 0$, the result that $|g(\xi)| < 1$ follows from a standard argument based on $g(m\xi) = f(g(\xi))$ (e.g., Stigum [19], Lemma 1). The second result also follows from the functional equation, after Sevastyanov transformation to render $q = 0$ (Stigum, Lemma 2). □

LEMMA 2. *For each* $\varepsilon > 0$, $\sup \{|f_n(e^{i\theta})| : n = 1, 2, \cdots; \varepsilon/c_n \leqq |\theta| \leqq \pi/L\} < 1$.

PROOF. $f_n(e^{it/c_n})$ converges to $g(-it)$ uniformly for t in a finite interval (Loève [15], page 191). Thus we can find a number $r \in (0, 1)$ and an N such $|f_n(e^{it/c_n})| \leqq r$ if $n \geqq N$ and $\varepsilon \leqq |t| \leqq m\varepsilon$. Now if $\varepsilon/c_k \leqq |\theta| \leqq \varepsilon/c_{k-1}$, then for $k \geqq N$, $|f_k(e^{i\theta})| \leqq r$. If $D_r = \{s : |s| \leqq r\}$, then the sequence $\{f_p\}$, $p \geqq 0$, converges uniformly on D_r to the constant function q, whence there exists an $R < 1$ such that $|f_p(w)| \leqq R$ for $|w| \leqq r$, and $p \geqq 0$. Hence $|f_{k+p}(e^{i\theta})| \leqq R$. Thus we have shown that

$|f_n(e^{i\theta})| \leqq R$ if $\varepsilon/c_n \leqq |\theta| \leqq \varepsilon/c_{N-1}$, $n \geqq N$. On the other hand, $\{e^{i\theta} : \varepsilon/c_{N-1} \leqq |\theta| \leqq \pi/L\}$ is a compact subset, with does not contain the Lth roots of unity, of the unit disc. It is thus easy to find an S such that $|f_n(e^{i\theta})| < S < 1$, $n \geqq 1$, and $\varepsilon/c_{N-1} \leqq |\theta| \leqq \pi/L$. □

In the sequel we shall need the sequence of intervals

$$J_k = \{\theta : \pi c_0 L^{-1} c_k^{-1} \leqq |\theta| \leqq \pi c_0 L^{-1} c_{k-1}^{-1}\}\,, \qquad k \geqq 1\,.$$

LEMMA 3. *If $\mu \equiv f'(q) \neq 0$, there exists a constant A (depending only on $a(s)$ and $f(s)$) such that $|a(f_n(e^{i\theta})) - a(q)| \leqq A\mu^{n-k}$ for $n \geqq k$ and all θ in J_k. If $f'(q) = 0$, then for each $\varepsilon > 0$ there exists a constant A_ε (depending only on ε, $a(s)$ and $f(s)$) such that $|a(f_n(e^{i\theta})) - a(0)| \leqq A_\varepsilon \varepsilon^{n-k}$ for $n \geqq k$ and all θ in J_k.*

PROOF. Consider first the case $a(s) = s$. By Lemma 2, $\bigcup_{k=1}^{\infty} \{f_k(e^{i\theta}) : \theta \in J_k\}$ is contained in a compact subset of $\{z : |z| < 1\}$. Now, on K, the sequence $(f_p(z) - q)\mu^{-p}$ converges uniformly; so that there exists a constant B such that $|f_p(z) - q| \leqq B\mu^p$ for $z \in K$ and $p \geqq 0$. Hence $|f_{k+p}(e^{i\theta}) - q| \leqq B\mu^p$ for all $\theta \in J_k$.

In the case of arbitrary $a(s)$, we have again by Lemma 2, that $\{f_n(e^{i\theta}) : \theta \in J_k, n \geqq k\}$ is contained in a compact K_1 of $\{z : |z| < 1\}$. The function $a(s)$ being continuously differentiable on K_1, we have $|a(w) - a(q)| \leqq C|w - q|$ for all $w \in K_1$. Thus $|a(f_n(e^{i\theta})) - a(q)| \leqq BC\mu^{n-k}$ if $n \geqq k$ and $\theta \in J_k$.

The second part of the lemma follows analogously after observing that in this case $q = 0$ and $f_p(z) = 0(\varepsilon^p)$ for any $\varepsilon > 0$, the 0 being uniform in z varying in a compact of $\{z : |z| < 1\}$. □

LEMMA 4. *The function $(1 - g(s))/s$ is slowly varying as $s \to 0+$, where $g(s) = E(\exp -sW)$, $s \geqq 0$.*

PROOF. Demonstrated in the course of proof of Theorem 1 of Seneta [17], pages 409–410. □

LEMMA 5. *There is function $V(s)$ slowly varying as $s \to 0+$, such that for all $s \in (0, 1)$, and all n, $1 - f_n(e^{-s/c_n}) \leqq sV(s)$.*

PROOF. Since $\{\exp -Z_n/c_n\}$ is a martingale when $Z_0 = 1$, then (the function $t \to t^s$ being concave for $s \in (0, 1)$) $\{\exp -sZ_n/c_n\}$ is a supermartingale for $s \in (0, 1)$. Then $f_n(e^{-s/c_n}) = E(e^{-sZ_n/c_n}) \geqq E(e^{-sW}) = g(s)$; where $1 - f_n(e^{-s/c_n}) \leqq 1 - g(s)$, and Lemma 4 yields the conclusion. □

LEMMA 6. *Let $\phi(\xi) = \int_0^\infty e^{-\xi t}\, d\mu(t)$ where ξ is complex with nonnegative real part, and $\mu(\cdot)$ a probability measure on $[0, \infty)$. Then there exists a universal constant C such that $|\phi(ia) - \phi(ib)| \leqq C(1 - \phi(|b - a|))$ for any pair (a, b) of real numbers.*

PROOF. $|\phi(ib) - \phi(ia)| = |\int_0^\infty (e^{ibt} - e^{iat})\, d\mu(t)| \leqq 2 \int_0^\infty |\sin((b - a)t/2)|\, d\mu(t)$. Thus $C \equiv 2 \sup\{|\sin(x/2)|/(1 - e^{-x}) : x \geqq 0\} < \infty$, will serve; for

$$|\phi(ib) - \phi(ia)| \leqq C \int_0^\infty (1 - e^{-|b-a|t})\, d\mu(t) = C(1 - \phi(|b - a|))\,. \qquad \square$$

LEMMA 7. *If* $\omega_{k,n}(\delta; f, a) = \sup\{|a(f_n(e^{i\theta_1})) - a(f_n(e^{i\theta_2}))| : |\theta_1 - \theta_2| \leqq \delta, \theta_i \in J_k\}$, *and* $\mu \equiv f'(q) \neq 0$, *then there exists a function* $V^*(x)$ *defined on* $(0, \infty)$ *and such that* (a) $V^*(x)$ *is slowly varying as* $x \to 0+$; (b) $V^*(x)$ *is bounded on every interval* (ε, ∞), $\varepsilon > 0$; (c) *for each* $n \geqq k$ *and* $\delta > 0$

$$\omega_{k,n}(\delta; f, a) \leqq c_k \delta \mu^{n-k} V^*(c_k \delta) .$$

PROOF. Consider first the case $a(s) = a_1(s) \equiv s$; and put $\omega_{k,n}(\delta; f) = \omega_{k,n}(\delta; f, a_1)$. If $\theta_1, \theta_2 \in J_k$, Lemma 6 yields

$$|f_k(e^{i\theta_1}) - f_k(e^{i\theta_2})| \leqq C(1 - f_k(e^{-\delta})) .$$

When $c_k \delta \leqq 1$, by Lemma 5, $1 - f_k(e^{-\delta}) \leqq c_k \delta V(c_k \delta)$. It then suffices to put $V_1(x) = CV(x)$ if $x \leqq 1$ and $V_1(x) = C$ if $x > 1$, to obtain $|f_k(e^{i\theta_1}) - f_k(e^{i\theta_2})| \leqq c_k \delta V_1(c_k \delta)$. Now, we know that there is a constant E_K such that $|f_p(z_1) - f_p(z_2)| \leqq E_K \mu^p |z_1 - z_2|$ if z_1 and z_2 vary in a compact subset K of $\{z : |z| < 1\}$. If K is a compact containing $\{f_k(e^{i\theta}) : \theta \in J_k\}$ and we put $V^*(x) = E_K V_1(x)$, then

$$|f_n(e^{i\theta_1}) - f_n(e^{i\theta_2})| \leqq c_k \delta \mu^{n-k} V^*(c_k \delta) .$$

For a general pgf $a(s)$, we observe that $\{f_n(e^{i\theta}) : n \geqq k, \theta \in J_k\}$ is contained in a compact K_1 of $\{z : |z| < 1\}$ in view of Lemma 2. If $F = \sup\{|a(z_1) - a(z_2)| : z_i \in K_1\}$, then $\omega_{k,n}(\delta; f, a) \leqq F\omega_{k,n}(\delta; f)$. □

LEMMA 8. *If* $\phi(\xi)$ *is the Fourier transform* (*characteristic function*) *of a finite measure*, $\mu(\cdot)$, *on* $\mathscr{R}$, *with* $\lim (|\xi| \to \infty)\phi(\xi) = q$, *and the sequence*

$$(1/2\pi) \int_{-\tau_n}^{\tau_n} (\phi(\xi) - q)e^{-it\xi}\, d\xi$$

converges uniformly to a (*necessarily continuous*) *function* $w(t)$ *on every interval* $[a, b]$ *not containing the origin, where* $\{\tau_n\}$ *is any fixed sequence of positive numbers with* $\tau_n \to \infty$ *as* $n \to \infty$, *then* $d\mu(t) = q\delta_0(t) + w(t)\, dt$ *where* $\delta_0(\cdot)$ *is the Dirac delta function* (*ascribing unit mass of the origin, zero mass elsewhere*).

PROOF (motivated by page 284 of [9]). By the uniform convergence

$$\begin{aligned}\int_a^b w(t)\, dt &= \int_a^b (\lim_{n\to\infty} (1/2\pi) \int_{-\tau_n}^{\tau_n} (\phi(\xi) - q)e^{-it\xi}\, d\xi)\, dt \\ &= \lim_{n\to\infty} (1/2\pi) \int_{-\tau_n}^{\tau_n} \{(\phi(\xi) - q)(e^{-i\xi a} - e^{-i\xi b})/i\xi\}\, d\xi \,; \\ &= \mu([a, b)) ,\end{aligned}$$

if a, b are continuity points of $\mu(\cdot)$ (Loève [15], page 186). □

LEMMA 9. *If* $0 < \alpha < \beta$, *the sequence of functions*

$$(1/2\pi) \int_{-\pi c_n/L}^{\pi c_n/L} (a(f_n(e^{i\xi/c_n})) - a(q))e^{-it\xi}\, d\xi$$

converges uniformly on $[\alpha, \beta]$ *to the function* $\sum_{k=1}^{\infty} a_k w_k(t)$.

PROOF. We verify the uniform convergence first, identifying the limit subsequently. Put $k_n(\xi, t) = (1/2\pi)(a(f_n(e^{i\xi/c_n})) - a(q))e^{-it\xi}$, and $I_{k,n}(t)$ for its integral over ξ such that $\xi/c_n \in J_{n-k}$ for $k = 0, 1, \cdots, n - 1$.

(a) We first examine the case where $0 < mf'(q) < 1$. Making use of the inequality $|\int_a^b h(\xi)e^{-it\xi}\,d\xi| \leqq |b-a|\,||h||_\infty$, we see since

$$I_{k,n}(t) = \int_{-\pi c_0 L^{-1} c_n/c_{n-k-1}}^{-\pi c_0 L^{-1} c_n/c_{n-k}} k_n(\xi, t)\,d\xi + \int_{\pi c_0 L^{-1} c_n/c_{n-k}}^{\pi c_0 L^{-1} c_n/c_{n-k-1}} k_n(\xi, t)\,d\xi$$

that $|I_{k,n}(t)| \leqq c_0 L^{-1}(c_n/c_{n-k-1})A\mu^k$ in view of Lemma 3.

Now, since $\pi c_0 L^{-1} c_n/c_{n-k} \uparrow \pi c_0 L^{-1} m^k$ as $n \to \infty$, with $(a(f_n(e^{i\xi/c_n})) - a(q)) \to a(g(-i\xi)) - a(q)$, uniformly in ξ for any finite fixed interval, it follows that

$$\lim_{n\to\infty} I_{k,n}(t) = (1/2\pi)\int_{-\pi c_0 L^{-1} m^{k+1}}^{-\pi c_0 L^{-1} m^k} (a(g(-i\xi)) - a(q))e^{-it\xi}\,d\xi + (1/2\pi)\int_{\pi c_0 L^{-1} m^k}^{\pi c_0 L^{-1} m^{k+1}} (a(g(-i\xi)) - a(q))e^{it\xi}\,d\xi\,.$$

Since $(c_n/c_{n-k-1}) \leqq m^{k+1}$, $\sum_{k=0}^\infty \sup_{n>k} ||I_{k,n}(t)||_\infty < \infty$, and Lebesgue's dominated convergence criterion, together with Weierstrass' M-criterion, shows that

$$\begin{aligned}\int_{-\pi c_0 L^{-1}}^{\pi c_0 L^{-1}} k_n(\xi, t)\,d\xi &+ \sum_{k=0}^{n-1} I_{k,n}(t)\\ &= \int_{-\pi c_0 L^{-1}}^{\pi c_0 L^{-1}} k_n(\xi, t)\,d\xi + \int_{-\pi L^{-1} c_n}^{-\pi c_0 L^{-1}} k_n(\xi, t)\,d\xi + \int_{\pi c_0 L^{-1}}^{\pi L^{-1} c_n} k_n(\xi, t)\,d\xi\\ &= \int_{-\pi L^{-1} c_n}^{\pi L^{-1} c_n} k_n(\xi, t)\,d\xi\end{aligned}$$

converges uniformly in all t to

$$\lim_{k\to\infty} (1/2\pi)\int_{-\tau_k}^{\tau_k} (a(g(-i\xi)) - a(q))e^{-i\xi t}\,d\xi\,,$$

(where $\tau_k = \pi c_0 L^{-1} m^k$), in which expression the convergence is also uniform in t.

(b) The case $f'(q) = 0$ (i.e., $p_0 = p_1 = 0$) is treated analogously.

(c) The most difficult case is $f'(q)m \geqq 1$; we shall employ the known inequality (derived via the device of, e.g., Zygmund [20], Section 2.21):

$$|\int_a^b h(\xi)e^{-it\xi}\,d\xi| \leqq ||h||_\infty \pi/|t| + \{(b-a)/2\}\omega(\pi|t|^{-1}, h, a, b)$$

where

$$\omega(\delta, h, a, b) = \sup\{|h(t_1) - h(t_2)| : a \leqq t_1 \leqq t_2 \leqq b \text{ and } t_2 - t_1 \leqq \delta\}\,.$$

To use the inequality, let us majorize the quantities $|I_{k,n}(t)|$ for values of t outside the interval $[-\varepsilon, \varepsilon]$, $\varepsilon > 0$. By Lemmas 3 and 7 we obtain

$$|I_{k,n}(t)| \leqq |t|^{-1}A\mu^k + (1/2)c_0 L^{-1}\pi|t|^{-1}m\mu^k V^*(c_{n-k}\pi|t|^{-1}c_n{}^{-1})$$

from which it follows easily that $\sum_{k=0}^\infty \sup_{n>k}\sup_{|t|\geqq\varepsilon} |I_{k,n}(t)| < \infty$. This yields the conclusion of (a), except that the convergence is now only uniform outside every neighbourhood of $t = 0$.

If we put $a(s) = s$ we now deduce from this conclusion and Lemmas 1 and 8 that W has a limit distribution with probability q at the origin and continuous density $w(t)$ over $(0, \infty)$. The result of Lemma 9 with arbitrary $a(s)$ follows again from Lemmas 1 and 8 by recognizing that $a(g(-i\xi))$ is the characteristic function of $a(q)\delta_0(t) + \sum_{k=1}^\infty a_k w_k(t)$. □

3. The local limit theorem. We now prove the theorem of Section 1. Initially consider the distribution $\{a_k\}$, $k \geqq 0$, to be such that $a_k = 0$ if $k \not\equiv j$

(mod L), where j is a fixed integer in $[1, L]$. If the process is of type (L, r) and if $jr^n \not\equiv y \pmod L$, it is easily seen that $P[Z_n = y] = 0$ (e.g., Dubuc [10]). Let $\{y_n\}$ be a sequence of positive integers such that $jr^n \equiv y_n \pmod L$ and $\lim_{n\to\infty} y_n/c_n = t > 0$. Then

$$P[Z_n = y_n] = (1/2\pi) \int_{-\pi}^{\pi} a(f_n(e^{i\theta}))e^{-iy_n\theta}\, d\theta\ .$$

If we put $\omega = \exp(2\pi i/L)$, it follows that $f(\omega s) = \omega^r f(s)$ for any complex number s of the unit disc, whence $f_n(\omega s) = \omega^{r^n} f_n(s)$ and $f_n(\omega^m s) = \omega^{mr^n} f_n(s)$. Then

$$\begin{aligned}\int_{-\pi}^{\pi} a(f_n(e^{i\theta}))e^{-iy_n\theta}\, d\theta &= \sum_{m=0}^{L-1} \int_{(2m-1)\pi L^{-1}}^{(2m+1)\pi L^{-1}} a(f_n(e^{i\theta}))e^{-iy_n\theta}\, d\theta \\ &= \sum_{m=0}^{L-1} \omega^{jmr^n - my_n} \int_{-\pi/L}^{\pi/L} a(f_n(e^{i\theta}))e^{-iy_n\theta}\, d\theta \\ &= L \int_{-\pi/L}^{\pi/L} a(f_n(e^{i\theta}))e^{-iy_n\theta}\, d\theta\ .\end{aligned}$$

If $q \neq 0$, then $p_0 = 0$, $r = 0$ and $y_n \equiv 0 \pmod L$. If $q = 0$ and $a(0) \neq 0$, then $j = L$ and $y_n \equiv 0 \pmod L$, whence $\int_{-\pi/L}^{\pi/L} a(q)e^{-iy_n\theta}\, d\theta = 0$. Thus

$$P[Z_n = y_n] = (L/2\pi) \int_{-\pi/L}^{\pi/L} (a(f_n(e^{i\theta})) - a(q))e^{-iy_n\theta}\, d\theta\ .$$

Lemma 9 yields the conclusion of the theorem.

The case of arbitrary initial distribution is now tractable by linearity. □

In the process where $0 < m < 1$ and $Z_0 = 1$ (an arbitrary initial distribution introduces complications), let $g(s)$, $s \in [0, 1]$ denote the pgf of the proper non-degenerate limit distribution as $n \to \infty$ of $P[Z_n = i \mid Z_n > 0]$, $i \geqq 1$ (see, e.g., [3] I.8). It is known from other contexts [18] that $c_n = \{1 - f_n(0)\}^{-1}$, $n \geqq 0$, provides a sequence with properties analogous to those in the supercritical case. Of present interest is the local limit result: if $\{y_n\}$ is a sequence of positive integers such that $y_n/c_n \to t > 0$ as $n \to \infty$, then $P[Z_n = i \mid Z_0 = y_n]$, $i \geqq 1$, has limit distribution with pgf $\exp\{-t(1 - g(s))\}$. This result, obtained in 1967 by Joffe and Spitzer [14] under the assumption $\sum p_k k \log k < \infty$ (see also [3], page 45), requires no substantial extension of their proof.

REFERENCES

[1] Athreya, K. B. (1971). On the absolute continuity of the limit random variable in the supercritical Galton-Watson branching process. *Proc. Amer. Math. Soc.* **30** 563-565.

[2] Athreya, K. B. and Ney, P. (1970). The local limit theorem and some related aspects of supercritical branching processes. *Trans. Amer. Math. Soc.* **152** 233-251.

[3] Athreya, K. B. and Ney, P. E. (1972). *Branching Processes.* Springer-Verlag, Berlin.

[4] Čistyakov, V. P. (1957). Local limit theorems for branching processes. *Teor. Veroiatnost. i Primenen.* **2** 360-374. (*Translated in Theor. Probability Appl.* **2** 345-364.)

[5] Čistyakov, V. P. (1965). Letter to the Editor. *Teor. Veroiatnost. i Primenen.* **10** 597-598. (*Translated in Theor. Probability Appl.* **10** 538-539.)

[6] Dubuc, S. (1970). La fonction de Green d'un processus de Galton-Watson. *Studia Math.* **34** 69-87.

[7] Dubuc, S. (1971). Problèmes relatifs à l'iteration des fonctions suggérés par la processus en cascade. *Ann. Inst. Fourier (Grenoble)* **21** 171-251.

[8] Dubuc, S. (1971). Processus de Galton-Watson surcritiques. *Séminaire d'Analyse Moderne No. 7.* Departement de Mathématiques, Université de Sherbrooke. Sherbrooke, Canada.

[9] DUBUC, S. (1971). La densité de la loi-limite d'un processus en cascade expansif. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **19** 281-290.

[10] DUBUC, S. (1974). États accessibles dans un processus de Galton-Watson. *Canad. Math. Bull.* **17** 111-113.

[11] HEYDE, C. C. (1970). Extension of a result of Seneta for the supercritical Galton-Watson process. *Ann. Math. Statist.* **41** 739-742.

[12] IMAI, H. (1968). Notes on a local limit theorem for discrete time Galton-Watson branching processes. *Ann. Inst. Statist. Math.* **20** 391-410.

[13] IMAI, H. (1973). Remarks to a local limit theorem for Galton-Watson processes. *Ann. Inst. Statist. Math.* **25** 453-455.

[14] JOFFE, A. and SPITZER, F. (1967). On multitype branching processes with $\rho \leqq 1$. *J. Math. Anal. Appl.* **19** 409-430.

[15] LOÈVE, M. (1963). *Probability Theory*, 3rd ed. Van Nostrand, Princeton.

[16] SENETA, E. (1968). On recent theorems concerning the supercritical Galton-Watson process. *Ann. Math. Statist.* **39** 2098-2102.

[17] SENETA, E. (1974). Regularly varying functions in the theory of simple branching processes. *Advances in Appl. Probability* **6** 408-420.

[18] SENETA, E. (1975). Characterization by functional equations of branching process limit laws. In *Statistical Distributions in Scientific Work* (G. P. Patil, S. Kotz, J. K. Ord, eds.) **3** Characterizations and Applications, 249-254. Reidel, Dordrecht.

[19] STIGUM, B. P. (1966). A theorem on the Galton-Watson process. *Ann. Math. Statist.* **37** 695-698.

[20] ZYGMUND, A. (1955). *Trigonometrical Series*. Dover, New York.

CENTRE DE RECHERCHES MATHÉMATIQUES
UNIVERSITÉ DE MONTRÉAL
CASE POSTALE 6128
MONTRÉAL H3C 3J7
CANADA

DEPARTMENT OF STATISTICS
AUSTRALIAN NATIONAL UNIVERSITY
P.O. BOX 4
CANBERRA, A.C.T. 2600
AUSTRALIA