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STRONG LIMIT THEOREMS FOR CERTAIN ARRAYS OF RANDOM VARIABLES[1]

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A lemma concerning real sequences is proved and applied to sequences of random variables (rv) $X_1, X_2 \cdots$ to determine conditions under which $\limsup_{n\to\infty} b_n^{-1} \sum_{m=1}^n f(m/n)X_m < \infty$ a.s. for all f in a particular collection of absolutely continuous functions and for nondecreasing positive real sequences $\{b_n\}$. Theorems in the case $b_n = (2n \log \log n)^{\frac{1}{2}}$ are proved for generalized Gaussian rv, for equinormed multiplicative systems and for certain martingale difference sequences.

1. Introduction. Let $X_1, X_2, \cdots$ be random variables (rv) defined on a probability space $(\Omega, \mathscr{F}, P)$. Let $\mathscr{C}$ be the set of real-valued continuous functions defined on $[0, 1]$. For $f \in \mathscr{C}$ and $n \geqq 1$, define the rv

$$W_n(f) \equiv \sum_{m=1}^n f(m/n)X_m . \tag{1}$$

In Sections 3 and 4, hypotheses about f and the rv $\{X_n\}$ will be presented which will determine an almost sure (a.s.) upper bound for the rv $\limsup_{n\to\infty} b_n^{-1}W_n(f)$, where $\{b_n\}$ is a nondecreasing positive real sequence. This problem has been examined in two of the author's previous papers (Tomkins (1971) and (1975)) which respectively examined the case in which $\{X_n\}$ are independent and in which $\{X_n\}$ form a martingale difference sequence. The theorems in this article deal with more general rv.

Earlier papers (for example, Gaposhkin (1965) and the two papers just cited) have tackled this problem with a two-step procedure: (1) for a suitably chosen integral sequence $\{n_k\}$, find an upper bound for $\limsup_{k\to\infty} b_{n_k}^{-1} W_{n_k}(f)$ and then (2) show that the rv $\{b_n^{-1}W_n, n_k < n < n_{k+1}\}$ do not vary significantly from $b_{n_k}^{-1} W_{n_k}(f)$. This method has been utilized successfully to prove limit theorems, such as laws of the iterated logarithm (LIL), for partial sums $S_n \equiv \sum_{j=1}^n X_j$. In the problem at hand, it is often easy enough to carry out step (1), but, lacking some Lévy-type inequality, it is somewhat more arduous to perform step (2).

The novelty of this paper lies in the fact that, it will be shown, step (2) may be replaced by a procedure which seeks only knowledge about the partial sums $\{S_n\}$; the new procedure will be detailed in Lemma 1 (in Section 2). That lemma is actually a result in the theory of real sequences but will be stated in terms of rv for purposes of application.

In Section 3, methods used by Gaposhkin (1969), Révész (1972) and Tomkins

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(1975) will be combined with Lemma 1 to determine the a.s. value of $\limsup_{n\to\infty} (2n \log\log n)^{-\frac{1}{2}} W_n(f)$ when $X_1, X_2, \cdots$ form an equinormed strongly multiplicative system. Section 4 will employ some known LIL and strong law of large numbers (SLLN) theorems in conjunction with Lemma 1 to produce results of the desired type for some rather general classes of rv.

2. Some definitions and a fundamental lemma. Let $\mathscr{P}$ and $\mathscr{A}$ be subsets of $\mathscr{C}$ denoting, respectively, the set of polynomials and the collection of absolutely continuous functions on [0, 1]. For $f \in \mathscr{A}$ let V_f be the total variation of f over [0, 1] and define the norm $N(f) \equiv V_f + |f(1)|$. Let $\mathscr{Q} \equiv \{f \in \mathscr{C}: N(f - p_i) \to 0$ as $i \to \infty$ for some $p_1, p_2 \cdots \in \mathscr{P}\}$. By a remark of Tomkins (1975), $\mathscr{P} \subseteqq \mathscr{Q} \subseteqq \mathscr{A}$.

Let $||f||_2$ be the L_2-norm of $f \in \mathscr{Q}$. Write "$a_n \sim b_n$" when $a_n/b_n \to 1$. Let $[x]$ be the integral part of x.

Note that, as shown in the proof of Theorem 1 of Tomkins (1975), if $p_i \in \mathscr{P}$ satisfy $N(f - p_i) \to 0$, then $||p_i||_2 \to ||f||_2$.

The following lemma is basic to the considerations of this paper.

LEMMA 1. *Let $X_1, X_2, \cdots$ be* rv *and let $\{b_n\}$ be a nondecreasing positive real sequence satisfying, for some $\beta > 0$,*

(2) $$\limsup_{n\to\infty} b_n^{-1}|X_1 + \cdots + X_n| \leqq \beta \quad \text{a.s.}$$

Let $f \in \mathscr{C}$ and define $W_n(f)$ by (1).

(i) *Then*

$$\limsup_{n\to\infty} b_n^{-1}|W_n(f)| \leqq \beta N(f) \quad \text{a.s.}$$

(ii) *Suppose $f \in \mathscr{Q}$,* i.e. *for some sequence $\{p_i\} \in \mathscr{P}$,*

(3) $$\lim_{i\to\infty} N(f - p_i) = 0 .$$

For $c > 1$ and $k \geqq 1$, let $n_k \equiv n_k(c) \equiv [c^k]$. Suppose that, for every $\gamma > 0$, there exist a number $\eta = \eta(\gamma) > 0$ and a positive real sequence $\{\alpha_i\}$, $i \geqq 1$, such that, for every $c \in (1, 1 + \eta)$,

(4) $$\limsup_{k\to\infty} \max_{n_{k-1} < n \leqq n_k} b_n^{-1}|X_{n_{k-1}+1} + \cdots + X_n| < \gamma \quad \text{a.s.}$$

and

(5) $$\limsup_{k\to\infty} b_{n_k}^{-1} W_{n_k}(p_i) \leqq \alpha_i \quad \text{a.s.} \qquad \textit{for all large } i.$$

Let $\alpha \equiv \liminf_{i\to\infty} \alpha_i$. Then

$$\limsup_{n\to\infty} b_n^{-1} W_n(f) \leqq \alpha\beta \quad \text{a.s.}$$

(iii) *Assume $f \in \mathscr{Q}$ and that* (3) *holds. Let $\{n_k\}$, $k \geqq 1$, be an increasing sequence of positive integers satisfying $n_{k+1} \sim n_k$. If* (5) *holds and if*

$$\lim_{k\to\infty} \max_{n_{k-1} < n \leqq n_k} b_n^{-1}|X_{n_{k-1}+1} + \cdots + X_n| = 0 \quad \text{a.s.},$$

then

$$\limsup_{n\to\infty} b_n^{-1} W_n(f) \leqq \beta \liminf_{i\to\infty} \alpha_i \quad \text{a.s.}$$

PROOF. Part (i) can be obtained by letting $a_{nm} = f(m/n)$ for $n \geqq 1$, $m \leqq n$ and applying Lemma 3 of Tomkins (1975).

Consider parts (ii) and (iii). In either case, it will be shown first that, for all large i,

$$\limsup_{n\to\infty} b_n^{-1} W_n(p_i) \leqq \alpha_i \beta \quad \text{a.s.} \tag{6}$$

Since we may replace X_k by $X_k \beta^{-1}$ and p_i by $\beta p_i \alpha_i^{-1}$, there is no harm in assuming $\alpha_i = \beta = 1$ for all i. Moreover, one may assume throughout that (5) holds for all $i \geqq 1$. Fix $i \geqq 1$, temporarily. Suppose $p_i(x) = \sum_{j=0}^{p} a_j x^j$. Since (6) is trivial if $a_j = 0$, $0 \leqq j \leqq p$, assume otherwise and let $A \equiv \sum_{j=0}^{p} |a_j| > 0$. For brevity, let $S_n = \sum_{m=1}^{n} X_m$, $n \geqq 1$. Note that $n_{k+1} \sim cn_k$ in case (ii), so $\lim_{k\to\infty} n_{k+1}/n_k \leqq c$ in either case.

Now, let $0 < \varepsilon < A$ and define $\gamma \equiv A^{-1}\varepsilon$. Choose c to satisfy

$$1 < c < 1 + \eta\,, \qquad c^p(1-\gamma) < 1 \quad \text{and} \quad 2(1+\varepsilon) \textstyle\sum_{j=0}^{p} |a_j|(1 - c^{-j}) < \varepsilon\,. \tag{7}$$

There exists an event $E \in \mathscr{F}$ such that $P(E) = 1$ and (2), (4) and (5) hold for all $\omega \in E$. For each $\omega \in E$, there is a number $K = K(\omega) > 0$ such that, for all $k \geqq K$,

$$\begin{aligned} &n_{k-1} < n_k\,, \qquad \max_{n_{k-1} < n \leqq n_k} b_n^{-1}|S_n - S_{n_{k-1}}| \leqq \gamma\,, \\ &W_{n_k}(p_i) \leqq (1+\varepsilon) b_{n_k}\,, \qquad \sup_{n \geqq n_k} b_n^{-1}|S_n| < 1 + \varepsilon\,. \end{aligned} \tag{8}$$

Assume $k > K$ hereafter. For brevity, define

$$I_k \equiv \{n : n_{k-1} < n \leqq n_k\}\,,$$
$$T_k \equiv \max_{n \in I_k} b_n^{-1} \left|\textstyle\sum_{m=1}^{n_{k-1}} \{p_i(m/n) - p_i(m/n_{k-1})\} X_m\right|$$

and

$$U_k \equiv \max_{n \in I_k} b_n^{-1} \left|\textstyle\sum_{n_{k-1} < m \leqq n} p_i(m/n) X_m\right|\,.$$

Moreover, for each $0 \leqq j \leqq p$, let

$$V_k(j) \equiv \max_{n \in I_k} b_n^{-1} n^{-j} \left|\textstyle\sum_{n_{k-1} < m \leqq n} m^j X_m\right|\,.$$

Suppose $0 \leqq j \leqq p$ and $n \in I_k$; let

$$P_n \equiv \textstyle\sum_{n_{k-1} \leqq m < n} \{m^j - (m+1)^j\} S_m\,,$$
$$Q_n \equiv n^j (S_n - S_{n_{k-1}})$$

and

$$R_n \equiv (n^j - n_{k-1}^j) S_{n_{k-1}}\,.$$

It is evident from (8) that

$$|P_n| \leqq (1+\varepsilon)(n^j - n_{k-1}^j) b_n\,,$$
$$|Q_n| \leqq \gamma n^j b_n$$

and

$$|R_n| \leqq (n^j - n_{k-1}^j)(1+\varepsilon) b_n\,.$$

Now, by rearrangement of terms, it is easy to see that, for $n \geqq 1$,

$$\textstyle\sum_{m=1}^{n} m^j X_m = \sum_{m=1}^{n-1} \{m^j - (m+1)^j\} S_m + n^j S_n\,. \tag{9}$$

From (9),

$$V_k(j) \leqq \max_{n\in I_k} b_n^{-1}n^{-j}(|P_n| + |Q_n| + |R_n) \leqq 2(1+\varepsilon)(1-(n_{k-1}/n_k)^j) + \gamma .$$

Hence, by (7),

$$\limsup_{k\to\infty} V_k(j) \leqq 2(1+\varepsilon)(1-c^{-j}) + \gamma < 3(1+\varepsilon)\gamma .$$

But $U_k \leqq \sum_{j=0}^p |a_j| V_k(j)$, so that

$$\limsup_{k\to\infty} U_k(\omega) < 3(1+\varepsilon)\gamma A = 3\varepsilon(1+\varepsilon) . \tag{10}$$

By (9) and (8),

$$\limsup_{n\to\infty} n^{-j} b_n^{-1} |\textstyle\sum_{m=1}^n m^j X_m| \leqq 2(1+\varepsilon) ,$$

from which it follows that

$$\begin{aligned} &\limsup_{k\to\infty} T_k(\omega) \\ &\quad\leqq \limsup_{k\to\infty} \textstyle\sum_{j=0}^p |a_j|(n_k{}^j - n_{k-1}^j) n_k{}^{-j} n_{k-1}^{-j} b_{n_{k-1}}^{-1} |\sum_{m=1}^{n_{k-1}} m^j X_m| \\ &\quad\leqq 2(1+\varepsilon) \textstyle\sum_{j=0}^p |a_j|(1-c^{-j}) < \varepsilon . \end{aligned} \tag{11}$$

Thus, for $\omega \in E$ and all large k, if $n \in I_k$ then, by (8), (10) and (11),

$$\begin{aligned} b_n{}^{-1} W_n(p_i) &\leqq b_n{}^{-1} W_{n_{k-1}} + b_n{}^{-1} |W_n(p_i) - W_{n_{k-1}}(p_i)| \\ &\leqq (1+\varepsilon) + T_k + U_k \leqq (1+\varepsilon) + \varepsilon + 3\varepsilon(1+\varepsilon) \end{aligned}$$

for all $0 < \varepsilon < A$. Hence (6) holds under the hypotheses of (ii) or (iii).

Hence, in either case, (6) and part (i) combine to yield, for all large i,

$$\begin{aligned} \limsup_{n\to\infty} b_n{}^{-1} W_n(f) &\leqq \limsup_{n\to\infty} b_n{}^{-1} W_n(p_i) + \limsup_{n\to\infty} b_n{}^{-1} W_n(f - p_i) \\ &\leqq \alpha_i \beta + N(f - p_i) \quad \text{a.s.} \end{aligned}$$

But (3) holds, so the proof is concluded by taking limits inferior as $i \to \infty$ on both sides. □

3. A LIL for multiplicative systems. Recall that sequence $X_1, X_2, \cdots$ of rv is called an *equinormed strongly multiplicative system* (ESMS) if $EX_n = 0$, $EX_n{}^2 = 1$ for all $n \geqq 1$ and $E(X_{i_1}^{r_1} X_{i_2}^{r_2} \cdots X_{i_k}^{r_k}) = \prod_{j=1}^k EX_{i_j}^{r_j}$ for all $1 \leqq i_1 \leqq i_2 \leqq \cdots \leqq i_k$, all sets $\{r_1, r_2, \cdots, r_k\}$ such that $r_j = 1$ or 2, $1 \leqq j \leqq k$, and all $k \geqq 1$.

THEOREM 1. *Let $X_1, X_2, \cdots$ be a uniformly bounded* ESMS.

(i) *For every $f \in \mathscr{C}$,*

$$\limsup_{n\to\infty} (2n \log\log n)^{-\frac12} \textstyle\sum_{m=1}^n f(m/n) X_m \geqq ||f||_2 \quad \text{a.s.}$$

(ii) *For every $f \in \mathscr{Q}$,*

$$\limsup_{n\to\infty} (2n \log\log n)^{-\frac12} \textstyle\sum_{m=1}^n f(m/n) X_m = ||f||_2 \quad \text{a.s.}$$

PROOF. Suppose $f \in \mathscr{C}$. Both parts are obvious if $||f||_2 = 0$, so assume $||f||_2 = 1$ without loss of generality.

By hypothesis, there exists $M > 0$ such that $|X_n| \leqq M$ a.s. for all n.

To prove (i), generous doses of some techniques due to Révész (1972) will be infused into the general method used in the proof of Theorem 2 of Tomkins (1975).

For $x \in [0, 1]$, let $I(x) = \int_0^x f^2(t)\,dt$. Let $0 < \varepsilon < 1$. Choose a positive integer c so large that

$$I(c^{-1}) < \varepsilon^2/(2+\varepsilon)^2 . \tag{12}$$

Define $\nu \equiv I(c^{-1})$ and note that $\nu < \varepsilon < 1$.

Now let $n_0 \equiv 0$. For all $k \geqq 1$, define $n_k = c^k$, and

$$I_k \equiv \{n \mid n \text{ is an integer},\ n_{k-1} < n \leqq n_k\} ,$$
$$U_k \equiv \textstyle\sum_{m=1}^{n_{k-1}} f(m/n_k)X_m ,$$
$$V_k \equiv \textstyle\sum_{m \in I_k} f(m/n_k)X_m ,$$
$$u_k^2 \equiv E(U_k^2) = \textstyle\sum_{m=1}^{n_{k-1}} f^2(m/n_k) ,$$

and

$$v_k^2 \equiv E(V_k^2) = \textstyle\sum_{m \in I_k} f^2(m/n_k) .$$

Clearly $u_k^2 + v_k^2 \sim n_k\|f\|_2^2 = n_k$. Indeed,

$$u_k^2/n_k \to \nu \qquad \text{and, hence,} \qquad v_k^2/n_k \to 1-\nu \qquad \text{as } k \to \infty . \tag{13}$$

Therefore,

$$\log\log u_k^2 \sim \log\log n_k , \qquad \text{if} \quad \nu > 0 . \tag{14}$$

Define $f^* \equiv \sup_{0 \leq x \leq 1} |f(x)|$ and, for $k \geqq 1$, $\lambda_k = (2\log\log n_k)^{\frac{1}{2}}/u_k$.

Suppose, first, that $\nu > 0$. Then $\lambda_k \to 0$ by (14). If k is so large that $4\lambda_k Mf^* \leqq 1$, $2\lambda_k M^3 f^* < \varepsilon$, and (using (13)) $u_k^2 < 4\nu n_k$, then Lemma 4 of Gaposhkin (1969) yields

$$\begin{aligned} P_k' &\equiv P[|U_k| > 2\varepsilon(2n_k \log\log n_k)^{\frac{1}{2}}] \\ &\leqq P[|U_k| > \varepsilon u_k \nu^{-\frac{1}{2}}(2\log\log n_k)^{\frac{1}{2}}] \\ &= P[\exp(\lambda_k|U_k|) \geqq \exp(\varepsilon\nu^{-\frac{1}{2}} \log\log n_k)] \\ &\leqq 2\exp\{-(\varepsilon\nu^{-\frac{1}{2}} - 1 - \varepsilon)\log\log n_k\} \\ &= 2(k\log c)^{-\varepsilon\nu^{-\frac{1}{2}}+1+\varepsilon} . \end{aligned}$$

But $\varepsilon\nu^{-\frac{1}{2}} - 1 - \varepsilon > 1$ from (12). Hence $\sum_{k=1}^{\infty} P_k' < \infty$ so that by the Borel–Cantelli lemma,

$$\limsup_{k\to\infty} \frac{|U_k|}{(2n_k \log\log n_k)^{\frac{1}{2}}} \leqq 2\varepsilon \quad \text{a.s.} \tag{15}$$

Note that (15) is trivial if $\nu = 0$, because $f \equiv 0$ on $[0, c^{-1}]$ in that case.

Now define a sequence of rv $\{\xi_m\}$ as follows:

$$\xi_m = f(m/n_k)X_m \qquad \text{if} \quad n_{k-1} < m \leqq n_k .$$

This sequence satisfies all the requirements of an ESMS except $E(\xi_m^2) = f^2(m/n_k)$, $m \in I_k$. Furthermore, $|\xi_m| \leqq Mf^*$ a.s, so $\{\xi_m\}$ is a uniformly bounded sequence too. Note that $V_k = \sum_{m \in I_k} \xi_m$.

For the remainder of the proof of (i), $i \equiv (-1)^{\frac{1}{2}}$. Following Révész (1972), for j, $k \geqq 1$ and all numbers t, define

$$\eta_k \equiv V_k/v_k ,$$
$$\alpha_k(t) \equiv \prod_{m \in I_k} (1 + it\xi_m v_k^{-1})$$

and

$$\beta_k \equiv v_k^{-2} \sum_{m \in I_k} \xi_m^2 .$$

Let $K = (1 + M^2)(f^*)^2$ and $K^* = 2^{-\frac{1}{2}}K$. Note that $E(\alpha_k(s)\alpha_{k+j}(t)) = 1$ for all real s, t.

For $m \geqq 1$, let $\zeta_m = \xi_m^2 - E(\xi_m^2)$. Then $|\zeta_m| \leqq K$ a.s. and $E(\zeta_{j_1} \zeta_{j_2} \cdots \zeta_{j_n}) = 0$ for all integers $0 < j_1 < j_2 < \cdots < j_n$.

Let $y_k = v_k^2(n_k - n_{k-1})^{-\frac{5}{6}}/2$ and $B = (1 - \nu)^2(1 - c^{-1})^{-2}/8$. In light of (13), $y_k^2 \sim (2B)(n_k - n_{k-1})^{\frac{1}{3}}$ and $2^{\frac{1}{2}}y_k(n_k - n_{k-1})^{-\frac{1}{6}} \sim 2^{-\frac{1}{2}}(1 - \nu)$. If k is so large that $y_k^2 > B(n_k - n_{k-1})^{\frac{1}{3}}$ and $2^{\frac{1}{2}}y_k < (n_k - n_{k-1})^{\frac{1}{6}}$, then, by Lemma 1 of Takahashi (1972),

$$\begin{aligned} P[|\beta_k - 1| &> K^*(n_k - n_{k-1})^{-\frac{1}{3}}] \\ &= P[|\textstyle\sum_{m \in I_k} \zeta_k| > K(2(n_k - n_{k-1}))^{\frac{1}{2}} y_k] \\ &< Ae^{-y_k^2} < A \exp(-B(n_k - n_{k-1})^{\frac{1}{3}}) \\ &\qquad \text{for some} \quad A > 0 \quad \text{independent of} \quad k . \end{aligned}$$

By following Révész' method, making a few obvious modifications, it turns out that

$$P[\eta_k \geqq ((2 - \varepsilon) \log\log (n_k - n_{k-1}))^{\frac{1}{2}} \text{ i.o.}] = 1 .$$

This is tantamount to

$$\limsup_{k\to\infty} \eta_k/(2 \log\log n_k)^{\frac{1}{2}} \geqq 1 - \varepsilon/2 \quad \text{a.s.}$$

Noting (13) and the fact that $\nu < \varepsilon$,

$$\text{(16)} \qquad \limsup_{k\to\infty} \frac{V_k}{(2n_k \log\log n_k)^{\frac{1}{2}}} \geqq (1 - \varepsilon)^{\frac{1}{2}}(1 - \varepsilon/2) \quad \text{a.s.}$$

(15) and (16) immediately yield (i).

Now Lemma 1 will be utilized to prove (ii). For $n \geqq 1$, let $S_n = \sum_{m=1}^{n} X_m$, $r_n = (2 \log\log n)^{\frac{1}{2}}$ and $b_n = n^{\frac{1}{2}}r_n$. Note that (2) holds with $\beta = 1$ by a theorem of Gaposhkin (1969).

For $c > 1$, to be specified later, let $n_k \equiv [c^k]$ for $k \geqq 1$. In addition, define

$$M_k^* \equiv \max_{n_{k-1} < n \leqq n_k} |S_n - S_{n_{k-1}}|$$

for all k so large that $n_{k-1} < n_k$. By Lemma 3 of Gaposhkin (1969), $A > 0$ exists such that

$$E \exp(\lambda M_k^*) \leqq 2 \exp(A\lambda^2(n_k - n_{k-1}))$$

for all $0 < \lambda < A$ and large k. Therefore, if $\varepsilon > 0$, and if k is so large that

$\varepsilon r_{n_k} < A(n_k - n_{k-1})^{\frac{1}{2}}$, we have

$$P[M_k^* \geqq \varepsilon b_{n_k}] = P[\exp(\varepsilon r_{n_k}(n_k - n_{k-1})^{-\frac{1}{2}} M_k^*) \geqq \exp(\varepsilon^2 r_{n_k}^2 (1 - n_{k-1}/n_k)^{-\frac{1}{2}})]$$
$$\leqq 2\exp\{-\varepsilon r_{n_k}^2 (1 - n_{k-1}/n_k)^{-\frac{1}{2}} + A\varepsilon^2 r_{n_k}^2\}$$
$$= 2\exp\{-(r_{n_k}^2/2)(2\varepsilon^2)((1 - n_{k-1}/n_k)^{-\frac{1}{2}} - A)\}.$$

But $(1 - n_{k-1}/n_k)^{-\frac{1}{2}} - A \to (1 - c^{-1})^{-\frac{1}{2}} - A$. Thus, if $c > 1$ is chosen so close to 1 that $(2\varepsilon^2)((1 - c^{-1})^{-\frac{1}{2}} - A) > 2$, then $P[M_k^* \geqq \varepsilon b_{n_k}] = 0(k^{-2})$. Hence $\sum_{k=1}^{\infty} P[M_k^* \geqq \varepsilon b_{n_k}] < \infty$, from which (4) follows readily.

Now let $p_i \in \mathscr{P}$ satisfy (3). Let $\varepsilon > 0$. For $i \geqq 1$, define

$$p_j^* \equiv \sup_{0 \leqq x \leqq 1} |p_i(x)|,$$
$$\alpha_i \equiv ||p_i||_2 + \varepsilon,$$
$$\eta_i = 2\alpha_i - 1 - \varepsilon,$$

and for $n \geqq 1$,

$$w_n^2(i) \equiv \sum_{m=1}^{n} p_i^2(m/n).$$

For $k \geqq 1$, let $\lambda_k(i) \equiv r_{n_k} w_{n_k}^{-1}(i)$. Hence

$$w_n^2(i) \sim n||p_i||_2^2,$$

$\lambda_k \to 0$ and $\alpha_i \to 1 + \varepsilon$. Moreover, $\eta_i \to 1 + \varepsilon$, so $\eta_i > 1$ for all large i.

Now $|X_n| \leqq M$ a.s. for some $M > 0$ and all $n \geqq 1$. Suppose k is so large that $4\lambda_k(i)p_i^*M \leqq 1$ and $2\lambda_k(i)p_i^*M^3 < \varepsilon$. Then, by Lemma 4 of Gaposhkin (1969),

$$P_k^*(i) \equiv P[W_{n_k}(p_i) \geqq \alpha_i w_{n_k}(i) r_{n_k}] = P[\lambda_k W_{n_k}(p_i) \geqq \alpha_i r_{n_k}^2]$$
$$\leqq \exp\{-\alpha_i r_{n_k}^2 + (1 + \varepsilon) r_{n_k}^2/2\} = \exp\{-\eta_i r_{n_k}^2/2\} = 0(k^{-\eta_i}).$$

If i is so large that $\eta_i > 1$, then $\sum_{k=1}^{\infty} P_k^*(i) < \infty$, so (5) holds by the Borel–Cantelli lemma. Since $\varepsilon > 0$ is arbitrary, Lemma 1 (ii) implies

$$\limsup_{n\to\infty} (2n \log\log n)^{-\frac{1}{2}} \sum_{m=1}^{n} f(m/n) X_m \leqq 1 \quad \text{a.s.}$$

In view of part (i), the theorem is proved. □

4. Some other consequences of Lemma 1. This section presents two one-sided LIL results and a general theorem about a.s. stability.

The first LIL holds for certain sequences of uniformly bounded rv and for some sequences of rv with weighted sums which are generalized Gaussian. Following Chow (1966), a rv X is called *generalized Gaussian* with parameter τ if $E\exp(tX) \leqq \exp(t^2\tau^2/2)$ for all real t.

THEOREM 2. *Let* $X_1, X_2, \cdots,$ *be* rv *and define* $S_{mn} = \sum_{j=m+1}^{m+n} X_j$ *for* $n \geqq 1$, $m \geqq 0$. *Let* $\beta > 0$. *Suppose, for all* $n \geqq 1$, *either* (a) S_{mn} *is generalized Gaussian with parameter* $(\beta n)^{\frac{1}{2}}$ *for each* $m \geqq 0$ *and, for each* $p \in \mathscr{P}$, $W_n(p)$ *is generalized Gaussian with parameter* $(\sum_{m=1}^{n} p^2(m/n))^{\frac{1}{2}}$, *or,* (b) $EX_n = 0$, $|X_n| \leqq \beta$ a.s., $P[|S_{0n}| \geqq t] \leqq 2\exp\{-t^2/(2\beta^2 n)\}$ *for all* $t > 0$,

$$(17) \qquad P[W_n(p) \geqq t(\sum_{m=1}^{n} p^2(m/n))^{\frac{1}{2}}] \leqq K\exp\{-t^2/2\}$$

for some $K > 0$, *all* $n \geqq 1$, *all* $t > 0$ *and all* $p \in \mathscr{P}$, *and*

$$E|S_{mn}|^\nu \leqq A_\nu n^{\nu/2} \tag{18}$$

for all $\nu \geqq 1$, $m \geqq 0$ and for some positive real sequence $\{A_\nu\}$. If $f \in \mathscr{Q}$, then

$$\limsup_{n\to\infty} (2n \log\log n)^{-\frac{1}{2}} \sum_{m=1}^n f(m/n)X_m \leqq \beta||f||_2 \quad \text{a.s.}$$

PROOF. For $n \geqq 1$, let $S_n = S_{0n}$ and $b_n = (2n \log\log n)^{\frac{1}{2}}$. Note that (2) holds in case (a) by Theorem 2.1 of Stout (1973) whereas Theorem 4.1 of Serfling (1970) implies (2) in case (b).

Now, if (a) holds, then (17) holds by expression (3) of Chow (1966), and (18) holds for $\nu > 2$ by Lemma 2.1 of Stout (1973). We shall assume only (17) and (18) hereinafter.

Let $\varepsilon > 0$ and choose $\delta < 1$ such that $(1 + \varepsilon)\delta > 1$. For $k \geqq 1$, define $n_k = [e^{k^\delta}]$ and $M_k = \max_{n_{k-1}<n\leqq n_k} b_{n_{k-1}}^{-1}|S_n - S_{n_{k-1}}|$. Note $n_{k+1} \sim n_k$. Choose $\nu > 2$ such that $(1 - \delta)\nu > 2$. Since (18) holds, an argument in the proof of Theorem 4.1 of Serfling (1970) establishes $\sum_{k=1}^\infty E(M_k^\nu) < \infty$. Hence $M_k \to 0$ a.s.

If $f \equiv 0$ on $[0, 1]$, then the theorem is trivial. Hence if $p_i \in \mathscr{P}$, $i \geqq 1$, satisfy (3), there is no harm in assuming $||p_i||_2 > 0$ for all $i \geqq 1$. For each $i \geqq 1$ and $k \geqq 1$, (17) implies

$$\begin{aligned} P[W_{n_k}(p_i) \geqq (2(1 + \varepsilon) \textstyle\sum_{m=1}^{n_k} p_i^2(m/n_k) \log\log n_k)^{\frac{1}{2}}] \\ \leqq K(\log n_k)^{-(1+\varepsilon)} \leqq K(k^{-(1+\varepsilon)\delta}) . \end{aligned}$$

But $\sum_{m=1}^{n_k} p_i^2(m/n_k) \sim n_k||p_i||_2^2$. Hence, by the Borel-Cantelli lemma, (5) holds with $\alpha_i \equiv (1 + \varepsilon)^{\frac{1}{2}}||p_i||_2$. But $||p_i||_2 \to ||f||_2$, so the theorem is a consequence of Lemma 1 (iii). □

The following theorem generalizes Theorem 1 of Tomkins (1975) and yet, because Lemma 1 is employed, its proof is shorter than that of the earlier result.

THEOREM 3. *Let* $(X_n, \mathscr{F}_n, n \geqq 1)$ *be a martingale difference sequence. Suppose a number* $N > 0$ *and a positive sequence* $c_n = o((\log\log n)^{-\frac{1}{2}})$ *such that, for all* $n \geqq N$ *and all real numbers* $a_1, a_2, \cdots, a_n$,

$$E\{\exp(t(\textstyle\sum_{m=1}^n a_m^2)^{-\frac{1}{2}} \sum_{m=1}^n a_m X_m)\} \leqq \exp\{(t^2/2)(1 + |t|c_n)\} \tag{19}$$

provided $|t|c_n \leqq 1$. *Then, for every* $f \in \mathscr{Q}$,

$$\limsup_{n\to\infty} (2n \log\log n)^{-\frac{1}{2}} \sum_{m=1}^n f(m/n)X_m \leqq ||f||_2 \quad \text{a.s.}$$

PROOF. Let $\varepsilon > 0$, $\beta = 1$ and $b_n = (2n \log\log n)^{\frac{1}{2}}$ for $n \geqq 1$. Taking $a_m = 1$ for $m \geqq 1$, Lemma 1 of Tomkins (1975) yields (2). Using (19) and taking $j = 0$ in the argument used to prove (9) of Tomkins (1975), (4) results, whereas the procedure used at the top of page 311 of Tomkins' paper (modified to make use of (19)) establishes (5) with $\alpha_i \equiv (1 + \varepsilon)||p_i||_2$. An application of Lemma 1 (ii) concludes the proof. □

The final theorem concerns a.s. stable rv (cf. Loève, page 252); it is an easy consequence of Lemma 1 (i).

THEOREM 4. *Let* $X_1, X_2, \cdots,$ *be any* rv *satisfying* $\lim_{n\to\infty} b_n^{-1}(X_1 + \cdots + X_n) = 0$ a.s. *for some nondecreasing positive sequence* $\{b_n\}$. *Then* $\lim_{n\to\infty} b_n^{-1} \sum_{m=1}^{n} f(m/n)X_m = 0$ a.s. *for every function f of bounded variation* (BV).

In particular, if $X_1, X_2, \cdots,$ *obey the* SLLN (i.e. $n^{-1}(X_1 + \cdots + X_n) \to 0$ a.s.), *then* $n^{-1} \sum_{m=1}^{n} f(m/n)X_m \to 0$ a.s. *for all f of* BV.

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