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THE INFINITE SECRETARY PROBLEM

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An infinite sequence of rankable individuals (rank 1 = best) arrive at times which are i.i.d., uniform on (0, 1). We, in effect, observe only their relative ranks as they arrive. We seek a stopping rule to minimize the mean of a prescribed positive increasing function, $q(\cdot)$, of the actual rank of the individual chosen.

Let $f(t)$ be the minimal mean among all stopping rules which are greater than t. Then $f(\cdot)$ is a solution to a certain differential equation which is derived and used to find an optimal stopping rule.

This problem is in a strong sense the "limit" of a corresponding sequence of "finite secretary problems" which have been examined by various authors. The limit of the "finite-problem" minimal risks is finite if and only if the differential equation has a solution, $f(\cdot)$, which is finite on $[0, 1)$ with $f(1^-) = \sup q(n)$. Usually, if such a solution exists, it is unique, in which case $f(0)$ is both the minimal risk for the infinite problem and the limit of the "finite-problem" minimal risks.

0. Introduction. A finite "secretary problem" is one in which a finite sequence of rankable individuals (rank 1 = best) arrive in random order but as they arrive only their relative ranks with respect to their predecessors can be observed. In one class of such problems an increasing loss function is prescribed and the object is to find a stopping rule which minimizes the expectation of this function of the absolute (actual) rank of the individual selected by the rule.

The study of the asymptotic form of the solutions, as the number of individuals goes to infinity, by Chow, Moriguti, Robbins and Samuels [2], and recently by Mucci [5, 6], together with a long-standing suggestion of Rubin [7] has stimulated us to formulate an infinite secretary problem, which is in a strong sense the "limit" of the corresponding sequence of finite problems putting the asymptotic results into a natural setting which permits their extension.

1. The infinite problem. In the finite problem—with, say, n individuals—the statement "the individuals arrive in random order" is generally understood to mean that each of the n is equally likely to arrive first, and each of the $n - 1$ remaining individuals is equally likely to arrive second, and so on. But it would be just as valid to say that the best individual is equally likely to arrive first or second or $\cdots$ or last, and the second best is equally likely to arrive in any of the $n - 1$ remaining positions, and so on. It happens that the former conception is more attractive for solving the finite problem. But it is only the latter

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one that suggests an analogous formulation of a problem with infinitely many individuals.

Letting U_i denote the arrival time of the ith best individual, $(i = 1, 2, \cdots)$, what we do is to simply take the U_i's to be independent and identically distributed (i.i.d.), each uniformly distributed on the interval [0, 1]. This is our "model" because all the other random variables which we shall define will be functions of the vector

$$\mathbf{U} = (U_1, U_2, \cdots) .$$

For each $t \in (0, 1]$, define

$$\mathbf{K}(t) = (K_1(t), K_2(t), \cdots) ,$$

where

$$K_1(t) = \min\{j : U_j \leqq t\}$$
$$K_{i+1}(t) = \min\{j > K_i(t) : U_j \leqq t\} ;$$

and

$$\mathbf{Z}(t) = (Z_1(t), Z_2(t) \cdots) ,$$

where

$$Z_i(t) = U_{K_i(t)} .$$

Thus $Z_i(t)$ is the arrival time of the individual who is ith best among those who arrive by time t.

The independence of $\mathbf{Z}(t)$ and $\mathbf{K}(t)$, and the fact that the $Z_i(t)$'s are i.i.d., uniform on $(0, t)$, are merely familiar properties of random samples from a uniform distribution on (0, 1).

The sequence $\mathbf{Z}(t)$ is to represent what we can observe up to time t, so we define

$$\mathscr{F}_t = \sigma\text{-field generated by } \mathbf{Z}(t) .$$

If $s < t$ then $\mathbf{Z}(s)$ is a function of $\mathbf{Z}(t)$: we simply replace U_j by $Z_j(t)$ in the definition of $K_i(s)$. Also $Z_i(t) \to Z_i(1) = U_i$ as $t \uparrow 1$. Thus the $\mathscr{F}_t$'s are increasing with t and $\bigvee_{t<1} \mathscr{F}_t = \mathscr{F}_1$, the σ-field generated by $\mathbf{U}$.

Let $\mathscr{C}$ be the class of all random variables τ satisfying

$$0 < \tau \leqq 1 \tag{1}$$

$$\tau < 1 \Rightarrow \tau \in \{U_i\} \tag{2}$$

$$\{\tau \leqq t\} \in \mathscr{F}_t \quad \forall\, t \leqq 1 . \tag{3}$$

This is the class of all stopping rules to be considered. (It is essential that $\mathscr{C}$ include rules which take the value one with positive probability, i.e. which "may not stop.")

For each $t \in (0, 1)$, define

$$\mathscr{C}_t = \{\tau \in \mathscr{C} : \tau > t \text{ a.s.}\} .$$

Clearly $\tau, \tau_1, \tau_2 \in \mathscr{C}$, $\tau_2 > \tau$ a.s. implies

$$\tau_1 I_{\{\tau_1 \leqq \tau\}} + \tau_2 I_{\{\tau_1 > \tau\}} \in \mathscr{C} .$$

However $\mathscr{C}$ is not closed under monotone limits so when we form such limits care will be required.

The following notation for the absolute and relative ranks, respectively, of "an individual ariving at time u" will be convenient:

$$\begin{aligned} X_u &= i \quad \text{if} \quad u = U_i \\ Y_u &= i \quad \text{if} \quad u = Z_i(u) \\ X_1 &= Y_1 = \infty . \end{aligned}$$

Here is the problem: Given a loss function $q(\cdot)$ with

$$\begin{gathered} 0 \leqq q(1) \leqq q(2) \cdots \\ q(\infty) \equiv \lim_{i\to\infty} q(i) \leqq \infty , \end{gathered}$$

find the minimal expected loss (minimal risk)

$$v \equiv \inf_{\tau\in\mathscr{C}} Eq(X_\tau) \leqq \infty , \tag{1.1}$$

and find, if possible, an optimal τ.

That this is really analogous to the finite secretary problem is partly confirmed by the fact that $\mathscr{C}$ contains the analogue of all the good rules in the finite problem—what we call the cutoff-point rules.

DEFINITION. τ is a *cutoff-point rule* if there is a sequence $0 < t_1 \leqq t_2 \leqq \cdots \leqq 1$ (which we call the *cutoff points*) such that

$$\begin{aligned} \tau &= \min\{U_i : U_i \geqq t_k, \text{ and } Y_{U_i} \leqq k\} \\ &= 1 \qquad \text{if no such } U_i . \end{aligned}$$

In other words τ stops at the first time in $[t_k, t_{k+1})$ at which there is an arrival of relative rank $\leqq k$, if there is such a time and if τ did not stop before time t_k.

The $Z_i(t_k)$'s are i.i.d., uniform on $(0, t_k)$, so

$$P(\tau > t_k) = P(Z_i(t_k) < t_i, i = 1, \cdots, k-1) = \textstyle\prod_{i=1}^{k-1} (t_i/t_k) . \tag{1.2}$$

Limits of sequences of cutoff points determine corresponding limits of rules and of risks as follows:

PROPOSITION 1.1. *Let* $\{t_k^{(n)}\}$: $n = 1, 2, \cdots$ *be cutoff-point sequences with limits* $t_k^{(\infty)} = \lim_n t_k^{(n)}$, *and let* τ_n: $n \leqq \infty$ *be the corresponding stopping rules. If* $t_1^{(\infty)} > 0$, *then* $X_{\tau_n} \to X_{\tau_\infty}$ a.s.; *hence* $Eq(X_{\tau_n}) \to Eq(X_{\tau_\infty})$ *if* $q(\cdot)$ *is bounded. A sufficient condition for* $t_1^{(\infty)} > 0$ *is* $\limsup Eq(X_{\tau_n}) < q(\infty)$.

PROOF. On $E_n = \{\tau_\infty < t_k^{(\infty)}$ and $\tau_m \neq \tau_\infty$ for some $m \geqq n\}$, there must be an arrival of relative rank $\leqq k$ in one of the intervals $[\inf_{m\geqq n} t_i^{(m)}, \sup_{m\geqq n} t_i^{(m)}]$, $i = 1, 2, \cdots, k$. The lengths of these intervals shrink to zero as $n \to \infty$; hence clearly $P(E_n) \to 0$ as $n \to \infty$ as long as $t_1^{(\infty)} > 0$. Thus $X_{\tau_n} = X_{\tau_\infty}$ for all n sufficiently large a.s. on $\{\tau_\infty < 1\}$.

Let $X_{[u,v]}$ denote the absolute rank of the best arrival in $[u, v]$. Let $T = \sup t_i^{(\infty)}$ and, for each k, let σ_k be the first time after T at which there is an arrival of relative rank $\leqq k$, if there is one; $\sigma_k = 1$ if not or if $T = 1$. Then $\limsup \tau_n \leqq \sigma_k$ so for any k and $n \geqq n(k)$ sufficiently large,

$$P(\tau_\infty = 1, \inf_{m\geqq n} X_{\tau_m} < x) \leqq P(\tau_\infty = 1, \inf_{m\geqq n} \tau_m < t_k^{(\infty)}) + P(X_{[t_k^{(\infty)},\sigma_k]} < x) .$$

For each k the first term on the right goes to zero as $n \to \infty$ by exactly the same argument as in the preceding paragraph. If $T = 1$ the second term is $1 - [t_k^{(\infty)}]^x$, while, if $T < 1$, $X_{[t_k^{(\infty)},\sigma_k]} \geqq Y_{\sigma_k}$ and, on $\{\sigma_k < 1\}$, Y_{σ_k} is uniformly distributed on $\{1, 2, \cdots, k\}$ (see Proposition 3.4). In both cases the second term goes to zero as $k \to \infty$. Thus $X_{\tau_n} \to X_{\tau_\infty} = \infty$ on $\{\tau_\infty = 1\}$ as well.

Convergence of the risks follows immediately by the bounded convergence theorem.

If $t_1^{(n)} \leqq \varepsilon$, then with probability $1 - \varepsilon$ the best arrival in $[0, t_1^{(n)}/\varepsilon]$ arrives in $[t_1^{(n)}, t_1^{(n)}/\varepsilon]$ in which case $t_1^{(n)} \leqq \tau_n \leqq t_1^{(n)}/\varepsilon$ and the expected loss is at least $Eq(K_1(t_1^{(n)}/\varepsilon))$. Thus

$$\limsup Eq(X_{\tau_n}) \geqq (1 - \varepsilon) \limsup Eq(K_1(t_1^{(n)}/\varepsilon)) .$$

But $u \downarrow 0 \Rightarrow K_1(u) \uparrow \infty$ a.s. so the right side is $(1 - \varepsilon)q(\infty)$ if $\liminf t_1^{(n)} = 0$. This completes the proof. □

2. Independence of the past and the future. In this section we set the stage for deriving the differential equation which is the analogue—and indeed the "limit," as Mucci [5, 6] showed—of the difference equations in the finite problem. As in the finite case, the key step is to show that the minimal risk (1.1) is achieved within the subclass of rules which, in effect, depend only on the arrival time and relative rank of the current arrival and not on the past. This is a trivial matter in the finite problem but requires some care here.

We need to generalize the notation of Section 1 as follows: For $0 < s < t \leqq 1$, define

$$\begin{aligned} \mathbf{K}(s, t) &= (K_1(s, t), K_2(s, t), \cdots) , \\ \mathbf{Z}(s, t) &= (Z_1(s, t), Z_2(s, t), \cdots) , \\ \mathbf{M}(s, t) &= (M_1(s, t), M_2(s, t), \cdots) , \\ \mathscr{F}_{s,t} &= \sigma\text{-field generated by} \quad (\mathbf{Z}(s, t), \mathbf{M}(s, t)) \end{aligned}$$

where

$$\begin{aligned} K_1(s, t) &= \min\{j : s < U_j \leqq t\} , \\ K_{i+1}(s, t) &= \min\{j > K_i(s, t) : s < U_j \leqq t\} , \\ Z_i(s, t) &= U_{K_i(s,t)} , \\ M_i(s, t) &= Y_{Z_i(s,t)} , \\ &= 1 + \textstyle\sum_j I_{\{K_j(s) < K_i(s,t)\}} + \sum_{k=1}^{i-1} I_{\{Z_k(s,t) < Z_i(s,t)\}} . \end{aligned}$$

Thus $Z_i(s, t)$ is the arrival time of the individual who is ith best among those who arrive in $(s, t]$, $M_i(s, t)$ is its relative rank at arrival time among *all* individuals who have so far arrived, and $K_i(s, t)$ is its absolute rank.

That $\mathbf{Z}(s)$ and $\mathbf{Z}(s, t)$ are independent and each is independent of $(\mathbf{K}(s), \mathbf{K}(s, t))$

are further familiar properties of random samples from a uniform distribution. From its definition, we see that $\mathbf{M}(s, t)$ is independent of $\mathbf{Z}(s)$. Thus we have

PROPOSITION 2.1. *For each s and t with $0 < s < t \leqq 1$, $\mathscr{F}_s$ and $\mathscr{F}_{s,t}$ are independent.*

In particular,

$$X_{Z_i(s,t)} = K_i(s, t) \quad \text{is independent of} \quad \mathscr{F}_s . \tag{2.1}$$

Also it is clear that

$$\mathscr{F}_t = \mathscr{F}_s \vee \mathscr{F}_{s,t} . \tag{2.2}$$

Now we define, for each $t \in [0, 1)$,

$$\gamma(t) = \operatorname{ess\,inf}_{\tau \in \mathscr{C}_t} E[q(X_\tau) \mid \mathscr{F}_t] \tag{2.3}$$

$$f(t) = \inf_{\tau \in \mathscr{C}_t} Eq(X_\tau) \tag{2.4}$$

$$g(t) = \inf_{\tau \in \mathscr{C}_t^*} Eq(X_\tau) \tag{2.5}$$

where $\mathscr{C}_t^*$ is the subclass of $\mathscr{C}_t$ consisting of those τ's which are measurable with respect to $\mathscr{F}_{t,1}$, hence independent of $\mathscr{F}_t$.

The main result of this section is

PROPOSITION 2.2. *For each $t \in (0, 1)$*

$$\gamma(t) \equiv f(t) = g(t) \quad \text{a.s.} \tag{2.6}$$

PROOF. First we shall verify that every $\tau > t$ can be expressed as a mixture $\{\tau_z\}$ of rules which are independent of $\mathscr{F}_t$, and that $E(q(X_\tau) \mid \mathscr{F}_t)$ is the corresponding mixture of $\{Eq(X_{\tau_z})\}$.

By (2.2), we can write

$$\tau = \tau(\mathbf{Z}(t), \mathbf{Z}(t, 1), \mathbf{M}(t, 1))$$

and define, for each $z = (z_1, z_2, \cdots)$: $0 < z_i \leqq t$,

$$\tau_z = \tau(z, Z(t, 1), \mathbf{M}(t, 1)) .$$

By measurability of sections, each τ_z is a random variable, which of course is greater than t, measurable with respect to $\mathscr{F}_{t,1}$, hence independent of $\mathscr{F}_t$ by Proposition 2.1 and satisfies parts (1) and (2) of the definition of $\mathscr{C}$ in Section 1. Moreover, since $\{\tau \leqq u\} \in \mathscr{F}_u$ for each $u > t$, we have, again by (2.2) and measurability of sections,

$$\{\tau_z \leqq u\} \in \mathscr{F}_{t,u} \subset \mathscr{F}_u ,$$

so τ_z satisfies part (3) as well and is therefore a stopping rule.

Now,

$$\begin{aligned} q(X_\tau) &= q(X_{\tau_{\mathbf{Z}(t)}}) \\ &= \textstyle\sum_i q(K_i(t, 1)) I_{\{\tau_{\mathbf{Z}(t)} = Z_i(t,1)\}} , \end{aligned}$$

which is a measurable function of $\mathbf{Z}(t)$ and $[\mathbf{K}(t, 1), \mathbf{M}(t, 1), \mathbf{Z}(t, 1)]$, the former being independent of the latter. So, by standard methods (see Breiman [1], page 80, Corollary 4.38), we conclude that a version of $E[q(X_\tau) \mid \mathscr{F}_t)$ has, on each set $\{\mathbf{Z}(t) = z\}$, the value

$$E[\textstyle\sum_i q(K_i(t, 1)) I_{\{\tau_z = Z_i(t,1)\}}] = Eq(X_{\tau_z}) .$$

Of course each $\tau_z \in \mathscr{C}_t^*$ so, by (2.5), we have shown that

$$\gamma(t) \geqq g(t) \quad \text{a.s.}$$

On the other hand, from the definitions (2.3)—(2.5),

$$g(t) \geqq f(t) \geqq E\gamma(t) .$$

Equations (2.6) follow immediately from these inequalities to complete the proof. □

If $\tau \in \mathscr{C}$, then $\tau I_{\{\tau > t\}} + I_{\{\tau \leqq t\}} \in \mathscr{C}_t$, so, by (2.3),

$$E[q(X_\tau) I_{\{\tau > t\}} \mid \mathscr{F}_t] \geqq \gamma(t) I_{\{\tau > t\}} \quad \text{a.s.};$$

hence we have an immediate corollary:

$$E[q(X_\tau) I_{\{\tau > t\}} \mid \mathscr{F}_t] \geqq f(t) I_{\{\tau > t\}} \quad \text{a.s.} \tag{2.7}$$

Conceivably, for some $q(\cdot)$, f, which is obviously always an increasing function, could be finite on $[0, t_0)$ and infinite on $(t_0, 1)$. But the next lemma and proposition show that this is impossible.

LEMMA 2.1. *If* $Eq(X_\tau) < \infty$ *and* $\operatorname{ess\,sup} \tau = s < t < 1$, *then there is a* σ *with* $\operatorname{ess\,sup} \sigma = t$ *and* $Eq(X_\sigma) = Eq(X_\tau)$.

PROOF. Informally, we want σ to be simply τ "applied" to the arrivals in $(t - s, t]$, rather than those in $(0, s]$ and "completely ignoring" all arrivals in $(0, t - s]$. Formally, let $U_i' = (U_i + 1 - t + s)_{\text{mod}\,1}$, $\mathbf{U}' = (U_1', U_2', \cdots)$, and primed objects (e.g., $\mathscr{F}_t'$, X_u', τ') be defined on $\mathbf{U}'$ exactly as the corresponding unprimed objects are defined on $\mathbf{U}$. Let $\sigma = \tau' + (t - s)$.

Of course $\mathbf{U}'$ has the same distribution as $\mathbf{U}$ so $Eq(X'_{\tau'}) = Eq(X_\tau)$. But σ is a stopping rule for the original process and $X'_{\tau'} = X_\sigma$. □

PROPOSITION 2.3. $v < \infty \Rightarrow f(t) < \infty \;\forall\, t \in [0, 1)$.

PROOF. By hypothesis there is a τ with $Eq(X_\tau) < \infty$. Then necessarily, by (2.7),

$$\tau \leqq \sup\{t : f(t) < \infty\} \quad \text{a.s.}$$

Now apply the previous lemma to conclude that this supremum must be one. □

Since, trivially, $v = \infty \Rightarrow f(t) \equiv \infty$ on $[0, 1)$, we have a dichotomy: f is either finite on all of $[0, 1)$ or infinite on all of $[0, 1)$ according as v is finite or infinite.

3. The differential equation for f. For each $i = 1, 2, \cdots$ and each $t \in (0, 1]$, define

$$R_i(t) = \textstyle\sum_{k=i}^{\infty} \binom{k-1}{i-1} q(k) t^i (1 - t)^{k-i} . \tag{3.1}$$

PROPOSITION 3.1. $E[q(X_{Z_i(t)}) \mid \mathscr{F}_t] \equiv R_i(t)$.

PROOF. $\{X_{Z_i(t)} = k\} = \{K_i(t) = k\}$, which is independent of $\mathscr{F}_t$, as we remarked earlier, and of course $K_i(t)$ has the indicated negative binomial distribution. □

Here are some easily verifiable properties of the $R_i(t)$'s which will be used frequently:

$$R_1(t_0) < \infty \Rightarrow R_i(t) < \infty \quad \forall\, i, t > t_0\,; \tag{3.2}$$

$$\text{for fixed } t > t_0\text{:} \quad R_i(t) \uparrow q(\infty) \quad \text{as} \quad i \uparrow \infty\,, \qquad R_i(t) < R_{i+1}(t) \quad \text{unless} \quad q(i) = q(\infty)\,; \tag{3.3}$$

$$\text{for fixed } i\text{:} \quad R_i(t) \quad \text{continuous on} \quad (t_0, 1]\,; \quad R_i(t) \downarrow \quad \text{and} \quad R_i(0^+) = q(\infty)\,, \quad R_i(t) \downarrow \quad \text{(strictly) on} \quad (t_0, 1]\text{—unless} \quad q(i) = q(\infty)\text{—and} \quad R_i(1^-) = q(i)\,. \tag{3.4}$$

This list slightly modifies Mucci's ([6], page 419); he considers only unbounded q's for which each $R_i(\cdot)$ is finite on (0, 1].

For convenience, define

$$R_\infty(1) \equiv q(\infty)\,. \tag{3.5}$$

PROPOSITION 3.2. *If* $Eq(X_\tau) < \infty$, *then*

$$E[q(X_\tau) \mid \mathscr{F}_\tau] = R_{Y_\tau}(\tau) \quad \text{a.s.} \tag{3.6}$$

PROOF. For $n = 1, 2, \cdots$, let $X_{n,k}$ and $Y_{n,k}$ be the absolute rank and the relative rank at arrival time, respectively, of the best arrival in the interval $J_{n,k} = ((k-1)/2^n, k/2^n]$, $k = 1, 2, \cdots, 2^n$. (Notice that the relative rank of this arrival at the later time $k/2^n$ necessarily remains $Y_{n,k}$.) Now let

$$\tau_n = \textstyle\sum_k k2^{-n} I_{\{\tau \in J_{n,k}\}}\,,$$
$$X_n = \textstyle\sum_k X_{n,k} I_{\{\tau \in J_{n,k}\}}\,,$$
$$Y_n = \textstyle\sum_k Y_{n,k} I_{\{\tau \in J_{n,k}\}}\,.$$

At each point ω in the event $\{\tau < 1\}$, τ is the arrival time of the best individual in some sufficiently small open neighborhood of $\tau(\omega)$. So not only does $\tau_n \downarrow \tau$, but also, on both $\{\tau < 1\}$ and $\{\tau = 1\}$, $X_n' \uparrow X_\tau$, $Y_n \uparrow Y_\tau$, and $q(X_n) \uparrow q(X_\tau)$. Thus $R_{Y_n}(\tau_n) \uparrow R_{Y_\tau}(\tau)$, by (3.3) and (3.4), and it is easy to see that $\bigcap_n \mathscr{F}_{\tau_n} = \mathscr{F}_\tau$. So by a standard martingale theorem (Chung [3], page 340),

$$E[q(X_n) \mid \mathscr{F}_{\tau_n}] \to_{\text{a.s.}} E[q(X_\tau) \mid \mathscr{F}_\tau]\,;$$

while, by Proposition 3.1,

$$\begin{aligned} E[q(X_n) \mid \mathscr{F}_{\tau_n}] &= \textstyle\sum_k E[q(X_n) \mid \mathscr{F}_{k/2^n}] I_{\{\tau_n = k/2^n\}} \\ &= \textstyle\sum_k R_{Y_n}(k/2^n) I_{\{\tau_n = k/2^n\}} \\ &= R_{Y_n}(\tau_n) \quad \text{a.s.} \end{aligned}$$

□

The preceding proposition shows that the "improper" question—What do we expect to lose if we choose an individual arriving at time t with relative rank i?—is not, after all, ambiguous, and the answer is $R_i(t)$.

It is trivial that $f(0^+) \geqq v$, but since not every $\tau \in \mathscr{C}$ is in $\mathscr{C}_t$ for some $t > 0$, it is not a priori clear that $f(0^+) = v$. Actually, more than this is true, namely:

PROPOSITION 3.3. *$f(t) \equiv v$ for all t sufficiently small.*

PROOF. $f(t)$ is increasing and $\leqq q(\infty)$ for all t, since $\tau \equiv 1 \in \mathscr{C}_i \; \forall \, t$.

The case $f \equiv q(\infty)$ is easy since, by (2.7), for any τ, as $t \downarrow 0$,

$$E[q(X_\tau)I_{\{\tau > t\}}] \geqq q(\infty)P(\tau > t) \to q(\infty) ,$$

so $v = q(\infty)$.

If $f(t_0) < q(\infty)$ we shall show that there is an s_0, $0 < s_0 < t_0$, and a $\tau' \in \mathscr{C}_{t_0}$ such that any τ can be improved on $\{\tau \leqq s_0\}$ by using τ' instead. This, of course, proves that $f(s_0) = v$.

There is certainly an s_0 such that, for any τ, $E[q(X_\tau)I_{\{\tau \leqq s_0\}}] > f(t_0)P(\tau \leqq s_0)$ because, on $\{\tau \leqq s\}$, we have $X_\tau \geqq X_{Z_1(s)}$, so by Proposition 3.1

$$E[q(X_\tau)I_{\{\tau \leqq s\}} \,|\, \mathscr{F}_s] \geqq R_1(s)I_{\{\tau \leqq s\}} ,$$

and by (3.4), $R_1(s) \uparrow q(\infty) > f(t_0)$ as $s \downarrow 0$.

Now, by Proposition 2.2, there is a $\tau' \in \mathscr{C}_{t_0}$, independent of $\mathscr{F}_{t_0}$, hence of $\mathscr{F}_{s_0}$, with $Eq(X_{\tau'})$ as close as we like to $f(t_0)$. Thus $\tau I_{\{\tau > s_0\}} + \tau' I_{\{\tau \leqq s_0\}}$, which is in $\mathscr{C}_{s_0}$ is better than τ. $\square$

Thus, to know f on $(0, 1)$ is to know v.

PROPOSITION 3.4. *Let $0 < s < t \leqq 1$; let r be a positive integer and let*

$$\sigma = \inf\{U_i \in (s, t] \colon Y_{U_i} \leqq r\} \qquad (3.7)$$
$$= 1 \qquad \text{if no such } U_i$$

then

$$P(Y_\sigma = j \,|\, \mathscr{F}_s) \equiv r^{-1} \qquad j = 1, 2, \cdots, r \quad \text{on} \quad \{\sigma < t\} . \qquad (3.8)$$

PROOF. Let N_j be the number of arrivals in $(s, t]$ of rank between those of the $j - 1$st and jth best arrivals in $(0, s]$; that is

$$N_j = \textstyle\sum_i I_{\{K_{j-1}(s) < K_i(s,t) < K_j(s)\}}$$

where $K_0(s) \equiv 0$. Then the N_j's are independent of $\mathscr{F}_s$ and i.i.d.; also independent of $\mathscr{F}_s$, the arrival at time σ, if there is one, is equally likely to be best, or second best, or $\cdots$ etc., of the $N_1 + \cdots + N_r$ best arrivals in $(s, t]$.

Thus on $\{\sigma \leqq t\}$, for $j = 1, 2, \cdots, r$,

$$\begin{aligned} P(Y_\sigma = j \,|\, \mathscr{F}_s) &= E[P(Y_\sigma = j \,|\, \mathscr{F}_s, \{N_i\}) \,|\, \mathscr{F}_s] \\ &= E[N_j / \textstyle\sum_{i=1}^r N_i \,|\, \sum_{i=1}^r N_i > 0] = 1/r . \end{aligned}$$ $\square$

THEOREM 3.1. *If $v < \infty$, then $f(t)$ is continuous on* $[0, 1)$ *and on this interval* *satisfies the differential equation*:

$$f'(t) = t^{-1} \sum_{j=1}^{\infty} [f(t) - R_j(t)]^+ . \tag{3.9}$$

(This is the differential equation obtained by Mucci [6].)

PROOF. Choose $0 < s < t < 1$ and let θ be the arrival time of the best arrival in $(s, t]$. For any $\tau > s$ we have, by Proposition 3.2, and formulas (2.7), (3.3), and (3.4),

$$\begin{aligned} Eq(X_\tau) &= ER_{Y_\tau}(\tau) \\ &\geqq E[R_{Y_\tau}(\tau) I_{\{\tau \leqq t\}} + f(t) I_{\{\tau > t\}}] \\ &\geqq E[R_{Y_\theta}(t) I_{\{\tau \leqq t\}} + f(t) I_{\{\tau > t\}}] \\ &\geqq E[R_{Y_\theta}(t) \wedge f(t)] . \end{aligned}$$

Thus, for $0 < s < t \leqq 1$,

$$f(s) \geqq E[R_{Y_\theta}(t) \wedge f(t)] . \tag{3.10}$$

Of course $Y_\theta \uparrow \infty$ as either $s \uparrow t$ or $t \downarrow s$, so both left and right continuity follow easily from (3.3) and (3.4) and the monotonicity of f.

Also we can easily see that f is constant on $[0, t_1]$ where $t_1 = \sup\{t: R_1(t) \geqq f(t)\}$, which agrees with (3.9).

Now let σ be as in (3.7), taking $t > t_1$, $r = \max\{j: R_j(t) < f(t)\}$ and s close enough to t so that we also have

$$r = \max\{j: R_j(u) < f(t)\} \qquad \forall\, u \in (s, t] . \tag{3.11}$$

(Properties (3.3) and (3.4) and continuity of f insure that this can be done.) Consider rules of the form

$$\rho = \sigma I_{\{\sigma \leqq t\}} + \tau I_{\{\sigma > t\}}$$

where τ is independent of $\mathscr{F}_t$ and as nearly optimal in $\mathscr{C}_t$ as we like. From (2.7), then (3.3) and (3.4) and Proposition 3.2, we have

$$\begin{aligned} f(s) &\leqq E[q(X_\sigma) I_{\{\sigma \leqq t\}} + f(t) I_{\{\sigma > t\}}] \\ &\leqq E[R_{Y_\sigma}(s) I_{\{\sigma \leqq t\}} + f(t) I_{\{\sigma > t\}}] \\ &= E[R_{Y_\sigma}(s) \wedge f(t)] . \end{aligned} \tag{3.12}$$

Now, for s close enough to t so that (3.11) holds,

$$E[R_{Y_\theta}(t) \wedge f(t)] = \sum_{j=1}^{r} (s/t)^{j-1}(1 - s/t) R_j(t) + (s/t)^r f(t)$$

(since the distribution of Y_θ is obviously geometric); and, using Proposition 3.4, and the fact that $Y_\sigma \leqq r$ if and only if $Y_\theta \leqq r$,

$$E[R_{Y_\sigma}(s) \wedge f(t)] = [1 - (s/t)^r] \sum_{j=1}^{r} R_j(s)/r + (s/t)^r f(t) .$$

It is easy to see that

$$\begin{aligned} \lim_{s \uparrow t} \{f(t) &- E[R_{Y_\theta}(t) \wedge f(t)]\}/(t - s) \\ &= \lim_{s \uparrow t} \{f(t) - E[R_{Y_\sigma}(s) \wedge f(t)]\}/(t - s) \\ &= t^{-1} \sum_{j=1}^{\infty} [f(t) - R_j(t)]^+ . \end{aligned}$$

Hence, by (3.10) and (3.12), (3.9) holds for the left derivative of f. But continuity of f insures its validity for the right derivative as well and the theorem is proved. □

4. An optimal rule. Ignore the trivial case: $q(\cdot)$ constant.

If $v < \infty$, then, by (3.3) and (3.4) and monotonicity of f, there are unique solutions $\{t_k : k = 1, 2, \cdots\}$ to the equations $R_k(t) = f(t)$, with $0 < t_1 < t_2 < \cdots \to 1$, unless $q(M) = q(\infty)$ for some M, in which case $0 < t_1 < \cdots < t_M = t_{M+1} = \cdots = 1$.

Let τ^* be the cutoff-point rule defined by the t_k's. That is

$$\begin{aligned} \tau^* &= \min\{u \in \{U_i\} : R_{Y_u}(u) \leqq f(u)\} \\ &= 1 \qquad \text{if no such } U_i\,. \end{aligned} \tag{4.1}$$

THEOREM 4.1. *τ^* is optimal whenever $v < \infty$.*

The proof of the theorem requires the following strengthening of (2.7):

PROPOSITION 4.1. *If $Eq(X_\sigma) < \infty$, then*

$$E[q(X_\sigma)I_{\{\sigma>\tau\}} \mid \mathscr{F}_\tau] \geqq f(\tau)I_{\{\sigma>\tau\}} \quad \text{a.s.} \tag{4.2}$$

PROOF. Use the same τ_n's as in the proof of Proposition 3.2 to conclude by the same martingale theorem:

$$E[q(X_\sigma)I_{\{\sigma>\tau_n\}} \mid \mathscr{F}_{\tau_n}] \to E[q(X_\sigma)I_{\{\sigma>\tau\}} \mid \mathscr{F}_\tau]\,,$$

while, by using (2.7) and monotonicity of f, we get

$$\begin{aligned} E[q(X_\sigma)I_{\{\sigma>\tau_n\}} \mid \mathscr{F}_{\tau_n}] &\geqq f(\tau_n)I_{\{\sigma>\tau_n\}} \\ &\geqq f(\tau)I_{\{\sigma>\tau_n\}} \\ &\to f(\tau)I_{\{\sigma>\tau\}}\,. \end{aligned}$$ □

Note also that, by (4.1), and Proposition 3.2,

$$E[q(X_{\tau^*}) \mid \mathscr{F}_{\tau^*}] = R_{Y_{\tau^*}}(\tau^*) \leqq f(\tau^*) \tag{4.3}$$

on $\{\tau^* < 1\}$.

PROOF OF THEOREM 4.1. For any rule τ with $Eq(X_\tau) < \infty$, the rule $\min(\tau^*, \tau)$ is at least as good, since by (4.2) and (4.3),

$$\begin{aligned} E[q(X_\tau) \mid \mathscr{F}_{\tau^*}]I_{\{\tau>\tau^*\}} &\geqq f(\tau^*)I_{\{\tau>\tau^*\}} \\ &\geqq E[q(X_{\tau^*}) \mid \mathscr{F}_{\tau^*}]I_{\{\tau>\tau^*\}}\,; \end{aligned}$$

moreover, for each k, on $\{\tau \in [t_{k-1}, t_k), \tau < \tau^*\}$, $R_{Y_\tau}(\tau) \geqq R_k(\tau) > R_k(t_k) = f(t_k)$, so τ can be strictly improved on this event by waiting until time t_t and then using a sufficiently good rule in $\mathscr{C}_{t_k}$.

Since, for any $T < 1$, there are only finitely many t_k's $\leqq T$, we conclude that for some $\tau_T \in \mathscr{C}_T$, the rule

$$\tau^* I_{\{\tau^* \leqq T\}} + \tau I_{\{\min(\tau^*, \tau) > T\}} + \tau_T I_{\{\tau \leqq T < \tau^*\}}$$

is at least as good as τ. Then, taking $T = T_n$ and $\tau = \tau_n$, where $T_n \uparrow 1$ and $Eq(X_{\tau_n}) \to v$, it follows that there are rules $\tau_{T_n} \in \mathscr{C}_{T_n}$ such that, with

$$\tau_n^* = \tau^* I_{\{\tau^* \leqq T_n\}} + \tau_{T_n} I_{\{\tau^* > T_n\}},$$

we have $Eq(X_{\tau_n^*}) \to v$.

Now $f(T_n) \to q(\infty)$ as $T_n \uparrow 1$, (see the next section) and, by (2.7),

$$Eq(X_{\tau_n^*}) \geqq Eq(X_{\tau^*}) I_{\{\tau^* \leqq T_n\}} + f(T_n) P(\tau^* > T_n);$$

so, if $q(\infty) = \infty$, $P(\tau^* < 1) = 1$ and $Eq(X_{\tau^*}) \leqq v$ (hence equality must hold), while, if $q(\infty) = Q < \infty$,

$$\begin{aligned} v &= Eq(X_{\tau^*}) I_{\{\tau^* < 1\}} + QP(\tau^* = 1) \\ &\equiv Eq(X_{\tau^*}) \end{aligned}$$

so again τ^* is optimal. □

5. Boundary conditions and finiteness of the minimal risk. In general, any solution to the differential equation (3.9) which is continuous where it is finite is one of the following kinds:

(1) Constant: $f(t) \equiv f(0) \leqq \infty$;
(2) Bounded: $f(0) < f(1) < \infty$;
(3) Finite unbounded: $f(t) < \infty \ \forall\, t < 1$,
$f(1^-) = \infty$;
(4) Explosive: $f(t) < \infty \ \forall\, t < T$, $0 < T < 1$,
$= \infty \ \forall\, t > T$.

Moreover, for any two solutions, f and g,

$$f(0) < g(0) < \infty \Rightarrow f(t) < g(t) \quad \forall\, t \ni f(t) < \infty .$$

Let f^* denote the solution to (3.9) which is the f for our problem—namely the one which satisfies (2.4).

It is easy to see that $f^*(1^-) = q(\infty)$. This is because if $\tau > t$ then $X_\tau \geqq K_1(t, 1)$, the rank of the best arrival in $(t, 1)$, and $Eq(K_1(t, 1)) = Eq(K_1(1 - t)) = R_1(1 - t) \uparrow q(\infty)$ as $t \uparrow 1$.

Here are some facts (recall the dichotomy for f^* at the end of Section 2).

PROPOSITION 5.1. *$R_1(t) = \infty$ for some $t \in (0, 1) \Rightarrow f^*(t) \equiv \infty$.*

In particular, if q grows exponentially, then v is infinite. Indeed, a somewhat stronger fact is true, namely:

PROPOSITION 5.2. *$\sum [\log q(k)]/k^2 = \infty \Rightarrow f^*(t) \equiv \infty$.*

PROPOSITION 5.3. *$q(\infty) = \infty$ and $q(k + 1)/q(k) = 1 + 0(k^{-1})$*—i.e. *$q(k)$ grows like a power of $k \Rightarrow v < \infty$ and f^* is the unique finite unbounded solution.*

PROPOSITION 5.4. *$q(\infty) < \infty \Rightarrow f^*$ is the unique bounded solution with $f(1^-) = q(\infty)$.*

The last of these is obvious from the remarks at the beginning of this section.

PROOF OF PROPOSITION 5.1. The hypothesis implies that $q(k) > (1 - t)^{-k/2}$ for infinitely many k's. Choose a subsequence $\{k_n\}$ of such k's with $k_{n+1} > 2k_n$. Let

$$\begin{aligned} r(k) &= (1-t)^{-k_n/2} \quad &\text{if } k_n \leqq k < k_{n+1} \\ &= 0 &\text{if } k < k_1 . \end{aligned}$$

The $r(k) \leqq q(k)$ for all k, since q is increasing, so

$$\begin{aligned} \sum [\log q(k)]/k^2 &\geqq \sum [\log r(k)]/k^2 \\ &\geqq -\tfrac{1}{2}\log(1-t)\sum_n (1 - k_n/k_{n+1}) \\ &= \infty . \end{aligned}$$

Apply Proposition 5.2. □

PROOF OF PROPOSITION 5.2. We have only to prove that $v < \infty$ implies convergence of the series whenever $q(\infty) = \infty$. The rule τ^* of Section 4, being optimal, satisfies

$$E[q(X_{\tau^*})I_{\{\tau^* > t\}}] = f(t)P(\tau^* > t)$$

for all t. This goes to zero as $t \uparrow 1$ since $P(\tau^* = 1) = 0$. In particular, at the cutoff points, $\{t_k\}$, of τ^*, $f(t_k) = R_k(t_k) > q(k)$, so, using formula (1.2) for $P(\tau^* > t_k)$ and taking logs, we conclude that

$$\log q(k) + k \log t_k^{-1} - \sum_{j=1}^{k} \log t_j^{-1} \to -\infty \qquad \text{as } k \uparrow \infty .$$

Let $a_k = \log t_k^{-1}$ and choose K so that, for all $k > K$,

$$\log q(k) \leqq \sum_{j=1}^{k} a_j - k a_k .$$

Then for all $N > K$,

$$\begin{aligned} \sum_{k=K}^{N} \frac{\log q(k)}{k(k-1)} &< \sum_{k=K}^{N} \sum_{j=1}^{k} \frac{a_j}{k(k-1)} - \sum_{k=K}^{N} \frac{k a_k}{k(k-1)} \\ &= \sum_{j=1}^{N} a_j \sum_{k=j\vee K}^{N} \frac{1}{k(k-1)} - \sum_{k=K}^{N} \frac{a_k}{k-1} \\ &= \sum_{j=1}^{K-1} \frac{a_j}{K-1} - \frac{1}{N}\sum_{j=1}^{N} a_j \\ &< \sum_{j=1}^{K-1} \frac{a_j}{K-1} . \end{aligned}$$

□

Finiteness of v in Proposition 5.3 is Theorem 4.1 of Mucci [6]. Uniqueness follows immediately from Corollary 2 of Proposition 5.6 and the corollary to Proposition 5.8 both stated and proved below.

The key to a further understanding of the relationship between the differential equation and the infinite secretary problem is to introduce four modified secretary problems:

(A) Truncated Problem: Loss is $q_M(k) = q(k)$ if $k \leqq M$
$\qquad = q(M)$ if $k > M$;

(B) Kickback Problem: Loss if $\tau = 1$ is $B < q(\infty)$;
(C) Hurry Up Problem: Loss if $\tau > T$ is ∞; $T < 1$;
(D) Inflated Cost Problem: Replace $q(k)$ by $R_k(T)$; $T < 1$.

The main results are:

PROPOSITION 5.5. *Let $v_{(M)}$ be the optimal expected cost for the Truncated Problem. Then $v_{(M)} \uparrow v$ whether v is finite or infinite.*

PROOF. If $q(\infty) < \infty$ the proposition is trivial. Likewise if $\lim v_{(M)} = \infty$ since $\lim v_{(M)} \leqq v$. So from now on we assume $\lim v_{(M)} < \infty = q(\infty)$.

Let τ_M be the cutoff-point rule based on $\{t_k^{(M)} : f_M(t_k^{(M)}) = R_k^{(M)}(t_k^{(M)})\}$ which we know, by Theorem 4.1, is optimal for the Truncated Problem. Choose a subsequence $\{M_n\}$ for which $t_k^{(M_n)}$ converges for each k. Then $\lim t_1^{(M_n)} > 0$ by our assumption and the last part of Proposition 1.1. Let τ_∞ be the rule based on the limits of the cutoff points. Then, using Proposition 1.1, for any N,

$$Eq_{M_n}(X_{\tau_{M_n}}) \geqq Eq_{M_n \wedge N}(X_{\tau_{M_n}}) \to Eq_N(X_{\tau_\infty}) \qquad \text{as } n \to \infty .$$

Thus, for all N

$$\lim v_{(M)} \geqq Eq_N(X_{\tau_\infty}) .$$

But by the monotone convergence theorem

$$Eq_N(X_{\tau_\infty}) \to Eq(X_{\tau_\infty}) \qquad \text{as } N \to \infty .$$

The limit is necessarily at least v since τ_∞ is a legitimate stopping rule. Thus $\lim v_{(M)} \geqq v$; the reverse inequality is trivial so the proposition is proved. □

COROLLARY. *For the sequence of N-individual problems with loss functions $q(i)$: $i \leqq N$, the sequence v_N of minimal expected losses converges to v.*

PROOF. Immediate since Mucci ([6], page 420) has shown that $\lim_{M\to\infty} v_{(M)} = \lim_{N\to\infty} v_N$. □

(This, incidentally, shows that Mucci's v and our v are always equal, though they are defined quite differently. His v is defined to be simply the limit of the v_N's.)

Gianini [4] has also proved equality of the two limits by a different method—one which, in effect, "imbeds" the finite problems in the infinite problem.

Further evidence that the infinite problem is the "limit" of the finite problems may be found in the papers of Gianini [4] and Mucci [5, 6]. Special cases are explicitly solved in [2], [5] and [6].

PROPOSITION 5.6. *The f for the Kickback Problem is f_B, the unique bounded solution to* (3.9) *with $f(1^-) = B$.*

PROOF. Theorem 3.1 remains valid. The rest is easy. □

COROLLARY 1. *Let v_B (which is of course $f_B(0)$) be the optimal expected cost for the Kickback Problem. Then $v_B \uparrow v$ as $B \uparrow q(\infty)$.*

PROOF. The Kickback Problem is squeezed between the original problem and the Truncated Problem. □

COROLLARY 2. *When* $q(\infty) = \infty$, *there is a finite unbounded solution if and only if* $v < \infty$. *Moreover*

$$v = \min\{A : f(0) = A \text{ and } f \text{ is a finite unbounded solution}\} .$$

PROOF. Existence of a finite unbounded solution guarantees that $\lim v_B < \infty$. And of course we already know that finiteness of v guarantees the existence of a finite unbounded solution, namely f^*. Minimality comes from Corollary 1. □

Let H and I be "the f's" for the Hurry Up Problem and the Inflated Cost Problem respectively. That is,

$$H(t) = \inf_{\tau>t} Eq(X_\tau)I_{\{\tau<T\}} + \infty \cdot P(\tau \geqq T)$$

($\infty \cdot 0 = 0$ of course), and

$$I(t) = \inf_{\tau>t} ER_{X_\tau}(T) .$$

PROPOSITION 5.7. *H is an infinite or explosive solution to* (3.9); *I is a solution to*

$$f'(t) = t^{-1} \textstyle\sum_{k=1}^{\infty} [f(t) - R_k(tT)]^+ ; \tag{5.1}$$

and $H(t) = I(t/T)$: *in other words the two problems are equivalent.*

PROOF. Theorem 3.1 holds for the Hurry Up Problem so the first assertion is easy. The second is immediate from

$$\textstyle\sum_{n=k}^{\infty} \binom{n-1}{k-1} R_n(T) t^k (1-t)^{n-k} = R_k(tT) . \tag{5.2}$$

Furthermore the transformation $g(t) = f(t/T)$ establishes a correspondence between solutions, f, to (5.1) and explosive solutions, g, to (3.9). This would prove the third assertion if we knew that H and I were the unique solutions satisfying their right-hand boundary conditions. Since we don't know this, we look at the problems more closely. By (5.2), $ER_{X_\tau}(T) = ER_{Y_\tau}(\tau T)$, from which it follows easily that we can transform each problem into the other simply by changing the time scale. (To be completely rigorous, we do need to truncate all τ's in the Hurry Up Problem to T: a minor point.) □

Now let $v_T = H(0) = I(0)$ denote the common minimal expected cost for the two problems.

PROPOSITION 5.8. *If $q(k)$ grows like a power of k—i.e.* $q(k+1)/q(k) = 1 + 0(k^{-1})$*—then* $v_T < \infty \ \forall T < 1$. *Moreover there is a cutoff-point rule,* τ *with* $ER_{X_\tau}(T) < \infty \ \forall T < 1$.

PROOF. The former assertion follows trivially from the latter.

For any cutoff-point rule τ with cutoff points $\{t_i\}$, monotonicity in (3.4) gives

$$\begin{aligned} Eq(X_\tau) &= \textstyle\sum Eq(X_\tau)I_{\{t_k \leqq \tau < t_{k+1}\}} \\ &\leqq \textstyle\sum R_k(t_k)P(t_k \leqq \tau < t_{k+1}) . \end{aligned}$$

if $q(k)$ grows like a power of k, Mucci ([6], Theorem 4.1) has shown that there is a set of cutoff points for which the even larger series $\sum R_k(t_k)P(\tau \geqq t_k)$ converges. This follows from the fact that in this case there is a positive integer M such that for all k, t,

$$R_k(t) \leqq e^{2M^2}t^{-2M}q(k) .$$

Combining this with (5.2) yields

$$R_k(tT) \leqq e^{2M^2}T^{-2M}R_k(t) .$$

Thus, using the same cutoff-point rule, we have

$$\begin{aligned} ER_{X_\tau}(T) &\leqq \sum R_k(t_k T)P(t_k \leqq \tau < t_{k+1}) \\ &\leqq e^{2M^2}T^{-2M} \sum R_k(t_k)P(t_k \leqq \tau < t_{k+1}) \\ &< \infty . \end{aligned}$$

□

COROLLARY. *$v_T \downarrow v$ as $T \uparrow 1$; hence*

$$v = \max \{A : f(0) = A \text{ and } f \text{ is a finite unbounded solution}\} .$$

PROOF. Immediate by the monotone convergence theorem since $R_k(T) \downarrow q(k)$ as $T \uparrow 1$ $\forall k$. □

There is still a gap to be closed. Can Proposition 5.3 be extended to include all $q(\cdot)$'s with $q(\infty) = \infty$ and $v < \infty$? In other words, is a finite unbounded solution to the differential equation—if it exists—always unique?

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