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A MULTIDIMENSIONAL RENEWAL THEOREM[1]

BY THOMAS A. RYAN, JR.

Pennsylvania State University

A multidimensional renewal theorem is proven which generalizes the well-known result of Pyke on Markov renewal processes. The result is a "natural" setting of the multidimensional renewal theorem, and has applications to age-dependent branching processes.

1. Introduction. In this paper we discuss a generalization of the multidimensional renewal theorem of Pyke (1961). The proofs are similar to those of Pyke and of Cheong (1968). The generalization is of interest because (1) it puts the multidimensional renewal theorem in its most natural mathematical setting, and (2) this form of the renewal equation plays an important role in multitype branching processes.

The theorems in this paper are essentially those in the author's doctoral thesis (Ryan (1968)). A similar generalization of the renewal theorem has been given by Mood (1971) under apparently stronger conditions, using a different method of proof.

Notation. When we say A is a matrix of measures, we mean A is an $r \times r$ matrix of distribution functions $a_{ij}(t)$ of finite measures such that $a_{ij}(0^-) = 0$. If A is a matrix of measures A^{*2} is a matrix whose ijth entry is $\sum_{k=1}^{r} a_{ik} * a_{kj}(t)$. Suppose A, B are matrices and ρ is a number. Then expressions such as $A \leqq B$, $A \leqq \rho$, $\int tA(dt)$ are meant to apply entrywise, i.e., $A \leqq \rho$ means $a_{ij} \leqq \rho$ for all ij, and $\int tA(dt)$ is a matrix whose ijth entry is $\int ta_{ij}(dt)$.

THEOREM 1. *If A is an $r \times r$ matrix of measures such that $\int tA(dt) < \infty$, $(A(\infty))^m > 0$ for some m, $A(\infty)$ has largest eigenvalue 1, $a_{ij}(0) < a_{ij}(\infty)$ for some ij, and $C(t)$ is an $r \times r$ matrix which is directly Riemann integrable, then*

$$R(t) = C(t) + A * R(t) \tag{1}$$

has a unique solution which is bounded on finite intervals. This solution satisfies

$$(R(t))_{ij} \to r_{ij} \quad as \quad t \to \infty$$

where

$$r_{ij} = \frac{\mathscr{A}_{i1}^{1}(\infty)}{\int t\mathscr{A}_{11}^{1}(dt)} \sum_k \mathscr{A}_{1k}^{1k}(\infty) \int C_{kj}(dt) .$$

(The $\mathscr{A}$'s are defined after Lemma 4.)

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COROLLARY 1. *If in Theorem* 1, $\int C(t)\,dt \geqq 0$ *and* $\int (C(t))_{ij}\,dt > 0$ *for some* ij *then* $r_{kj} > 0$ *for all* k.

COROLLARY 2. *If* A *and* C *are the same as in Theorem* 1, *except that* $\int tA(dt) \not< \infty$, i.e., *at least one entry* a_{ij} *has infinite mean, then* (1) *has a unique solution which is bounded on infinite intervals, and* $(R(t))_{ij} \to 0$ *as* $t \to \infty$.

2. Proof of the renewal theorem. The proof of the following proposition is routine.

PROPOSITION 1. *If* M *and* N *are* $r \times r$ *matrices of nonnegative numbers such that* $N > 0$, *and* $M \leqq N$ *with at least one entry strictly less than, that* $M^m < N^m$, *for all* $m \geqq 3$.

PROPOSITION 2. (a) *If* A *and* B *are both* $r \times r$ *matrices of measures then* $A * B(t) \leqq A(t)B(t)$ *and* $A * B(\infty) = A(\infty)B(\infty)$.

(b) *If* A *is a matrix of measures such that* $\int tA(dt) < \infty$ *then there is a constant* c *such that* $\int tA^{*n}(dt) \leqq ncA^{*n}(\infty)$, *for all* n.

PROOF. (a) $(A * B)_{ij} = \sum a_{ik} * b_{kj}$ and $(A(t)B(t))_{ij} = \sum a_{ik}(t)b_{kj}(t)$. Let X, Y be random variables with distributions $a_{ik}(t)/a_{ik}(\infty)$ and $b_{kj}(t)/b_{kj}(\infty)$, respectively. Then $(a_{ik}/a_{ik}(\infty)) * (b_{kj}/b_{kj}(\infty))(t) = \Pr\{X + Y \leqq t\} \leqq \Pr\{X \leqq t\}\Pr\{Y \leqq t\} = (a_{ik}/t)/a_{ik}(\infty))(b_{kj}(t)/b_{kj}(\infty))$, and the first statement follows. Also, $1 = \Pr\{X + Y \leqq \infty\} = (a_{ik}/a_{ik}(\infty)) * (b_{kj}/b_{kj}(\infty))(\infty) = (a_{ik} * b_{kj}(\infty))/(a_{ik}(\infty)b_{kj}(\infty))$ and the second statement follows.

(b) $(\int tA^{*n}(dt))_{i_0i_n} = \sum \int ta_{i_0i_1} * \cdots * a_{i_{n-1}i_n}(dt)$. Consider one integral in the sum. For $1 \leqq k \leqq n$, let Y_k be a random variable with distribution $a_{i_{k-1}i_k}(t)/a_k$, where $a_k = a_{i_{k-1}i_k}(\infty)$. Let $d = \max\{E(Y_1), \cdots, E(Y_n)\}$. Then $nd \geqq E(Y_1 + \cdots + Y_n) = \int t(a_{i_0i_1}/a_1) * \cdots * (a_{i_{n-1}i_n}/a_n)(dt) = (\int ta_{i_0i_1} * \cdots * a_{i_{n-1}i_n}(dt))/(a_1 \cdots a_n)$. Hence $\int ta_{i_0i_1} * \cdots * a_{i_{n-1}i_n}(dt) \leqq nda_{i_0i_1}(\infty) \cdots a_{i_{n-1}i_n}(\infty)$. Summing, we get $(\int tA^{*n}(dt))_{i_0i_n} \leqq nc((A(\infty))^n)_{i_0i_n}$ where c is the maximum of all the d's. Using part (a), we get $(\int tA^{*n}(dt)) \leqq nc(A(\infty))^n = ncA^{*n}(\infty)$.

If N is an $r \times r$ matrix of nonnegative numbers such that $N^m > 0$ for some integer m, then the Perron–Frobenius theory says that N has an eigenvalue which is positive and exceeds all other eigenvalues in absolute value.

LEMMA 1. *Suppose* N *is an* $r \times r$ *matrix of nonnegative numbers such that* $N^m > 0$ *for some integer* m, *and* N *has largest eigenvalue* 1. *Suppose* M *is an* $r \times r$ *matrix of nonnegative numbers such that* $M \leqq N$, *with at least one entry strictly less than. Then there are numbers* c *and* $0 < \rho < 1$ *such that* $M^n \leqq c\rho^n$, *for all* n.

PROOF. By Proposition 1, $(M^m)^3 < (N^m)^3$. Thus $M^{3m} < uN^{3m}$ for some $u < 1$. Consider M^ν where $\nu = 3mn + k$ and $k < 3m$. Then $M^\nu = (M^{3m})^nM^k \leqq (uN^{3m})^nN^k = u^nN^\nu$. Since N has largest eigenvalue 1 its powers are bounded from the Perron–Frobenius theory. Thus, using $\rho = u^{1/3m}$, we can find a number c such that $M^\nu \leqq \rho^\nu c$.

If A is a matrix $Z_i[A]$ is the matrix A with all entries in row i and all entries in column i replaced by zeros.

LEMMA 2. *Suppose A is a matrix of measures such that* $\int tA(dt) < \infty$, $(A(\infty))^m > 0$ *for some integer m, and $A(\infty)$ has largest eigenvalue* 1. *Then* $\sum_{n=1}^{\infty} \int t(Z_i[A])^{*n}(dt) < \infty$, *for all i.*

PROOF. Since $\int tZ_i[A](dt) < \infty$, by Proposition 2, there is a constant d such that $\int t(Z_i[A])^{*n}(dt) \leqq nd(Z_i[A])^{*n}(\infty) = nd(Z_i[A(\infty)])^n$, for all n. Since $Z_i[A(\infty)] \leqq A(\infty)$ with at least one entry strictly less than, Lemma 1 yields constants c and $\rho < 1$ such that $(Z_i[A(\infty)])^n \leqq c\rho^n$, for all n. Hence $\sum \int t(Z_i[A])^{*n}(dt) \leqq \sum ndc\rho^n$, which converges.

LEMMA 3. *Suppose A is a matrix of measures such that $(A(\infty))^m > 0$ for some m, $A(\infty)$ has largest eigenvalue* 1, *and $a_{ij}(0) < a_{ij}(\infty)$ for some i, j. Then, for all t, $A^{*n}(t) \to 0$ as $n \to \infty$.*

PROOF. Since $a_{ij}(0) < a_{ij}(\infty)$, there is some $t_0 > 0$ such that $a_{ij}(t_0) < a_{ij}(\infty)$. From this and the fact that $(A(\infty))^m > 0$ it is easy to see that $(A^{*(m+1)}(t_0))_{ij} < (A^{*(m+1)}(\infty))_{ij}$. Then, as in Proposition 1, it is easy to show that $A^{*3(m+1)}(t_0) < A^{*3(m+1)}(\infty)$.

Consider any two distribution functions (possibly defective) $F(t)$, $G(t)$. Let X, Y be random variables with distributions $F(t)/F(\infty)$, $G(t)/G(\infty)$ respectively. Suppose $F(t_0) < F(\infty)$ and $G(t_0) < G(\infty)$. Then $\Pr\{X + Y > 2t_0\} > 0$ and it follows that $F * G(2t_0) < F * G(\infty)$. Using this and the fact that $A^{*3(m+1)}(t_0) < A^{*3(m+1)}(\infty)$ we have, routinely, that $A^{*3k(m+1)}(kt_0) < A^{*3k(m+1)}(\infty)$, for all k.

Consider any fixed t. Choose $k > t/t_0$. We can write any n in the form $n = 3ks(m+1) + r$ for some $r < 3k(m+1)$. By Proposition 2, $A^{*3ks(m+1)}(t) \leqq (A^{*3k(m+1)}(t))^s \leqq (A^{*3k(m+1)}(kt_0))^s \leqq (\rho A^{*3k(m+1)}(\infty))^s$, for some $\rho < 1$. Hence $A^{*n}(t) \leqq A^{*3ks(m+1)}(t)A^{*r}(t) \leqq \rho^s(A(\infty))^n$. Since the powers of $A(\infty)$ are bounded, we have $A^{*n}(t) \to 0$ as $n \to \infty$.

LEMMA 4. *Suppose A is a matrix of measures as in Lemma* 3. *Then* $\sum_{n=0}^{\infty} A^{*n}(t)$ *converges for all t.*

PROOF. Consider a fixed t. By Lemma 3, $A^{*n}(t) \to 0$ as $n \to \infty$. So, $(A^{*m}(t))_{ij} < a_{ij}(\infty)$, for some m, i, j. Using Lemma 1 on the matrices $A^{*m}(t)$ and $A(\infty)$ we get $(A^{*m}(t))^n < c\rho^n$, $\rho < 1$ and thus $\sum_{n=0}^{\infty} A^{*mn}(t)$ converges. Similarly, $\sum_{n=0}^{\infty} A^{*(m+k)n}(t)$ converges for all $k = 0, \cdots, m-1$ and the lemma follows.

If A is a matrix of measures, define $\mathscr{A}_{ij}^{r_1 \cdots r_s}$ as the sum of all products (of any length n) of the form $a_{ik_1} * a_{k_1k_2} * \cdots * a_{k_{n-2}k_{n-1}} * a_{k_{n-1}j}$ where none of the $k_1, \cdots, k_{n-1}$ are equal to $r_1, r_2, \cdots, r_s$. Thus $\mathscr{A}_{ij}^{r_1 \cdots r_s} = \sum_{n=1}^{\infty} \sum_{k_1 \cdots k_{n-1} \neq r_1 \cdots r_s} a_{ik_1} * \cdots * a_{k_{n-1}j}$. It is easy to see that this series converges when $(A(\infty))^m > 0$ for some m, and $A(\infty)$ has largest eigenvalue 1, since, for any $n \geqq 2$, $\sum_{k_1 \cdots k_{n-1} \neq r_1 \cdots r_s} a_{ik_1} * \cdots * a_{k_{n-1}j}(t) \leqq \sum_{k_1 \cdots k_{n-1} \neq r_1 \cdots r_s} a_{ik_1} * \cdots * a_{k_{n-1}j}(\infty) \leqq (A(\infty)(Z_{r_1}[A(\infty)])^{n-2}A(\infty))_{ij} \leqq c\rho^{n-2}$, $\rho < 1$ by Lemma 1.

LEMMA 5. *Suppose A is an $r \times r$ matrix of measures as in Lemma* 3. *Then*

$$\begin{aligned}(\textstyle\sum_{n=2}^{\infty} A^{*n})_{ij} = {} & \mathscr{A}_{i1}^{1} * \textstyle\sum_{n=0}^{\infty} (\mathscr{A}_{11}^{1})^{*n} * \mathscr{A}_{1j}^{1j} \\ & + \mathscr{A}_{i2}^{12} * \textstyle\sum_{n=0}^{\infty} (\mathscr{A}_{22}^{12})^{*n} * \mathscr{A}_{2j}^{12j} + \cdots \\ & + \mathscr{A}_{ir}^{12\cdots r} * \textstyle\sum_{n=0}^{\infty} (\mathscr{A}_{rr}^{12\cdots r})^{*n} * \mathscr{A}_{rj}^{12\cdots r} .\end{aligned}$$

PROOF. Notice that on the right side of the equation, each of the r summands, $\mathscr{A}_{is}^{1\cdots s} * \sum_{n=0}^{\infty} (\mathscr{A}_{ss}^{1\cdots s})^{*n} * \mathscr{A}_{sj}^{1\cdots sj}$, consists of the sum of all terms $a_{ik_1} * \cdots * a_{k_{n-1}j}$ (for any length n) such that at least one of the subscripts $k_1, \cdots, k_{n-1}$ is an s but none is a $1, 2, \cdots, s-1$. It is fairly clear that the terms in the sum $(\sum_{n=2}^{\infty} A^{*n})_{ij} = \sum_n \sum_{k_1\cdots k_{n-1}} a_{ik_1} * \cdots * a_{k_{n-1}j}$ can be rearranged and collected into these r categories.

LEMMA 6. *Suppose A is a matrix of measures as in Lemma* 3. *Then* $\sum_{n=0}^{\infty} (\int e^{\alpha t} A(dt))^n < \infty$, *for all* $\alpha < 0$.

PROOF. If $\alpha < 0$, then $\int e^{\alpha t} A(dt) \leqq A(\infty)$ with $\int e^{\alpha t} a_{ij}(dt) < a_{ij}(\infty)$ for any entry such that $a_{ij}(0) < a_{ij}(\infty)$. Then by Lemma 1, $(\int e^{\alpha t} A(dt))^n \leqq c\rho^n$ where $\rho < 1$. Thus the series converges.

LEMMA 7. *Suppose A is a matrix of measures as in Lemma* 3. *Then $\mathscr{A}_{11}^{1}(dt)$ is a probability measure.*

PROOF. Taking transforms of the decomposition in Lemma 5, we get $\int e^{\alpha t} (\sum_{n=2}^{\infty} A^{*n})_{ij}(dt) \geqq \int e^{\alpha t} \mathscr{A}_{i1}^{1} * \sum_{n=0}^{\infty} (\mathscr{A}_{11}^{1})^{*n} * \mathscr{A}_{1j}^{1j}(dt)$. From Proposition 2a, the left side of this inequality is equal to $(\sum_{n=2}^{\infty} (\int e^{\alpha t} A(dt))^n)_{ij}$ and the right side is equal to $(\int e^{\alpha t} \mathscr{A}_{i1}^{1}(dt))(\sum_{n=0}^{\infty} (\int e^{\alpha t} \mathscr{A}_{11}^{1}(dt))^n)(\int e^{\alpha t} \mathscr{A}_{1i}^{1j}(dt))$. By Lemma 6, $\sum (\int e^{\alpha t} A(dt))^n$ converges for $\alpha < 0$. Thus $\sum (\int e^{\alpha t} \mathscr{A}_{11}^{1}(dt))^n$ converges, and consequently $\int e^{\alpha t} \mathscr{A}_{11}^{1}(dt) < 1$ for all $\alpha < 0$. By the Lebesgue dominated convergence theorem, $\int e^{\alpha t} \mathscr{A}_{11}^{1}(dt) \leqq 1$ for $\alpha = 0$ and so $\mathscr{A}_{11}^{1}(\infty) \leqq 1$.

From the fact that, for some n, $(A(\infty))^n > 0$, we can see that $a_{1k_1} * a_{k_1k_2} * \cdots * a_{k_{n-1}1}(\infty) > 0$ for some sequence $k_1, k_2, \cdots, k_{n-1} \neq 1$ with at least one subscript among $k_1, \cdots, k_{n-1}$ equal to 2. Then if $s \geqq 2$, $\mathscr{A}_{11}^{1} = \mathscr{A}_{11}^{1\cdots s} + a_{1k_1} * a_{k_1k_2} * \cdots * a_{k_{n-1}1} + \mu$, for some measure μ. Hence $\mathscr{A}_{11}^{1\cdots s}(\infty) < \mathscr{A}_{11}^{1}(\infty) \leqq 1$. Then $\sum (\mathscr{A}_{11}^{1\cdots s})^{*n}(\infty) < \infty$ for all $s \geqq 2$. Since $A(\infty)$ has largest eigenvalue 1, $A^{*n}(\infty)$ approaches a non-zero limit and $\sum A^{*n}(\infty)$ diverges. Using the decomposition of A^{*n} given in Lemma 5, we see that $\sum (\mathscr{A}_{11}^{1})^{*n}(\infty)$ must diverge. And since we already showed that $\mathscr{A}_{11}^{1}(\infty) \leqq 1$, it must equal 1.

PROOF OF THEOREM 1. First we show uniqueness. Suppose R_1 and R_2 are two solutions bounded on finite intervals. Then $R_1 - R_2 = A * (R_1 - R_2)$. Hence $(R_1 - R_2)(t) = A^{*n} * (R_1 - R_2)(t) \leqq A^{*n}(t)(R_1 - R_2)(\infty) \to 0$, as $n \to \infty$, from Lemma 3. Thus $(R_1 - R_2)(t) = 0$ for all t.

We now show that a formal solution to the renewal equation is given by $R(t) = (\sum_{n\geq 0} A^{*n}) * C(t)$. Thus, $C + A * R = C + A * (\sum_{n\geq 1} A^{*n}) * C = C + (\sum_{n\geq 0} A^{*n}) * C = R$. The boundedness of $R(t)$ is obvious.

Finally, we investigate the limit of $R(t)$. Now, $R = (\sum_{n\geqq 0} A^{*n}) * C = C + A * C + (\sum_{n\geqq 2} A^{*n}) * C$. Since C is directly Riemann integrable and $A(\infty) < \infty$, $C(t) + A * C(t) \to 0$ as $t \to \infty$.

Decompose $((\sum_{n\geqq 2} A^{*n}) * C)_{ij}$ as in Lemma 5, and consider the first term. Now, $\sum_k \mathscr{A}_{i1}^{1} * \sum_n (\mathscr{A}_{11}^{1})^{*n} * \mathscr{A}_{1k}^{1k} * C_{kj} = \sum_k [C_{kj} * \sum_n (\mathscr{A}_{11}^{1})^{*n}] * \mathscr{A}_{i1}^{1} * \mathscr{A}_{1k}^{1k}$. Consider the term in square brackets. Now, it is clear that $\mathscr{A}_{11}^{1} = \sum_{ij\neq 1} a_{i1} * (\sum (Z_1[A])^{*n})_{ij} * a_{j1}$. By Lemma 2, $\sum (Z_1[A])^{*n}$ has finite mean, and thus $\mathscr{A}_{11}^{1}$ has finite mean. By Lemma 7, $\mathscr{A}_{11}^{1}$ is a probability measure. Then, since C_{kj} is directly Riemann integrable, the (one-dimensional) renewal theorem says $C_{kj} * \sum (\mathscr{A}_{11}^{1})^{*n}(t) \to d_{kj}$ boundedly. So, the first term in our decomposition approaches the finite limit $\sum_k d_{kj} \mathscr{A}_{i1}^{1}(\infty) \mathscr{A}_{1k}^{1k}(\infty)$. All the other terms in the decomposition involve measures $\mathscr{A}_{ss}^{1\cdots s}$ with $s \geqq 2$. In the proof of Lemma 7, we showed that $\mathscr{A}_{11}^{12\cdots s}(\infty) < 1$, for any $s \geqq 2$. Renumbering states, we get $\mathscr{A}_{ss}^{12\cdots s}(\infty) < 1$ for any $s \geqq 2$. It then follows that the 2nd through rth terms in the decomposition tend to zero as $t \to \infty$. (Use an analysis similar to that for the first term.)

3. Application to branching processes. There are r types of particles. A particle of type i lives a lifetime which has distribution $G_i(t)$. The particle may have offspring of any type, anytime throughout its life. Let $Q_{ij}(t) = E$ (number of type j particles born to a type i particle by age t). Both (1) the number of offspring a particle has in dt, and (2) whether or not the particle dies in dt may depend on the number, types and times of birth of its previous offspring, and on the particle's age in dt. All particles behave independently, and all particles of a given type behave according to the same distribution. Let $M_{ij}(t) = E$ (number of type j particles alive at time $t \mid$ the process started at time 0 with one particle of type i and age 0) and let $M(t) = (M_{ij}(t))$.

THEOREM 2. *Suppose* $(Q(\infty))^m > 0$ *for some* m, $Q(\infty)$ *has largest eigenvalue less than* 1, Q_{ij} *is nonlattice for all* i, j, $Q_{ij}(0) < Q_{ij}(\infty)$ *for some* i, j, *and* $G_i(0) < 1$ *for all* i. *If there exists on* α *(necessarily positive) such that the largest eigenvalue of* $\int_0^\infty e^{\alpha t} Q(dt)$ *is* 1, $\int te^{\alpha t} Q(dt) < \infty$ *and* $\int te^{\alpha t} G_i(dt) < \infty$ *for all* i, *then*

$$M(t) \approx e^{-\alpha t} C\,, \qquad as \quad t \to \infty$$

where C *is an* $r \times r$ *matrix with* $C > 0$, *and* $\approx$ *means the ratios of the corresponding entries on both sides tends to* 1.

PROOF. $M_{ij}(t)$ is the sum of the expected number of zeroth generation type j particles alive at t plus the sum, over all the intervals $d\tau$, of the expected number of type j particles alive at t which are descended from first generation particles born in $d\tau$. This yields the equation:

$$M_{ij}(t) = \delta_{ij}[1 - G_i(t)] + \int_{0-}^{t+} [\textstyle\sum_k Q_{ik}(d\tau) M_{kj}(t - \tau)]\,. \tag{2}$$

Since the imbedded Galton–Watson process is subcritical, the finiteness of $M(t)$ is obvious. Multiplying equation (2) by $e^{\alpha t}$, we get

$$e^{\alpha t} M_{ij}(t) = e^{\alpha t}\delta_{ij}[1 - G_i(t)] + \int_{0-}^{t+} \textstyle\sum_k e^{\alpha\tau} Q_{ik}(d\tau) e^{\alpha(t-\tau)} M_{kj}(t - \tau)\,. \tag{3}$$

Define the following matrices: $R_{ij}(t) = e^{\alpha t} M_{ij}(t)$, $C_{ij}(t) = e^{\alpha t}\delta_{ij}[1 - G_i(t)]$ and $A_{ij}(t) = e^{\alpha t} Q_{ij}(t)$. Then we can rewrite equation (3) as

$$R(t) = C(t) + A * R(t) .$$

The conditions of Theorem 1 are routinely verified for $A(t)$ and $C(t)$ and Theorem 2 follows.

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DEPARTMENT OF STATISTICS
219 POND LABORATORY
PENNSYLVANIA STATE UNIVERSITY
UNIVERSITY PARK, PA. 16802