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UNIFORM TAUBERIAN THEOREMS AND THEIR APPLICATIONS TO RENEWAL THEORY AND FIRST PASSAGE PROBLEMS[1]

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In this paper, we prove an analogue of the classical renewal theorem for the case where there is no drift. Our proof depends on a uniform version of Spitzer's well-known theorem on ladder epochs and ladder variables, and we obtain this uniform result by using uniform Tauberian theorems. Some further applications of these uniform Tauberian theorems to other problems in renewal theory and first passage times are also given.

1. Introduction and summary. Let $X_1, X_2, \cdots$ be i.i.d. random variables, $S_n = X_1 + \cdots + X_n$. For $a > 0$, define

$$T(a) = \inf\{n \geqq 1 : S_n > a\}. \tag{1.1}$$

The renewal theorem asserts that if $EX_1 = \mu > 0$, then

$$ET(a) \sim a/\mu \qquad \text{as } a \to \infty. \tag{1.2}$$

As to the case $\mu = 0$, it follows from a result of Spitzer [10] that if $0 < EX_1^2 < \infty$, then

$$\begin{aligned} ET^\gamma(a) &< \infty \quad \text{if } \gamma < \tfrac{1}{2}, \\ &= \infty \quad \text{if } \gamma \geqq \tfrac{1}{2}. \end{aligned} \tag{1.3}$$

In this case, letting $EX_1^2 = \sigma^2$, it is well known (cf. [11]) that as $a \to \infty$,

$$\sigma^2 T(a)/a^2 \to_{\mathscr{D}} \tau = \inf\{t : W(t) > 1\}, \tag{1.4}$$

where $W(t)$, $t \geqq 0$, is the standard Wiener process and "$\to_{\mathscr{D}}$" denotes convergence in distribution. As an analogue of the classical renewal theorem (1.2) for the case $\mu > 0$, it is natural to ask if the following statement holds in the present case of zero mean:

$$\lim_{a\to\infty} E(\sigma^2 T(a)/a^2)^\gamma = E\tau^\gamma \qquad \text{for } 0 < \gamma < \tfrac{1}{2}. \tag{1.5}$$

The answer to this question turns out to be affirmative as will be shown in Section 4 below, where our method of proof in fact gives a stronger version of (1.5). In this connection, we shall also obtain a slight refinement of (1.4).

Let us now consider the case $\mu = 0$ and $\sigma = 1$. Define the ladder epoch

$$N = \inf\{n \geqq 1 : S_n > 0\}. \tag{1.6}$$

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Spitzer [10] has proved that as $n \to \infty$,

(1.7) $$P[N > n] \sim c(n\pi)^{-\frac{1}{2}}$$

where $c = \exp\{\sum_1^\infty k^{-1}(P[S_k \leqq 0] - \frac{1}{2})\}$. He has also shown that the series defining c is indeed convergent, so $0 < c < \infty$. Our proof of (1.5) depends on the following uniform version of Spitzer's result:

THEOREM 1. *Let $\mathscr{F}$ be a family of distribution functions with mean* 0, *variance* 1 *and satisfying the following uniform square integrability condition*:

(1.8) $$\sup_{F \in \mathscr{F}} \int_{|x|>a} x^2\, dF(x) \to 0 \qquad \text{as } a \to \infty .$$

Let $X_1, X_2, \cdots$ be i.i.d. *with a common distribution function $F \in \mathscr{F}$, $S_n = X_1 + \cdots + X_n$, and let N be the ladder epoch as defined by* (1.6). *Let $c_F = \exp\{\sum_1^\infty k^{-1}(P_F[S_k \leqq 0] - \frac{1}{2})\}$, where P_F denotes the probability measure under which X_1 has distribution function F. Then $\inf_{F \in \mathscr{F}} c_F > 0$, $\sup_{F \in \mathscr{F}} c_F < \infty$ and*

(1.9) $$\sup_{F \in \mathscr{F}} |(n\pi)^{\frac{1}{2}} P_F[N > n] - c_F| \to 0 \qquad \text{as } n \to \infty .$$

Furthermore, the following uniform integrability condition is satisfied:

(1.10) $$\sup_{F \in \mathscr{F}} \int_{[S_N > a]} S_N\, dP_F \to 0 \qquad \text{as } a \to \infty .$$

We shall prove Theorem 1 in Section 3 below. Our proof illustrates an application of certain uniform versions of Tauberian theorems which will be listed in Section 2 for reference. We shall also give a number of other applications of uniform Tauberian theorems to renewal theory, one of which is the following uniform version of (1.2):

THEOREM 2. *Let $\mathscr{F}$ be a family of distribution functions such that the following two conditions are satisfied*:

(1.11 a) $$\inf_{F \in \mathscr{F}} \int_{-\infty}^{\infty} x\, dF(x) > 0 ;$$

(1.11 b) $$\sup_{F \in \mathscr{F}} \int_a^\infty x\, dF(x) \to 0 \qquad \text{as } a \to \infty .$$

Let $X, X_1, X_2, \cdots$ be i.i.d. *random variables with a common distribution function $F \in \mathscr{F}$ and let $\mu_F = E_F X$, where E_F denotes the expectation corresponding to the probability distribution under which X has distribution function F. Set $S_n = X_1 + \cdots + X_n$ and define $T(a)$ by* (1.1). *Then as $a \to \infty$,*

(1.12) $$E(T(a)/a) \to \mu_F^{-1} \qquad \textit{uniformly for } F \in \mathscr{F} .$$

Furthermore, defining the ladder epoch N by (1.6), *we have $\sup_{F \in \mathscr{F}} E_F N < \infty$, and the uniform integrability condition* (1.10) *is satisfied by the ladder variable S_N.*

The proof of Theorem 2 will be given in Section 6 where we shall also give other uniform versions of the elementary renewal theorem. Since there is a close connection between the strong law of large numbers and (1.2), it is interesting to compare Theorem 2 with Chung's uniform version of the strong law of large numbers (cf. [2]) which states that if $\tilde{\mathscr{F}}$ is a family of distribution functions satisfying

$$\sup_{F\in\tilde{\mathscr{F}}} \int_{|x|>a} |x|\, dF(x) \to 0 \qquad \text{as } a \to \infty\,, \tag{1.13}$$

then given any $\varepsilon > 0$,

$$\sup_{F\in\tilde{\mathscr{F}}} P_F[\sup_{j\geqq m} |j^{-1}S_j - \mu_F| \geqq \varepsilon] \to 0 \qquad \text{as } m \to \infty\,. \tag{1.14}$$

We note that while Chung's result requires the uniform integrability condition (1.13), our result only requires the uniform integrability of X^+ plus the obvious condition $\inf_{F\in\mathscr{F}} \mu_F > 0$. In [6], Lorden has proved that if $\mu_F > 0$, then for all $a \geqq 0$,

$$E_F(S_{T(a)} - a) \leqq E_F(X^+)^2/\mu_F\,. \tag{1.15}$$

Therefore if we replace (1.11 b) by the stronger assumption

$$\sup_{F\in\mathscr{F}} \int_0^\infty x^2\, dF(x) < \infty\,, \tag{1.11 b$'$}$$

then (1.12) follows easily from (1.15) and Wald's lemma. Uniform Tauberian theorems, however, enable us to obtain the uniform convergence in (1.12) without resort to second moment conditions.

A motivation for studying uniform renewal theorems of the type like Theorem 2 and Theorem 8 below lies in the investigation of robust properties of certain sequential procedures in statistics. In evaluating a sequential statistical procedure, we would like to investigate its expected sample size not only for the distribution assumed by the theoretical model, but also for small deviations from the idealized theoretical model. Unfortunately the exact expression of the expected sample size is in most cases unwieldy and we often have to resort to asymptotic approximations. To judge the performance of the procedure from the asymptotic estimates, it is comforting to know that these asymptotic estimates at least converge at a uniform rate over the family of all distribution functions that deviate slightly from the idealized model. Of course, for practical applications, it is desirable to have also an estimate of this uniform rate and we should substantiate our asymptotic results with actual Monte Carlo studies. Our investigations in robust sequential procedures will be reported in another paper, and it is some of these investigations that first led us to the uniform renewal theorems discussed in this paper.

2. Uniform Tauberian theorems. Throughout this section, we shall let $I\,(\neq \varnothing)$ denote an index set. It will be convenient to describe a measure μ on $[0, \infty)$ by its (possibly improper) distribution function which we shall also denote by μ. In the subsequent sections, we shall make use of some of the following uniform Tauberian theorems.

THEOREM 3. *Let $\{\mu_i, i \in I\}$ be a family of measures on $[0, \infty)$ such that*

$$f_i(s) = \int_0^\infty e^{-st}\, d\mu_i(t) \tag{2.1}$$

is finite for all $s > 0$ and $i \in I$. If there exists $\alpha \geqq 0$ such that as $s \downarrow 0$,

$$s^\alpha f_i(s) \to A_i \qquad \text{uniformly for} \quad i \in I\,, \tag{2.2}$$

and $\sup_i A_i < \infty$, *then as* $t \to \infty$,

$$t^{-\alpha}\mu_i(t) \to A_i/\Gamma(\alpha+1) \qquad \textit{uniformly for} \quad i \in I. \tag{2.3}$$

THEOREM 4. *Let* $\{g_i, i \in I\}$ *be a family of real-valued functions on* (δ, ∞) *such that for each* $i \in I$, g_i *has a second derivative on* (δ, ∞). *Suppose for some real number* α *and some positive number* M,

(i) $x^{-\alpha}g_i(x) \to A_i$ *uniformly for* $i \in I$ *as* $x \to \infty$ *and* $\sup_i |A_i| < \infty$;
(ii) *for each* $i \in I$, *either* $\sup_{x>\delta} x^{2-\alpha}g_i''(x) < M$ *or*

$$\inf_{x>\delta} x^{2-\alpha}g_i''(x) > -M.$$

Then as $x \to \infty$,

$$x^{1-\alpha}g_i'(x) \to \alpha A_i \qquad \textit{uniformly for} \quad i \in I. \tag{2.4}$$

THEOREM 5. *Let* $\alpha \geqq 0$ *and let* $\{h_i, i \in I\}$ *be a family of continuously differentiable functions on* (δ, ∞) *satisfying the following conditions*:

(i) $x^{-\alpha}\int_0^x h_i(t)\,dt \to A_i$ *uniformly for* $i \in I$ *as* $x \to \infty$ *and* $\sup_i |A_i| < \infty$;
(ii) h_i *is monotone for each* $i \in I$.

Then as $x \to \infty$,

$$x^{1-\alpha}h_i(x) \to \alpha A_i \qquad \textit{uniformly for} \quad i \in I. \tag{2.5}$$

The same conclusion (2.5) *still holds if we replace condition* (ii) *by*:

(ii)′ *For each* $i \in I$, *either* $(xh_i(x))' \geqq 0$ *for all* $x > \delta$ *or* $(xh_i(x))' \leqq 0$ *for all* $x < \delta$.

THEOREM 6. *For each* $i \in I$, *let* $(q_n^{(i)})_{n\geqq 0}$ *be a sequence of nonnegative numbers such that*

$$Q_i(t) = \sum_{n=0}^{\infty} q_n^{(i)}t^n \tag{2.6}$$

converges for $0 \leqq t < 1$. *Suppose for some* $\alpha \geqq 0$, *as* $t \uparrow 1$,

$$(1-t)^{\alpha}Q_i(t) \to A_i \qquad \textit{uniformly for} \quad i \in I, \tag{2.7}$$

and $\sup_i A_i < \infty$. *Then as* $n \to \infty$,

$$n^{-\alpha}(q_0^{(i)} + \cdots + q_n^{(i)}) \to A_i/\Gamma(\alpha+1) \qquad \textit{uniformly for} \quad i \in I. \tag{2.8}$$

Furthermore, if for each $i \in I$, *the sequence* $(q_n^{(i)})$ *is monotone and* $\alpha > 0$, *then as* $n \to \infty$,

$$n^{1-\alpha}q_n^{(i)} \to A_i/\Gamma(\alpha) \qquad \textit{uniformly for} \quad i \in I. \tag{2.9}$$

When I is a singleton, these theorems are well known and their proofs (see [12], pages 189–195) can be easily modified to prove the above uniform versions when I is an arbitrary index set. We remark that Theorem 6 corresponds to the discrete case of Theorems 3 and 5. In fact, letting μ_i in Theorem 3 be the atomic measure putting mass $q_n^{(i)}$ at the point n, $n = 0, 1, 2, \cdots$, the conclusion (2.8) is an immediate consequence of Theorem 3. In the subsequent sections, in addition to the above uniform Tauberian theorems, we shall also make use of the following uniform Abelian theorem. Its proof is straightforward (see [12], pages 181–182 for the case where I is a singleton).

THEOREM 7. (i) *Let* $\{\mu_i, i \in I\}$ *be a family of real-valued functions on* $[0, \infty)$ *such that for each* $T > 0$, $\sup_i \sup_{0 \leqq t \leqq T} |\mu_i(t)| < \infty$ *and* μ_i *is of bounded variation on* $[0, T]$ *for all* $i \in I$. *Suppose for some* $\alpha \geqq 0$, (2.3) *is satisfied as* $t \to \infty$ *and* $\sup_i |A_i| < \infty$. *Then the improper Riemann–Stieltjes integral* $f_i(s)$ *defined by* (2.1) *converges for all* $s > 0$ *and* $i \in I$ *and* (2.2) *holds as* $s \downarrow 0$.

(ii) *For each* $i \in I$, *let* $(q_n^{(i)})_{n \geqq 0}$ *be a sequence of real numbers such that* $\sup_i \sup_{1 \leqq n \leqq N} |q_n^{(i)}| < \infty$ *for every positive integer* N. *Suppose for some* $\alpha > 0$, (2.9) *is satisfied as* $n \to \infty$ *and* $\sup_i |A_i| < \infty$. *Then the series* $Q_i(t)$ *defined by* (2.6) *converges for* $0 \leqq t < 1$ *and* (2.7) *holds for* $t \uparrow 1$.

3. Proof of Theorem 1. Throughout this section, we shall use the same notations and assumptions as in Theorem 1 which is stated in Section 1. We first prove some preliminary lemmas.

LEMMA 1. *Let* $\sigma_n^2(F) = \operatorname{Var}_F (X_1 I_{[|X_1| \leqq n^{\frac{1}{2}}]})$, $n = 1, 2, \cdots$ *and let* Φ *denote the distribution function of the standard normal distribution. The series*

$$\sum_1^\infty n^{-1} \sup_x |P_F[n^{-\frac{1}{2}} S_n \leqq x] - \Phi(x/\sigma_n(F))| \tag{3.1}$$

converges uniformly for $F \in \mathscr{F}$. *Consequently the series*

$$\sum_1^\infty (t^n/n)(P_F[S_n \leqq 0] - \tfrac{1}{2}) \tag{3.2}$$

converges uniformly for $F \in \mathscr{F}$ *and* $t \in [0, 1]$, *and* $\inf_{F \in \mathscr{F}} c_F > 0$, $\sup_{F \in \mathscr{F}} c_F < \infty$.

PROOF. The uniform convergence of the series (3.1) is a uniform version of the result of Friedman, Katz and Koopman [4]. Set

$$X_{k,n} = X_k I_{[|X_k| \leqq n^{\frac{1}{2}}]}, \qquad S_{n,n} = X_{1,n} + \cdots + X_{n,n}.$$

We shall show that as $k \to \infty$,

$$\sup_{F \in \mathscr{F}} \sum_{n=k}^\infty n^{-\frac{1}{2}} |E_F X_{1,n}| \to 0 ; \tag{3.3}$$

$$\sup_{F \in \mathscr{F}} \sum_{n=k}^\infty n^{-\frac{3}{2}} E_F |X_{1,n}|^3 \to 0 ; \tag{3.4}$$

$$\sup_{F \in \mathscr{F}} \sum_{n=k}^\infty P_F[|X_1| > n^{\frac{1}{2}}] \to 0 ; \tag{3.5}$$

$$\sigma_k^2(F) \to 1 \quad \text{uniformly for} \quad F \in \mathscr{F}; \tag{3.6}$$

$$\sup_{F \in \mathscr{F}} \sum_{n=k}^\infty n^{-1} \sup_x |\Phi((x - n^{\frac{1}{2}} E_F X_{1,n})/\sigma_n(F)) - \Phi(x/\sigma_n(F))| \to 0 . \tag{3.7}$$

To prove (3.3), we note that for $k \geqq 2$,

$$\begin{aligned} \sum_{n=k}^\infty n^{-\frac{1}{2}} |E_F X_{1,n}| &\leqq \int_{|X_1| \geqq k^{\frac{1}{2}}} |X_1| (\textstyle\sum_{2 \leqq n \leqq X_1^2} n^{-\frac{1}{2}}) \, dP_F \\ &\leqq 2 \int_{|X_1| \geqq k^{\frac{1}{2}}} |X_1|^2 \, dP_F \\ &\to 0 \quad \text{uniformly for} \quad F \in \mathscr{F} \quad \text{as} \quad k \to \infty \quad \text{by (1.8)} . \end{aligned}$$

To prove (3.4), we have for $k \geqq 3$,

$$\begin{aligned}\textstyle\sum_{n=k}^{\infty} n^{-\frac{3}{2}} E_F|X_{1,n}|^3 &\leqq \textstyle\sum_{n=k}^{\infty} n^{-\frac{3}{2}} E_F|X_1|^3 I_{[|X_1| \leqq k^{\frac{1}{2}}]} \\ &\quad + \textstyle\sum_{n=k}^{\infty} n^{-\frac{3}{2}} \sum_{j=k}^{n} j^{\frac{3}{2}} P_F[j-1 < X_1^2 \leqq j] \\ &\leqq 2(k-1)^{-\frac{1}{2}}\{k^{\frac{1}{2}} E_F X_1^2 I_{[k^{\frac{1}{4}} \leqq |X_1| \leqq k^{\frac{1}{2}}]} + k^{\frac{1}{4}} E_F X_1^2 I_{[|X_1| \leqq k^{\frac{1}{4}}]}\} \\ &\quad + 3 \textstyle\sum_{j=k}^{\infty} j P_F[j-1 < X_1^2 \leqq j] \\ &\rightarrow 0 \qquad \text{uniformly for} \quad F \in \mathscr{F} \quad \text{as} \quad k \rightarrow \infty .\end{aligned}$$

The relations (3.5) and (3.6) are immediate from (1.8). We now prove (3.7). By the mean value theorem, for all x,

$$\begin{aligned}|\Phi((x - n^{\frac{1}{2}} E_F X_{1,n})/\sigma_n(F)) - \Phi(x/\sigma_n(F))| &\leqq (2\pi)^{-\frac{1}{2}} n^{\frac{1}{2}} |E_F X_{1,n}|/\sigma_n(F) \\ &\leqq n^{\frac{1}{2}} |E_F X_{1,n}| \qquad \text{for} \quad n \geqq n_0 ,\end{aligned}$$

where n_0 is independent of F in view of (3.6). Therefore (3.7) follows from (3.3). Making use of (3.3), (3.4), (3.5), (3.6) and (3.7), we can follow the same argument as in the proof of Theorem 1 of [4] to prove the uniform convergence of the series (3.1). (The use of Rosén's theorem in [4] is totally unnecessary, as our proof of (3.7) shows. This point is also mentioned in the note added in proof of [4].)

LEMMA 2. *Let* $q_n(F) = P_F[N > n]$ *for* $n = 0, 1, \cdots$. *Then as* $t \uparrow 1$,

$$(1 - t)^{\frac{1}{2}} \textstyle\sum_{n=0}^{\infty} q_n(F) t^n \rightarrow c_F \qquad \textit{uniformly for} \quad F \in \mathscr{F} . \tag{3.8}$$

PROOF. It is well known (cf. [10]) that

$$(1 - t)^{\frac{1}{2}} \textstyle\sum_{n=0}^{\infty} q_n(F) t^n = \exp\{\sum_1^{\infty} (t^n/n)(P_F[S_n \leqq 0] - \frac{1}{2})\} . \tag{3.9}$$

By Lemma 1, the series (3.2) converges uniformly for $F \in \mathscr{F}$ and $t \in [0, 1]$. It therefore easily follows that as $t \uparrow 1$,

$$\textstyle\sum_1^{\infty} (t^n/n)(P_F[S_n \leqq 0] - \frac{1}{2}) \rightarrow \sum_1^{\infty} n^{-1}(P_F[S_n \leqq 0] - \frac{1}{2}) \qquad \text{uniformly for} \quad F \in \mathscr{F} . \tag{3.10}$$

The desired conclusion (3.8) then follows from (3.9) and (3.10).

LEMMA 3. *As* $n \rightarrow \infty$,

$$P_F[n^{-\frac{1}{2}} S_n \leqq x] \rightarrow \Phi(x) \qquad \textit{uniformly for} \quad F \in \mathscr{F} \quad \textit{and} \quad x \in (-\infty, \infty) ; \tag{3.11}$$

$$E_F(n^{-\frac{1}{2}} S_n)^- \rightarrow (2/\pi)^{\frac{1}{2}} \qquad \textit{uniformly for} \quad F \in \mathscr{F} . \tag{3.12}$$

PROOF. The relation (3.11) is due to Parzen ([8], page 38). To prove (3.12), we note that $P_F[|n^{-\frac{1}{2}} S_n| > x] \leqq x^{-2}$ for all n and $F \in \mathscr{F}$. Hence given $\varepsilon > 0$, we can choose $B > 0$ such that $\int_B^{\infty} P_F[n^{-\frac{1}{2}} S_n < -x]\, dx < \varepsilon$ and $\int_B^{\infty} \Phi(-x)\, dx < \varepsilon$ for all $F \in \mathscr{F}$ and $n \geqq 1$. Therefore writing

$$E_F(n^{-\frac{1}{2}} S_n)^- = \int_0^{\infty} P_F[n^{-\frac{1}{2}} S_n < -x]\, dx = \int_0^B + \int_B^{\infty} ,$$

we easily obtain (3.12) from (3.11).

LEMMA 4. *As* $n \to \infty$,

$$E_F S_n I_{[N>n]} \to -2^{-\frac{1}{2}} c_F \quad \textit{uniformly for} \quad F \in \mathscr{F}; \tag{3.13}$$

$$E_F S_N I_{[N \geqq n]} \to 0 \quad \textit{uniformly for} \quad F \in \mathscr{F}. \tag{3.14}$$

PROOF. Let $a_n(F) = E_F S_n I_{[N>n]}$, $A_F(t) = \sum_{n=0}^\infty a_n(F) t^n$. Then it is well known (cf. [10], page 159) that the sequence $(a_n(F))_{n \geqq 0}$ is negative and monotone and

$$A_F(t) = (\sum_1^\infty k^{-1} t^k E_F S_k I_{[S_k \leqq 0]}) \exp\{\sum_1^\infty (t^k/k) P_F[S_k \leqq 0]\}.$$

Since $\sup_{F \in \mathscr{F}} c_F < \infty$, to prove (3.13) using Theorem 6, we need only show that as $t \uparrow 1$,

$$(1 - t) A_F(t) \to -2^{-\frac{1}{2}} c_F \quad \text{uniformly for} \quad F \in \mathscr{F}. \tag{3.15}$$

We now prove (3.15). By (3.10), we have as $t \uparrow 1$,

$$\begin{aligned}(1-t)^{\frac{1}{2}} \exp\{\sum_1^\infty (t^k/k) P_F[S_k \leqq 0]\} &= \exp\{\sum_1^\infty (t^k/k)(P_F[S_k \leqq 0] - \tfrac{1}{2})\} \\ &\to c_F \quad \text{uniformly for} \quad F \in \mathscr{F}.\end{aligned} \tag{3.16}$$

We note that $E_F S_k^-/k \leqq E_F|X_1| \leqq 1$ for all k and $F \in \mathscr{F}$ and by (3.12),

$$k^{\frac{1}{2}}(E_F S_k^-/k) \to (2/\pi)^{\frac{1}{2}} \quad \text{uniformly for} \quad F \in \mathscr{F}.$$

Therefore by Theorem 7, as $t \uparrow 1$,

$$(1-t)^{\frac{1}{2}} \sum_1^\infty (t^k/k) E_F S_k^- \to 2^{-\frac{1}{2}} \quad \text{uniformly for} \quad F \in \mathscr{F}. \tag{3.17}$$

From (3.16) and (3.17), the desired conclusion (3.15) follows.

To prove (3.14), we note that

$$\begin{aligned} E_F S_N I_{[N \geqq n]} &= \lim_{m\to\infty} \sum_{j=n}^m (E_F S_j I_{[N>j-1]} - E_F S_j I_{[N>j]}) \\ &= \lim_{m\to\infty} \sum_{j=n}^m (E_F S_{j-1} I_{[N>j-1]} - E_F S_j I_{[N>j]}) \\ &= E_F S_{n-1} I_{[N>n-1]} - \lim_{m\to\infty} E_F S_m I_{[N>m]} \\ &= E_F S_{n-1} I_{[N>n-1]} + 2^{-\frac{1}{2}} c_F. \end{aligned} \tag{3.18}$$

From (3.13) and (3.18), (3.14) follows immediately.

PROOF OF THEOREM 1. Defining $q_n(F) = P_F[N > n]$ as in Lemma 2, since $\sup_{F \in \mathscr{F}} c_F < \infty$ and $(q_n(F))_{n \geqq 0}$ is monotone and nonnegative, (1.9) is an immediate consequence of Lemma 2 and Theorem 6. We now prove (1.10). By (3.14), given $\varepsilon > 0$, we can choose m large enough such that $E_F S_N I_{[N \geqq m]} \leqq \varepsilon$ for all $F \in \mathscr{F}$. Then

$$\begin{aligned} E_F S_N I_{[S_N > a]} &\leqq \varepsilon + \sum_{n=1}^m E_F S_n I_{[N=n, S_n > a]} \\ &\leqq \varepsilon + \sum_{n=1}^m E_F S_n I_{[S_n > a]} \leqq \varepsilon + \sum_{n=1}^m n a^{-1}. \end{aligned}$$

4. A uniform renewal theorem when there is no drift. In this section, we shall prove the analogue (1.5) of the elementary renewal theorem for the case of zero mean. To do this, we shall divide the random variables $X_1, X_2, \cdots$ into large blocks and take their sum within each block to form new i.i.d. random variables $X_1', X_2', \cdots$. We then apply Theorem 1 to analyze the successive ladder epochs defined in terms of $X_1', X_2', \cdots$. This argument turns out to give not only (1.5), but with a little more work, also its uniform version as stated in the following theorem.

THEOREM 8. *Let $\mathscr{F}$ be a family of distribution functions with mean* 0, *variance* 1 *and satisfying the uniform square integrability condition* (1.8). *Let $X_1, X_2, \cdots$ be* i.i.d. *with a common distribution function $F \in \mathscr{F}$, $S_n = X_1 + \cdots + X_n$ and define $T(a)$ as in* (1.1). *Then letting Φ denote the distribution function of the standard normal distribution, we have as $a \to \infty$,*

$$P_F[T(a)/a^2 > u] \to 2\Phi(u^{-\frac{1}{2}}) - 1 \quad \textit{uniformly for} \quad F \in \mathscr{F} \quad \textit{and} \quad u > 0\,; \tag{4.1}$$

$$\textit{for any} \quad 0 < \gamma < \tfrac{1}{2}\,, \quad E_F(T(a)/a^2)^\gamma \to \gamma \textstyle\int_0^\infty u^{\gamma-1}(2\Phi(u^{-\frac{1}{2}}) - 1)\,du \quad \textit{uniformly for} \quad F \in \mathscr{F}\,. \tag{4.2}$$

PROOF. To prove (4.1), define successive ladder epochs $M_1 = \inf\{n \geqq 1: S_n > 0\}$, $M_2 = \inf\{n > N_1: S_n - S_{N_1} > 0\}$, etc. Let $Z_i = S_{M_i} - S_{M_{i-1}}$ ($M_0 = S_0 = 0$), $N_i = M_i - M_{i-1}$, $\tau(a) = \inf\{n \geqq 1: Z_1 + \cdots + Z_n > a\}$. By the uniform integrability condition (1.10), Chung's uniform law of large numbers (see (1.14)) is applicable, and so for any $\varepsilon > 0$,

$$P_F[|n^{-1}(Z_1 + \cdots + Z_n) - 2^{-\frac{1}{2}}c_F| > \varepsilon] \to 0 \quad \text{uniformly for} \quad F \in \mathscr{F}\,. \tag{4.3}$$

As an immediate corollary of this, we obtain that as $a \to \infty$,

$$P_F[|(\tau(a)/a) - (2^{\frac{1}{2}}/c_F)| > \varepsilon] \to 0 \quad \text{uniformly for} \quad F \in \mathscr{F}\,. \tag{4.4}$$

By Theorem 1, $\sup_{F \in \mathscr{F}} |(n\pi)^{\frac{1}{2}} P_F[N_1 > n] - c_F| \to 0$ as $n \to \infty$, $\sup_{F \in \mathscr{F}} c_F < \infty$ and $\inf_{F \in \mathscr{F}} c_F > 0$. Hence by Theorem 9(i) in the next section, we have as $n \to \infty$,

$$P_F[(N_1 + \cdots + N_n)/(\tfrac{1}{2}c_F^2 n^2) > u] \to 2\Phi(u^{-\frac{1}{2}}) - 1 \quad \text{uniformly for} \quad F \in \mathscr{F} \quad \text{and} \quad u > 0\,. \tag{4.5}$$

Take any $0 < \varepsilon < 1$. We note that

$$\begin{aligned} P_F[T(a) > a^2u] &\leqq P_F[\tau(a) < 2^{\frac{1}{2}}(1+\varepsilon)a/c_F,\ N_1 + \cdots + N_{[2^{\frac{1}{2}}(1+\varepsilon)a/c_F]} > a^2u] \\ &\qquad + P_F[\tau(a) \geqq 2^{\frac{1}{2}}(1+\varepsilon)a/c_F] \\ &\leqq P_F[a^{-2}(N_1 + \cdots + N_{[2^{\frac{1}{2}}(1+\varepsilon)a/c_F]}) > u] \\ &\qquad + P_F[\tau(a) \geqq 2^{\frac{1}{2}}(1+\varepsilon)a/c_F] \\ &\to 2\Phi((1+\varepsilon)u^{-\frac{1}{2}}) - 1 \quad \text{uniformly for} \quad F \in \mathscr{F} \quad \text{and} \quad u > 0\,. \end{aligned}$$

The last relation follows from (4.4) and (4.5). In a similar way,

$$\begin{aligned} &P_F[T(a) > a^2u] \\ &\quad \geqq P_F[a^{-2}(N_1 + \cdots + N_{[2^{\frac{1}{2}}(1-\varepsilon)a/c_F]}) > u] - P_F[\tau(a) \leqq 2^{\frac{1}{2}}(1-\varepsilon)a/c_F] \\ &\quad \to 2\Phi((1-\varepsilon)u^{-\frac{1}{2}}) - 1 \quad \text{uniformly for} \quad F \in \mathscr{F} \quad \text{and} \quad u > 0\,. \end{aligned}$$

Since ε is arbitrary, (4.1) is established.

To prove (4.2), since $T(a) \leqq T(a')$ if $a < a'$, we can without loss of generality restrict ourselves to integral values of a. Let

$$X_n(a) = (X_{(n-1)a^2+1} + \cdots + X_{na^2})/a\,, \qquad S_n(a) = X_1(a) + \cdots + X_n(a)\,.$$

Define the successive ladder epochs $M_0(a) = 0$, $M_1(a) = \inf\{n \geqq 1 : S_n(a) > 0\}$, $M_2(a) = \inf\{n > M_1(a) : S_n(a) - S_{M_1(a)}(a) > 0\}$, etc. Set $N_i(a) = M_i(a) - M_{i-1}(a)$, $Z_i(a) = S_{M_i(a)}(a) - S_{M_{i-1}(a)}(a)$. Choose $\delta > 0$ such that $\gamma + \delta < \frac{1}{2}$. It is easy to see that for any $u > 1$,

$$\begin{aligned}
&P_F[T(a) > a^2u] \\
&\qquad = P_F[\max_{n \leqq a^2u} S_n/a \leqq 1] \leqq P_F[\max_{j \leqq u} S_j(a) \leqq 1] \\
(4.6)\qquad &\qquad \leqq P_F[N_1(a) + \cdots + N_{[u^\delta]}(a) \leqq u,\ Z_1(a) + \cdots + Z_{[u^\delta]}(a) \leqq 1] \\
&\qquad\qquad + P_F[N_1(a) + \cdots + N_{[u^\delta]}(a) > u] \\
&\qquad \leqq P_F[Z_1(a) + \cdots + Z_{[u^\delta]}(a) \leqq 1] \\
&\qquad\qquad + P_F[N_1(a) + \cdots + N_{[u^\delta]}(a) > u]\,.
\end{aligned}$$

We note that $E_F X_1(a) = 0$, $E_F X_1^2(a) = 1$ and by Lemma 5 below,

$$(4.7) \qquad \sup_{a \geqq 1, F \in \mathscr{F}} E_F X_1^2(a) I_{[|X_1(a)| > x]} \to 0 \qquad \text{as} \quad x \to \infty\,.$$

Therefore letting $c_a(F) = \exp\{\sum_1^\infty k^{-1}(P_F[S_k(a) \leqq 0] - \frac{1}{2})\}$, we obtain by Theorem 1 that $\sup_{a \geqq 1, F \in \mathscr{F}} c_a(F) < \infty$, $\inf_{a \geqq 1, F \in \mathscr{F}} c_a(F) > 0$, and as $n \to \infty$,

$$(4.8) \qquad (n\pi)^{\frac{1}{2}} P_F[N_1(a) > n] \to c_a(F) \qquad \text{uniformly for} \quad a \geqq 1 \quad \text{and} \quad F \in \mathscr{F}\,.$$

Hence we have a constant $c > 0$ and a positive integer n_0 such that

$$(4.9) \qquad P_F[N_1(a) > n] \leqq c(n\pi)^{-\frac{1}{2}} \qquad \text{for all} \quad n \geqq n_0\,, \quad a \geqq 1 \quad \text{and} \quad F \in \mathscr{F}\,.$$

In view of (4.9), we can choose constants $\theta > 0$ and ζ and construct a new probability space $(\Omega, \mathscr{B}, P)$ on which we define a random variable Y_1 and a family $\{N_1(a, F) : a \geqq 1, F \in \mathscr{F}\}$ of random variables such that the following conditions are satisfied:

(4.10a) $\quad Y_1$ has a stable distribution with exponent $\frac{1}{2}$;

(4.10b) $\quad N_1(a, F) < \theta Y_1 + \zeta \qquad$ for all $a \geqq 1$ and $F \in \mathscr{F}$;

(4.10c) $\quad$ the distribution function of $N_1(a, F)$ is $P_F[N_1(a) \leqq x]$, $-\infty < x < \infty$.

(See the proof of Theorem 9(i) in Section 5 below and [5], pages 144–145.) Letting $Y_1, Y_2, \cdots$ be i.i.d. random variables on $(\Omega, \mathscr{B}, P)$, we then obtain for all $a \geqq 1$ and $F \in \mathscr{F}$,

$$\begin{aligned}
&P_F[N_1(a) + \cdots + N_{[u^\delta]}(a) > u] \\
(4.11)\qquad &\qquad \leqq P[\theta(Y_1 + \cdots + Y_{[u^\delta]}) + \zeta[u^\delta] > u] \\
&\qquad = P[Y_1 > (u - \zeta[u^\delta])/(\theta[u^\delta]^2)]\,.
\end{aligned}$$

The last equality in (4.11) follows from the fact that $(Y_1 + \cdots + Y_{[u^\delta]})/[u^\delta]^2$

has the same distribution as Y_1. Noting that $P[Y_1 > x] \sim Ax^{-\frac{1}{2}}$ as $x \to \infty$ for some $A > 0$, we obtain from (4.11) that

$$P_F[N_1(a) + \cdots + N_{[u^\delta]}(a) > u] \leqq 2A\theta^{\frac{1}{2}}u^{\delta-\frac{1}{2}} \quad \text{for all} \quad u \geqq u_0, \quad a \geqq 1 \quad \text{and} \quad F \in \mathscr{F}. \tag{4.12}$$

By (3.11), $P_F[Z_1(a) > 1] \geqq P_F[X_1(a) > 1] = P_F[S_{a^2} > a] \geqq 1 - \Phi(2)$ for all $F \in \mathscr{F}$ and $a \geqq a_0$ (where $a_0 \geqq 1$ is independent of F). Therefore $P_F[Z_1(a) \leqq 1] \leqq \Phi(2) = p < 1$, and so for all $a \geqq a_0$, $u > 1$ and $F \in \mathscr{F}$,

$$P_F[Z_1(a) + \cdots + Z_{[u^\delta]}(a) \leqq 1] \leqq (P_F[Z_1(a) \leqq 1])^{[u^\delta]} \leqq p^{[u^\delta]}. \tag{4.13}$$

By (4.6), (4.12) and (4.13), since $\gamma + \delta < \frac{1}{2}$, for any given $\varepsilon > 0$, we can choose $B \geqq u_0$ such that

$$\gamma \int_B^\infty u^{\gamma-1}P_F[T(a) > a^2u]\,du < \varepsilon \qquad \text{for all} \quad a \geqq a_0 \quad \text{and} \quad F \in \mathscr{F}. \tag{4.14}$$

By (4.1), we obtain that as $a \to \infty$,

$$\gamma \int_0^B u^{\gamma-1}P_F[T(a) > a^2u]\,du \to \gamma \int_0^B u^{\gamma-1}(2\Phi(u^{-\frac{1}{2}}) - 1)\,du \quad \text{uniformly for} \quad F \in \mathscr{F}. \tag{4.15}$$

From (4.14) and (4.15), the desired conclusion (4.2) follows.

LEMMA 5. *With the same notations and assumptions as in Theorem* 8, *we have as* $x \to \infty$,

$$\sup_{n \geqq 1, F \in \mathscr{F}} E_F(n^{-\frac{1}{2}}S_n)^2 I_{[|n^{-\frac{1}{2}}S_n| > x]} \to 0. \tag{4.16}$$

PROOF. Given $\varepsilon > 0$, we can choose c large enough such that $\int_{|x| \geqq c} x^2\,dF(x) \leqq \varepsilon$ for all $F \in \mathscr{F}$. Let $X_i' = X_i I_{[|X_i| < c]}$, $X_i'' = X_i - X_i'$, $S_n' = X_1' + \cdots + X_n'$, $S_n'' = X_1'' + \cdots + X_n''$. Clearly $S_n = (S_n' - E_F S_n') + (S_n'' - E_F S_n'')$. For all $F \in \mathscr{F}$,

$$n^{-1}E_F(S_n'' - E_F S_n'')^2 \leqq E_F(X_1'')^2 \leqq \varepsilon. \tag{4.17}$$

By the Marcinkiewicz-Zygmund inequality (cf. [7]), there exists a universal constant K such that for all $F \in \mathscr{F}$,

$$E_F(S_n' - E_F S_n')^4 \leqq K E_F\{\textstyle\sum_1^n (X_i' - E_F X_i')^2\}^2 \leqq 16Kc^4n^2.$$

Therefore by the Schwarz inequality,

$$\begin{aligned} n^{-1}E_F(S_n' - E_F S_n')^2 I_{[|n^{-\frac{1}{2}}S_n| > x]} &\leqq 4K^{\frac{1}{2}}c^2 P_F^{\frac{1}{2}}[|n^{-\frac{1}{2}}S_n| > x] \\ &\leqq 4K^{\frac{1}{2}}c^2x^{-1} \to 0 \qquad \text{as} \quad x \to \infty. \end{aligned} \tag{4.18}$$

The second inequality above follows from the Chebyshev inequality. From (4.17) and (4.18), the desired conclusion (4.16) follows.

5. A uniform limit theorem for distributions in the domain of attraction of a stable law.

THEOREM 9. *Let* $0 < \alpha < 1$ *and let* $\mathscr{F}$ *be a family of distribution functions such that* $F(0-) = 0$ *and as* $x \to \infty$,

$$x^\alpha(1 - F(x)) \to A_F/\Gamma(1 - \alpha) \qquad \textit{uniformly for} \quad F \in \mathscr{F}, \tag{5.1}$$

where $\sup_{F\in\mathscr{F}} A_F < \infty$ *and* $\inf_{F\in\mathscr{F}} A_F > 0$. *Let* $X_1, X_2, \cdots$ *be* i.i.d. *random variables with a common distribution function* $F \in \mathscr{F}$, *and set* $S_n = X_1 + \cdots + X_n$ $(S_0 = 0)$. *Let* G_α *be the distribution function of the stable distribution with Laplace transform* $f(\lambda) = \exp(-\lambda^\alpha)$, $\lambda > 0$.

(i) *As* $n \to \infty$,

$$P_F[(nA_F)^{-1/\alpha} S_n \leqq x] \to G_\alpha(x) \quad \text{uniformly for} \quad F \in \mathscr{F} \quad \text{and} \quad x \geqq 0\,. \tag{5.2}$$

(ii) *Let* $U_F(x) = \sum_{n=0}^\infty P_F[S_n \leqq x]$ *denote the renewal function. Then as* $t \to \infty$,

$$t^{-\alpha} U_F(t) \to (A_F \Gamma(1+\alpha))^{-1} \quad \text{uniformly for} \quad F \in \mathscr{F}\,. \tag{5.3}$$

PROOF. To prove (i), we make use of an argument due to Lévy ([5], pages 144–145). For each $F \in \mathscr{F}$, define $F^{-1}(t) = \sup\{x : F(x) \leqq t\}$, $0 \leqq t < 1$. Since $x^\alpha(1 - G_\alpha(x)) \to 1/\Gamma(1-\alpha)$ as $x \to \infty$, it follows from (5.1) that given any $1 > \delta > 0$, there exists $\eta > 0$ such that

$$1 - G_\alpha(x/((1-\delta)A_F^{1/\alpha})) \leqq 1 - F(x) \leqq 1 - G_\alpha(x/((1+\delta)A_F^{1/\alpha})) \tag{5.4}$$

for all $x \geqq \eta$ and $F \in \mathscr{F}$. Setting $\underline{G}_\alpha(x) = G_\alpha(x/((1+\delta)A_F^{1/\alpha}))$ and $\bar{G}_\alpha(x) = G_\alpha(x/((1-\delta)A_F^{1/\alpha}))$, we obtain from (5.4) that $\bar{G}_\alpha(x) \geqq F(x) \geqq \underline{G}_\alpha(x)$ for $x \geqq \eta$ and $F \in \mathscr{F}$, and therefore

$$\bar{G}_\alpha^{-1}(t) \leqq F^{-1}(t) \leqq \underline{G}_\alpha^{-1}(t) \quad \text{if} \quad t \geqq \bar{G}_\alpha(\eta)\,. \tag{5.5}$$

Let $\theta = \underline{G}_\alpha^{-1}(\bar{G}_\alpha(\eta))$. Clearly $\theta \geqq \eta$ and if $t < \bar{G}_\alpha(\eta)$, then $0 \leqq F^{-1}(t) \leqq \theta$ and $0 \leqq \underline{G}_\alpha^{-1}(t) \leqq \theta$. It then follows from (5.5) that

$$\bar{G}_\alpha^{-1}(t) - \theta \leqq F^{-1}(t) \leqq \underline{G}_\alpha^{-1}(t) + \theta \quad \text{for all} \quad t \in [0, 1) \quad \text{and} \quad F \in \mathscr{F}\,. \tag{5.6}$$

Set $Y = G_\alpha^{-1}$ and $Z(F) = F^{-1}$. Then Y and $Z(F)$ are random variables on the probability space $([0, 1), \mathscr{B}, P)$ where $\mathscr{B}$ denotes the class of Borel sets and P is the Lebesgue measure. Also the distribution functions of $Z(F)$ and Y are F and G_α respectively. Therefore from (5.6), we obtain that for all $F \in \mathscr{F}$,

$$(1-\delta)A_F^{1/\alpha} Y - \theta \leqq Z(F) \leqq (1+\delta)A_F^{1/\alpha} Y + \theta\,. \tag{5.7}$$

Letting $Y, Y_1, Y_2, \cdots$ be i.i.d. random variables on $([0, 1), \mathscr{B}, P)$, we then obtain that for all $F \in \mathscr{F}$ and $x \geqq 0$,

$$\begin{aligned} &P[(nA_F)^{-1/\alpha}\{(1+\delta)A_F^{1/\alpha} \textstyle\sum_{i=1}^n Y_i + n\theta\} \leqq x] \\ &\quad \leqq P_F[(nA_F)^{-1/\alpha} S_n \leqq x] \\ &\quad \leqq P[(nA_F)^{-1/\alpha}\{(1-\delta)A_F^{1/\alpha} \textstyle\sum_{i=1}^n Y_i - n\theta\} \leqq x]\,. \end{aligned} \tag{5.8}$$

Noting that $(nA_F)^{-1/\alpha}(n\theta) \to 0$ uniformly for $F \in \mathscr{F}$, we have

$$\begin{aligned} &P[(nA_F)^{-1/\alpha}\{(1+\delta)A_F^{1/\alpha} \textstyle\sum_{i=1}^n Y_i + n\theta\} \leqq x] \\ &\qquad = P[(1+\delta)Y_1 + (nA_F)^{-1/\alpha} n\theta \leqq x] \\ &\qquad \to P[Y_1 \leqq x/(1+\delta)] \quad \text{uniformly for} \quad F \in \mathscr{F} \quad \text{and} \quad x \geqq 0\,. \end{aligned}$$

Likewise the last term of (5.8) converges to $P[Y_1 \leqq x/(1-\delta)]$ uniformly for $F \in \mathscr{F}$ and $x \geqq 0$. Hence the desired conclusion (5.2) follows.

To prove (ii), let $\varphi_F(\lambda) = \int_0^\infty e^{-\lambda x}\, dF(x)$, $\Psi_F(\lambda) = \int_0^\infty e^{-\lambda x}\, dU_F(x)$, $\omega_F(x) = \int_0^\infty e^{-\lambda x}(1 - F(x))\, dx$. Then $\omega_F(\lambda) = (1-\varphi_F(\lambda))/\lambda$ and $\Psi_F = 1/(1-\varphi_F)$ (cf. [3], page 446). Since $\sup_{F\in\mathscr{F}} A_F < \infty$, it then follows from (5.1) and Theorem 7 that as $\lambda \downarrow 0$,

(5.9) $$\lambda^{1-\alpha}\omega_F(\lambda) \to A_F \qquad \text{uniformly for} \quad F \in \mathscr{F}.$$

Since $\inf_{F\in\mathscr{F}} A_F > 0$, this implies that as $\lambda \downarrow 0$,

(5.10) $$\begin{aligned}\lambda^\alpha \Psi_F(\lambda) &= \lambda^\alpha/(1-\varphi_F(\lambda)) \\ &= 1/(\lambda^{1-\alpha}\omega_F(\lambda)) \to A_F^{-1} \qquad \text{uniformly for} \quad F \in \mathscr{F}.\end{aligned}$$

The desired conclusion (5.3) then follows from Theorem 3.

6. Uniform versions of classical renewal theorems. Theorem 2 stated in Section 1 is a uniform version of the renewal theorem (1.2). We now make use of Theorem 3 to provide its proof.

PROOF OF THEOREM 2. We first assume that $P_F[X \geqq 0] = 1$ for all $F \in \mathscr{F}$. In this case, obviously $P_F[N = 1] = 1$ and (1.10) holds. To prove (1.12), we let $U_F(x) = E_F T(x) = \sum_{n=0}^\infty P_F[S_n \leqq x]$ $(S_0 = 0)$. Let $\varphi_F(\lambda) = \int_0^\infty e^{-\lambda x}\, dF(x)$, $\Psi_F(\lambda) = \int_0^\infty e^{-\lambda x}\, dU_F(x)$, $\omega_F(\lambda) = (1-\varphi_F(\lambda))/\lambda$ for $\lambda > 0$. Define $\nu_F(t) = \int_0^t (1 - F(x))\, dx$, $x \geqq 0$. Then by (1.11b), as $t \to \infty$,

(6.1) $$\nu_F(t) \to \mu_F \qquad \text{uniformly for} \quad F \in \mathscr{F},$$

and $\sup_{F\in\mathscr{F}} \mu_F < \infty$. Since ω_F is the Laplace transform of the measure ν_F, it follows by Theorem 7 that as $\lambda \downarrow 0$,

(6.2) $$\omega_F(\lambda) \to \mu_F \qquad \text{uniformly for} \quad F \in \mathscr{F}.$$

Noting that $\inf_{F\in\mathscr{F}} \mu_F > 0$, this implies that as $\lambda \downarrow 0$.

(6.3) $$\lambda\Psi_F(\lambda) = \lambda/(1-\varphi_F(\lambda)) = 1/\omega_F(\lambda) \to \mu_F^{-1} \qquad \text{uniformly for} \quad F \in \mathscr{F}.$$

Hence as $a \to \infty$, $U_F(a)/a \to \mu_F^{-1}$ uniformly for $F \in \mathscr{F}$ by Theorem 3.

We now consider the general case and drop the assumption that $P_F[X \geqq 0] = 1$. We first prove that $\sup_{F\in\mathscr{F}} E_F N < \infty$. It is well known that

(6.4) $$E_F N = \exp\{\textstyle\sum_1^\infty n^{-1} P_F[S_n \leqq 0]\}$$

(cf. [10]). By (1.11a), there exists $\delta > 0$ such that $\mu_F \geqq 2\delta$ for all $F \in \mathscr{F}$. By (1.11b), we can choose $c > 1$ such that $E_F X I_{[X \geqq c]} < \delta$ for all $F \in \mathscr{F}$. For $i = 1, \cdots, n$, let $X_i' = X_i I_{[|X_i| \leqq cn]}$, $X_i'' = X_i - X_i'$ and define $S_n' = X_1' + \cdots + X_n'$, $S_n'' = X_1'' + \cdots + X_n''$. Then for $i = 1, \cdots, n$,

(6.5) $$E_F X_i' = E_F X I_{[|X| \leqq cn]} \geqq E_F X I_{[X \leqq cn]} \geqq \delta \qquad \text{for all} \quad F \in \mathscr{F}.$$

We note that

(6.6) $$\begin{aligned}\textstyle\sum_1^\infty n^{-1} P_F[S_n'' \neq 0] &\leqq \textstyle\sum_1^\infty n^{-1} P_F[X_i'' \neq 0 \ \text{ for some } \ i = 1, \cdots, n] \\ &\leqq \textstyle\sum_1^\infty P_F[|X| > cn] \leqq E_F|X|/c\,.\end{aligned}$$

Using (6.5) and the Chebyshev inequality, we obtain that

$$\begin{aligned}
&\textstyle\sum_1^\infty n^{-1}P_F[S_n' \leqq 0] \\
&\quad = \textstyle\sum_1^\infty n^{-1}P_F[S_n' - ES_n' \leqq -\delta n] \\
&\quad \leqq \delta^{-2} \textstyle\sum_1^\infty n^{-2}E_F X^2 I_{[|X| \leqq cn]} \qquad (6.7)\\
&\quad = \delta^{-2}\{\textstyle\sum_1^\infty n^{-2} \int_{|X|\leqq 1} X^2\, dP_F + \int_{|X|>1} X^2(\sum_{cn \geqq |X|} n^{-2})\, dP_F\} \\
&\quad \leqq \delta^{-2}\{\textstyle\sum_1^\infty n^{-2} + c^2 + cE_F|X|\}\,.
\end{aligned}$$

Since (1.11b) implies $\sup_{F\in\mathscr{F}} E_F X^+ < \infty$, it follows from (1.11a) that $\sup_{F\in\mathscr{F}} E_F X^- < \infty$. Hence $\sup_{F\in\mathscr{F}} E_F|X| < \infty$, and therefore from (6.4), (6.6) and (6.7), we obtain that $\sup_{F\in\mathscr{F}} E_F N < \infty$.

To prove the uniform integrability condition (1.10) for S_N, we note that

$$\begin{aligned}
E_F S_N I_{[S_N > a]} &\leqq \textstyle\sum_{n=1}^\infty \int_{[N=n, S_N>a]} X_n^+\, dP_F \leqq \sum_{n=1}^\infty \int_{[N\geqq n, X_n > a]} X_n^+\, dP_F \\
&= \textstyle\sum_{n=1}^\infty P_F[N \geqq n] \int_{[X>a]} X\, dP_F = (E_F N) \int_{[X>a]} X\, dP_F
\end{aligned}$$

and apply condition (1.11b).

We now prove (1.12). Define successive ladder indices $M_1 = \inf\{n \geqq 1 : S_n > 0\}$, $M_2 = \inf\{n > N_1 : S_n - S_{N_1} > 0\}$, etc., and let $Z_i = S_{M_i} - S_{M_{i-1}}$ ($M_0 = S_0 = 0$), $N_i = M_i - M_{i-1}$. Set $\tau(a) = \inf\{n \geqq 1 : Z_1 + \cdots + Z_n > a\}$. Then $T(a) = N_1 + \cdots + N_{\tau(a)}$ and by Wald's lemma,

$$E_F T(a) = (E_F N_1)(E_F \tau(a))\,. \tag{6.8}$$

We note that $P_F[Z_1 > 0] = 1$, $\inf_{F\in\mathscr{F}} E_F Z_1 \geqq \inf_{F\in\mathscr{F}} E_F X^+ > 0$ and by (1.10), $\sup_{F\in\mathscr{F}} E_F Z_1 I_{[Z_1 > x]} \to 0$ as $x \to \infty$. Hence applying the theorem which we have established for the nonnegative case, we obtain that as $a \to \infty$,

$$E_F(\tau(a)/a) \to (E_F Z_1)^{-1} \qquad \text{uniformly for} \quad F \in \mathscr{F}\,. \tag{6.9}$$

Since $E_F Z_1 = \mu_F E_F N_1$ by Wald's lemma, the desired conclusion (1.12) follows from (6.8) and (6.9).

In the literature on renewal theory, instead of the first passage time $T(a)$ defined in (1.1), one often considers the random variable

$$\Lambda(a) = \textstyle\sum_{n=1}^\infty I_{[S_n \leqq a]}\,. \tag{6.10}$$

The function $1 + E\Lambda(a) = 1 + \sum_{n=1}^\infty P[S_n \leqq a]$ is called the renewal function and it is well known (cf. [9]) that if $X_1, X_2, \cdots$ are i.i.d. with $EX_1 = \mu > 0$ and $E(X_1^-)^2 < \infty$, then

$$E\Lambda(a) \sim a/\mu \qquad \text{as } a \to \infty\,. \tag{6.11}$$

This result is commonly called the elementary renewal theorem. In general, if $E(X_1^-)^{r+1} < \infty$, then $E\Lambda^r(a) < \infty$ and

$$E\Lambda^r(a) \sim (a/\mu)^r \qquad \text{as } a \to \infty\,. \tag{6.12}$$

By making use of Paley-type inequalities on the tail distribution of sample sums (cf. [1]), we obtain the following uniform version of (6.12).

THEOREM 10. *Let $r > 0$ and let $\mathscr{F}$ be a family of distribution functions satisfying the following conditions*:

(6.13a) $$\inf_{F \in \mathscr{F}} \int_{-\infty}^{\infty} x \, dF(x) > 0 \,;$$

(6.13b) $$\sup_{F \in \mathscr{F}} \int_{-\infty}^{0} |x|^{r+1} \, dF(x) < \infty \,;$$

(6.13c) $$\sup_{F \in \mathscr{F}} \int_{0}^{\infty} x^{\eta} \, dF(x) < \infty \qquad \textit{for some} \quad \eta > 1 \,.$$

Let $X, X_1, X_2, \cdots$ be i.i.d. *random variables with a common distribution function $F \in \mathscr{F}$ and let $\mu_F = E_F X$. Set $S_n = X_1 + \cdots + X_n$ and define $\Lambda(a)$ by* (6.10). *Then as $a \to \infty$*,

(6.14) $$E_F(\Lambda(a)/a)^r \to \mu_F^{-r} \qquad \textit{uniformly for} \quad F \in \mathscr{F} \,.$$

The above conclusion (6.14) *remains true if we replace $\Lambda(a)$ by $T(a)$, where $T(a)$ is the first passage time defined in* (1.1).

PROOF. Given any $0 < \delta < 1$, define $L(\delta, F) = \sup\{n \geqq 1 : S_n \leqq \delta n \mu_F\} = \sup\{n \geqq 1 : \sum_1^n (X_i - \mu_F) \leqq -(1 - \delta) n \mu_F\}$ ($\sup \varnothing = 0$). Without loss of generality, we can assume that in (6.13c), $\eta \leqq \min(2, r + 1)$. Then as proved in [1], there exists a universal constant A_r depending only on r such that

(6.15) $$E_F L^r(\delta, F) \leqq A_r\{E_F(X^-/(1 - \delta)\mu_F)^{r+1} + E_F^{r/(\eta-1)}(|X|/(1 - \delta)\mu_F)^{\eta}\} \,.$$

(See Theorem 1 and Lemma 2 of [1].) Since $\inf_{F \in \mathscr{F}} \mu_F > 0$, it follows from conditions (6.13b) and (6.13c) together with (6.15) that $\sup_{F \in \mathscr{F}} E_F L^r(\delta, F) < \infty$. Therefore

(6.16) $$\sup_{0 \leqq s \leqq r} \sup_{F \in \mathscr{F}} E_F L^s(\delta, F) < \infty \,.$$

We note that if $n \geqq \max\{L(\delta, F) + 1, (\delta \mu_F)^{-1} a\}$, then $S_n > \delta n \mu_F \geqq a$, and so

(6.17) $$\Lambda(a) \leqq L(\delta, F) + (\delta \mu_F)^{-1} a$$

for all $F \in \mathscr{F}$. Writing $r = k + \varepsilon$ where k is a nonnegative integer and $0 \leqq \varepsilon < 1$, it follows from (6.17) that

(6.18) $$\Lambda^r(a) \leqq (L(\delta, F) + (\delta \mu_F)^{-1} a)^k (L^{\varepsilon}(\delta, F) + (\delta \mu_F)^{-\varepsilon} a^{\varepsilon}) \,.$$

Applying the binomial expansion to $(L(\delta, F) + (\delta \mu_F)^{-1} a)^k$ in (6.18) and making use of (6.13a) and (6.16), we can choose $a_0 > 1$ such that

(6.19) $$E_F(\Lambda(a)/a)^r \leqq (1 - \delta) + (\delta \mu_F)^{-r} \qquad \text{for all} \quad a \geqq a_0 \quad \text{and} \quad F \in \mathscr{F} \,.$$

Clearly $\Lambda(a) \geqq T(a) - 1$. By Lemma 6 below, as $a \to \infty$,

(6.20) $$P_F[T(a) > \delta \mu_F^{-1} a] \to 1 \qquad \text{uniformly for} \quad F \in \mathscr{F} \,.$$

Therefore we can choose $a_1 > 1$ such that

(6.21) $$\begin{aligned} E_F T^r(a) &\geqq (\delta \mu_F^{-1} a)^r P_F[T(a) > \delta \mu_F^{-1} a] \\ &\geqq (\delta \mu_F^{-1} a)^r \delta \qquad \text{for all} \quad a \geqq a_1 \quad \text{and} \quad F \in \mathscr{F} \,. \end{aligned}$$

The desired conclusion then follows from (6.19) and (6.21).

LEMMA 6. *Let $\tilde{\mathscr{F}}$ be a family of distribution functions satisfying the uniform integrability condition* (1.13). *Let $X, X_1, X_2, \cdots$ be* i.i.d. *random variables with a common distribution function $F \in \tilde{\mathscr{F}}$ and let $\mu_F = E_F X$. If $\inf_{F \in \tilde{\mathscr{F}}} \mu_F > 0$, then for any $\varepsilon > 0$,*

$$(6.22) \qquad \sup_{F \in \tilde{\mathscr{F}}} P_F[|T(a)/a - \mu_F^{-1}| > \varepsilon] \to 0 \qquad as \quad a \to \infty ,$$

where $T(a)$ is as defined in (1.1).

PROOF. Take any $\delta > 0$. By Chung's uniform version (1.14) of the strong law of large numbers and the assumption that $\inf_{F \in \tilde{\mathscr{F}}} \mu_F > 0$, we can choose a_0 such that for all $a \geqq a_0$ and $F \in \tilde{\mathscr{F}}$,

$$(6.23) \qquad P_F[\sup_{j \geqq \delta a} (j^{-1}S_j - \mu_F) \geqq \varepsilon \mu_F^2] \leqq \delta .$$

Since $(S_j, j \geqq 1)$ is a submartingale, so is $(S_j^+, j \geqq 1)$ and therefore

$$(6.24) \qquad P_F[\max_{j \leqq \delta a} S_j > a] \leqq a^{-1} E_F S_{[\delta a]}^+ \leqq \delta E_F X^+ .$$

Hence for all $a \geqq a_0$ and $F \in \tilde{\mathscr{F}}$,

$$\begin{aligned} P_F[T(a) < (\mu_F^{-1} - \varepsilon)a] &\leqq P_F[\max_{j \leqq \delta a} S_j > a] + P_F[\max_{\delta a \leqq j \leqq (\mu_F^{-1} - \varepsilon)a} S_j > a] \\ &\leqq \delta E_F X^+ + P[\sup_{j \geqq \delta a} (j^{-1}S_j - \mu_F) > \varepsilon \mu_F^2] \\ &\leqq \delta E_F X^+ + \delta . \end{aligned}$$

From (1.14), it also follows that

$$\begin{aligned} P_F[T(a) > (\mu_F^{-1} + \varepsilon)a] &= P_F[\max_{j \leqq (\mu_F^{-1} + \varepsilon)a} S_j \leqq a] \\ &\leqq P_F[S_{[(\mu_F^{-1} + \varepsilon)a]} \leqq a] \\ &\to 0 \qquad \text{uniformly for} \quad F \in \tilde{\mathscr{F}} \quad \text{as} \quad a \to \infty . \end{aligned}$$

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