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ON r-QUICK CONVERGENCE AND A CONJECTURE OF STRASSEN[1]

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In this paper, we prove a conjecture of Strassen on the set of r-quick limit points of the normalized linearly interpolated sample sum process in $C[0, 1]$. We give the best possible moment conditions for this conjecture to hold by finding the r-quick analogue of the classical law of the iterated logarithm and its converse. The proof is based on an r-quick version of Strassen's strong invariance principle and a theorem on the r-quick limit set of a semi-stable Gaussian process. Some applications of Strassen's conjecture are given. We also consider the notion of r-quick convergence related to the law of large numbers and outline some statistical applications to indicate the usefulness of this concept.

1. Introduction.

In [16], Strassen introduced the notion of r-quick limit points of a sequence of real-valued random variables θ_n. For any real number c, define the random variable

$$T_c = \sup\{n \geqq 1 : \theta_n \geqq c\} \quad (\sup \emptyset = 0)\,, \tag{1.1}$$

i.e., T_c is the last time when θ_n exceeds c. We note that the statement

$$\limsup_{n\to\infty} \theta_n = y \quad \text{a.s.} \tag{1.2}$$

can be expressed in terms of T_c as follows:

$$\begin{aligned} P[T_c < \infty] &= 1 \quad \text{if} \quad c > y\,, \\ &= 0 \quad \text{if} \quad c < y\,. \end{aligned} \tag{1.3}$$

This observation leads Strassen to give the following definition.

DEFINITION 1. Let θ_n be a sequence of real-valued random variables. For any real number c, define T_c as in (1.1). Let $r > 0$ and y be a real constant. Then

$$\limsup_{n\to\infty} \theta_n = y \quad (r\text{-quickly}) \tag{1.4}$$

if and only if the following two conditions hold:

$$ET_c^{\,r} < \infty \quad \text{for} \quad c > y\,, \tag{1.5a}$$

$$ET_c^{\,r} = \infty \quad \text{for} \quad c < y\,. \tag{1.5b}$$

In other words, (1.4) holds if and only if $y = \sup\{c : ET_c^{\,r} = \infty\}$. We shall also

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say that $\limsup_{n\to\infty} \theta_n \leqq y$ (r-quickly) if (1.5a) holds, and $\limsup_{n\to\infty} \theta_n \geqq y$ (r-quickly) if (1.5b) holds. Therefore we say that $\limsup_{n\to\infty} \theta_n < \infty$ (r-quickly) if there exists a real constant c for which $ET_c^r < \infty$, and we say that $\limsup_{n\to\infty} \theta_n = \infty$ if otherwise. Likewise if $ET_c^r < \infty$ for all real constants c, then we write $\lim_{n\to\infty} \theta_n = -\infty$ (r-quickly).

The following theorem on the r-quick lim sup of sample sums was proved by Strassen in [16].

THEOREM 1 (Strassen). *Let $X_1, X_2, \cdots$ be a sequence of* i.i.d. *real-valued random variables such that $EX_1 = 0$ and $EX_1^2 = 1$. Let $r > 0$, $p > 2(r+1)$ and $E|X_1|^p < \infty$. Let $S_n = X_1 + \cdots + X_n$, $S_0 = 0$. Then*

$$\limsup_{n\to\infty} (2n \log n)^{-\frac{1}{2}} S_n = r^{\frac{1}{2}} \quad (r\text{-quickly}) . \tag{1.6}$$

In an earlier paper [15], Strassen proved his well-known functional form of the law of the iterated logarithm. Let $C[0, 1]$ be the Banach space of continuous functions $f: [0, 1] \to (-\infty, \infty)$ with $\|f\|_C = \max_t |f(t)|$. Letting $S_0 = 0$, $S_n = X_1 + \cdots + X_n$, where $X_1, X_2, \cdots$ are i.i.d. real-valued random variables with mean 0 and variance 1, define

$$\begin{aligned} \eta_n(t) &= (2n \log\log n)^{-\frac{1}{2}} S_i \quad \text{at} \quad t = i/n \quad (i = 0, 1, \cdots, n) , \\ \eta_n &\text{ is linear on } [(i-1)/n, i/n] , \qquad i = 1, \cdots, n . \end{aligned} \tag{1.7}$$

Then with probability 1, $(\eta_n)_{n\geqq 3}$ is relatively compact in $C[0, 1]$ and its set of limit points in $C[0, 1]$ is

$$\begin{aligned} K_0 = \{h \in C[0, 1] &: h(0) = 0, \ h \text{ is absolutely continuous and} \\ &\textstyle\int_0^1 (h'(t))^2 \, dt \leqq 1\} . \end{aligned} \tag{1.8}$$

Strassen's result (1.6) led him to consider an r-quick analogue of the above functional form of the law of the iterated logarithm. In analogy with Definition 1, he defined r-quick limit points in $C[0, 1]$ and stated a conjecture about the functional form of Theorem 1 (cf. [16] page 319).

DEFINITION 2. Let M be a metric space, endowed with its σ-algebra of Borel sets. Let (ζ_n) be a sequence of random variables taking values in M. Then (ζ_n) is said to be *r-quickly relatively compact* in M if for every $\varepsilon > 0$, there is a finite union U of ε-spheres in M such that

$$E(\sup\{n : \zeta_n \notin U\})^r < \infty . \tag{1.9}$$

An element x of M is called an *r-quick limit point* of (ζ_n) in M if for any open neighborhood V of x,

$$E(\sup\{n : \zeta_n \in V\})^r = \infty . \tag{1.10}$$

STRASSEN'S CONJECTURE. With the same assumptions and notations as in Theorem 1, define

$$\begin{aligned} \zeta_n(t) &= (2n \log n)^{-\frac{1}{2}} S_i \quad \text{at} \quad t = i/n \quad (i = 0, 1, \cdots, n) , \\ \zeta_n &\text{ is linear on } [(i-1)/n, i/n] , \qquad i = 1, \cdots, n . \end{aligned} \tag{1.11}$$

Then the sequence $(\zeta_n)_{n\geqq 2}$ is r-quickly relatively compact in $C[0, 1]$ and the set of its r-quick limit points in $C[0, 1]$ is

$$(1.12) \qquad r^{\frac{1}{2}}K_0 = \{h \in C[0,1] : h(0) = 0,\ h \text{ is absolutely continuous and } \textstyle\int_0^1 (h'(t))^2\,dt \leqq r\}\,.$$

There is in fact a misprint in [16] where the factor $\frac{1}{2}$ is erroneously appended to the integral in (1.12). In Section 4 below, we shall give a proof of Strassen's conjecture. In fact we shall prove the following stronger result.

THEOREM 2. *Let $r > 0$ and let $X_1, X_2, \cdots$ be a sequence of* i.i.d. *real-valued random variables such that $EX_1 = 0$, $EX_1^2 = 1$ and*

$$(1.13) \qquad E|X_1|^{2(r+1)}(\log^+ |X_1| + 1)^{-(r+1)} < \infty\,.$$

Define ζ_n as in (1.11) *and $r^{\frac{1}{2}}K_0$ as in* (1.12). *Then for every $\varepsilon > 0$, letting U denote the open ε-neighborhood of $r^{\frac{1}{2}}K_0$,*

$$(1.14) \qquad E(\sup\{n : \zeta_n \notin U\})^r < \infty\,,$$

and so the sequence $(\zeta_n)_{n\geqq 2}$ is r-quickly relatively compact in $C[0, 1]$. The set of its r-quick limit points in $C[0, 1]$ is $r^{\frac{1}{2}}K_0$.

In Theorem 2 above, the terminology "open ε-neighborhood" U of a set A in $C[0, 1]$ refers to the set $\{x \in C[0,1] : \inf_{y\in A} \|x - y\|_C < \varepsilon\}$, and we define the closed ε-neighborhood of A to be $\{x \in C[0,1] : \inf_{y\in A} \|x - y\|_C \leqq \varepsilon\}$. Likewise we shall define the open (or closed) ε-neighborhood of a set A in a general metric space M. As the following theorem shows, the moment condition (1.13), which is weaker than Strassen's condition $E|X_1|^p < \infty$ for some $p > 2(r + 1)$, is in fact the best possible.

THEOREM 3. *Suppose $X_1, X_2, \cdots$ are* i.i.d. *real-valued random variables and $S_n = X_1 + \cdots + X_n$. Then for any $r > 0$,*

$$(1.15) \qquad \limsup_{n\to\infty} (n \log n)^{-\frac{1}{2}}|S_n| < \infty \quad (r\text{-quickly}) \iff EX_1 = 0 \quad \text{and} \quad E|X_1|^{2(r+1)}(\log^+ |X_1| + 1)^{-(r+1)} < \infty\,.$$

In this case,

$$(1.16) \qquad \limsup_{n\to\infty} (2n \log n)^{-\frac{1}{2}} S_n = (rEX_1^2)^{\frac{1}{2}} \quad (r\text{-quickly})\,.$$

It is interesting to compare Theorem 3 with the corresponding result for the a.s. lim sup of normalized sample sums. As is well known,

$$(1.17) \qquad EX_1 = 0 \quad \text{and} \quad EX_1^2 = \sigma^2 \iff \limsup_{n\to\infty} (2n \log\log n)^{-\frac{1}{2}} S_n = \sigma \quad \text{a.s.}$$

However, under the conditions of (1.17), we have for every $r > 0$,

$$(1.18) \qquad \limsup_{n\to\infty} (2n \log\log n)^{-\frac{1}{2}} S_n = \infty \quad (r\text{-quickly})$$

(cf. [14], [16]). The proof of Theorem 3 will be given in Section 2 below.

In Section 3, we shall obtain an analogue of Theorem 2 for Brownian motion (and in fact more generally for a semi-stable Gaussian process) which we shall use in Section 4 to prove Strassen's conjecture. We shall study the properties of r-quick limit sets and give some applications of Theorem 2 in Section 5. In Section 6, we shall consider the somewhat simpler notion of r-quick convergence related to the law of large numbers and outline some statistical applications of these concepts.

2. Proof of Theorem 3. The proof of Theorem 3 makes use of the following result of [5] (see Theorem 3 on page 438 of that paper): Let $p > 2$ and $X_1, X_2, \cdots$ be i.i.d. real-valued random variables and $S_n = X_1 + \cdots + X_n$. If $EX_1 = 0$, $EX_1^2 = \sigma^2$ and $E|X_1|^p(\log^+ |X_1| + 1)^{-p/2} < \infty$, then for any $\varepsilon > \sigma(p-2)^{\frac{1}{2}}$,

$$\sum n^{p/2-2}P[|S_n| \geqq \varepsilon(n \log n)^{\frac{1}{2}}] < \infty\,, \tag{2.1}$$

$$\sum n^{p/2-2}P[\sup_{k \geqq n} (k \log k)^{-\frac{1}{2}}|S_k| \geqq \varepsilon] < \infty\,. \tag{2.2}$$

Conversely, if for some $\varepsilon > 0$, either (2.1) or (2.2) holds, then $EX_1 = 0$ and $E|X_1|^p(\log^+ |X_1| + 1)^{-p/2} < \infty$.

We now prove the equivalence (1.15) in Theorem 3. First assume that $EX_1 = 0$ and $E|X_1|^{2(r+1)}(\log^+ |X_1| + 1)^{-(r+1)} < \infty$. Let $c > (2rEX_1^2)^{\frac{1}{2}}$. Then by (2.2),

$$\sum n^{(r+1)-2}P[\sup_{k \geqq n} (k \log k)^{-\frac{1}{2}}|S_k| \geqq c] < \infty\,, \tag{2.3}$$

i.e., $\sum n^{r-1}P[T_c \geqq n] < \infty$, where

$$T_c = \sup\{n \geqq 1 : (n \log n)^{-\frac{1}{2}}|S_n| \geqq c\}\,. \tag{2.4}$$

Hence $ET_c^r < \infty$ and $\limsup_{n\to\infty} (n \log n)^{-\frac{1}{2}}|S_n| \leqq (2rEX_1^2)^{\frac{1}{2}}$ (r-quickly).

Conversely, if $\limsup_{n\to\infty} (n \log n)^{-\frac{1}{2}}|S_n| < \infty$ (r-quickly), then there exists a positive constant c for which $ET_c^r < \infty$, where T_c is as defined in (2.4). Therefore $\sum n^{r-1}P[T_c \geqq n] < \infty$, i.e., (2.3) holds. Hence $EX_1 = 0$ and $E|X_1|^{2(r+1)}(\log^+ |X_1| + 1)^{-(r+1)} < \infty$.

Thus we have proved (1.15). The relation (1.16) follows as an easy corollary of Theorem 2 (see Example 1 in Section 5 below).

3. The r-quick limit set of a semi-stable Gaussian process. In [13], Oodaira has proved a functional form of the law of the iterated logarithm for a continuous real-valued Gaussian process $X(t)$, $t \geqq 0$, with $X(0) = 0$, $EX(t) = 0$ and continuous covariance kernel $R(s, t) = EX(s)X(t)$ satisfying conditions (3.2) and (3.3) below. Let

$$Z_n(t) = X(nt)/(2n^\lambda \log\log n)^{\frac{1}{2}}\,, \qquad t \in [0, 1]\,,$$

where λ is as given by (3.2). Oodaira's result states that with probability 1, the sequence $(Z_n)_{n\geqq 3}$ is relatively compact in $C[0, 1]$ and its set of limit points is contained in the set

$$K = \{h \in H(R_1) : ||h||_H \leqq 1\}\,, \tag{3.1}$$

where $H(R_1)$ is the reproducing kernel Hilbert space corresponding to the kernel $R(s, t)$, $0 \leqq s, t \leqq 1$, and $||\cdot||_H$ denotes the norm of $H(R_1)$, i.e., K is the unit ball of $H(R_1)$. Under additional assumptions on $R(s, t)$, the set of limit points of $(Z_n)_{n\geq 3}$ in $C[0, 1]$ in fact coincides with K. We remark that although Oodaira's original theorem imposes a somewhat stronger condition in place of (3.3), condition (3.3) in fact would suffice, as can be proved by using Theorem 1 of [6]. The following theorem gives an r-quick version of Oodaira's result.

THEOREM 4. *Let $X(t)$, $t \geqq 0$, be a continuous real-valued Gaussian process with $X(0) = 0$, $EX(t) = 0$ and continuous covariance kernel $R(s, t) = EX(s)X(t)$ such that the following two conditions are satisfied:*

(3.2) *There exists $\lambda > 0$ such that $R(\theta s, \theta t) = \theta^\lambda R(s, t)$ for all $\theta > 0$ and $s, t \geqq 0$,* i.e., *the process is semi-stable.*

(3.3) *There exists a positive nondecreasing function ψ on $[0, 1]$ such that $\int_0^\infty \psi(e^{-u^2})\, du < \infty$ and for all $s, t \in [0, 1]$, $E(X(s) - X(t))^2 \leqq \psi^2(|t - s|)$.*

Define K as in (3.1) *and set*

$$Y_n(t) = X(nt)/(2n^\lambda \log n)^{\frac{1}{2}}\,, \qquad t \in [0, 1]\,. \tag{3.4}$$

Then for every $r > 0$, letting $(r^{\frac{1}{2}}K)_\varepsilon$ denote the closed ε-neighborhood of $r^{\frac{1}{2}}K$ in $C[0, 1]$, we have

$$E(\sup\{n : Y_n \notin (r^{\frac{1}{2}}K)_\varepsilon\})^r < \infty \qquad \textit{for all} \quad \varepsilon > 0\,, \tag{3.5}$$

and so the sequence $(Y_n)_{n\geqq 2}$ is r-quickly relatively compact in $C[0, 1]$. The set of its r-quick limit points is $r^{\frac{1}{2}}K$.

REMARK. The notion of semi-stable Gaussian processes was introduced by Lamperti [11] whose results imply that if $R(s, t)$ satisfies $R(\theta s, \theta t) = v(\theta)R(s, t)$ for all $\theta, s, t \geqq 0$, where $v(\theta)$ is a positive function such that $v(\theta) \uparrow \infty$ (which is in fact the condition stated by Oodaira in [13]), then $v(\theta)$ is of the form θ^λ with $\lambda > 0$. By a theorem of Fernique [3], condition (3.3) implies that the processes $Y_n(t)$, $0 \leqq t \leqq 1$, are continuous with probability 1.

LEMMA 1. *Suppose $U(t)$, $t \in [0, 1]$, is a continuous real-valued Gaussian process with mean 0 and continuous covariance $R(s, t)$. Let $(X_n(t))_{n\geqq 1}$ be a sequence of Gaussian processes defined on the same probability space and having the same distribution as the process $U(t)$ and let $V_n(t) = X_n(t)/(2 \log n)^{\frac{1}{2}}$. Let $\{e_j(t), t \in [0, 1]\}$ be a complete orthonormal system in the reproducing kernel Hilbert space $H(R)$ and let $\varphi_n : H(R) \to L_2(X_n)$ be the isometric isomorphism (defined by $\varphi_n(R(t, \cdot)) = X_n(t)$) between $H(R)$ and the closed linear manifold $L_2(X_n)$ spanned by $\{X_n(t), t \in [0, 1]\}$. Let $\xi_n^{(j)} = \varphi_n(e_j)$ be the Gaussian random variable corresponding to e_j. Then given $\varepsilon > 0$, $\rho > 0$ and $k_0 \geqq 1$, there exists $k = k(\varepsilon, \rho, k_0)$ such that $k \geqq k_0$ and*

$$P[||V_n - (2 \log n)^{-\frac{1}{2}} \textstyle\sum_{j=1}^k \xi_n^{(j)} e_j||_C > \varepsilon] = o(n^{-\rho})\,. \tag{3.6}$$

PROOF. Take α such that $2\alpha\varepsilon^2 > \rho$ and choose $k \geqq k_0$ such that $P[||X_1 - \sum_{j=1}^k \xi_1^{(j)} e_j||_C \leqq 1] = q > \frac{1}{2}$ and $\log(q/(1 - q)) > 24\alpha$. Then by a theorem of

Fernique [4], $E\exp\{\alpha||X_1 - \sum_{j=1}^k \xi_1^{(j)}e_j||_C^2\} < \infty$. Hence

$$P[||X_n - \sum_{j=1}^k \xi_n^{(j)}e_j||_C > \varepsilon(2\log n)^{\frac{1}{2}}] \leqq \{\exp(-2\alpha\varepsilon^2\log n)\}E\exp(\alpha||X_1 - \sum_{j=1}^k \xi_1^{(j)}e_j||_C^2) = o(n^{-\rho}).$$

PROOF OF THEOREM 4. Let $\frac{1}{2} > \varepsilon > 0$, $\rho > r + 1$ and set

$$\delta = 1 + r^{\frac{1}{2}}\max_{0\leqq t\leqq 1} R^{\frac{1}{2}}(t, t). \tag{3.7}$$

Define $L(\varepsilon) = \sup\{n \geqq 1 : Y_n \notin (r^{\frac{1}{2}}K)_{4\delta\varepsilon}\}$. We now proceed to show

$$EL^r(\varepsilon) < \infty. \tag{3.8}$$

Choose $\alpha > 1$ sufficiently close to 1 such that $\alpha/(\alpha - 1)$ is an integer and the following two inequalities hold:

$$\varepsilon(\alpha^{\lambda/2} - 1)^{-1} > \rho^{\frac{1}{2}}\{\max_{0\leqq t\leqq 1} R^{\frac{1}{2}}(t, t) + 4\int_1^\infty \psi(2^{-u^2})\,du\}, \tag{3.9a}$$

$$\tfrac{1}{2}\varepsilon\alpha^{-\lambda/2} > \rho^{\frac{1}{2}}\{\psi(2(\alpha-1)/\alpha) + 4\int_1^\infty \psi(q^{-u^2})\,du\} \quad \text{for some integer } q > 1. \tag{3.9b}$$

In what follows, we shall let $Y_u = Y_{[u]}$ for any positive number u. We note that

$$\begin{aligned} \sum_{n=1}^\infty \alpha^{rn}P[L(\varepsilon) \geqq \alpha^n] &= \sum_{n=1}^\infty \alpha^{rn}P[Y_m \notin (r^{\frac{1}{2}}K)_{4\delta\varepsilon} \text{ for some } m \geqq \alpha^n] \\ &\leqq \sum_{n=1}^\infty \alpha^{rn}\sum_{i=n}^\infty P[Y_m \notin (r^{\frac{1}{2}}K)_{4\delta\varepsilon} \text{ for some } \alpha^i \leqq m < \alpha^{i+1}] \\ &\leqq \sum_{n=1}^\infty \alpha^{rn}\sum_{i=n}^\infty P[Y_{\alpha^i} \notin (r^{\frac{1}{2}}K)_{2\delta\varepsilon}] \\ &\quad + \sum_{n=1}^\infty \alpha^{rn}\sum_{i=n}^\infty P[\max_{\alpha^i\leqq m<\alpha^{i+1}} ||Y_m - Y_{\alpha^i}||_C > 2\delta\varepsilon]. \end{aligned} \tag{3.10}$$

Set $X_n(t) = n^{-\lambda/2}X(nt)$, $0 \leqq t \leqq 1$. Then X_n has the same distribution as X_1. Define e_j and $\xi_n^{(j)}$ as in Lemma 1 and choose k such that (3.6) holds. We note that if $x \in H(R_1)$ and $||x||_H \leqq (1 + \varepsilon)r^{\frac{1}{2}}$, then $x \in (r^{\frac{1}{2}}K)_{\delta\varepsilon}$ since $(1 + \varepsilon)^{-1}x \in r^{\frac{1}{2}}K$ and $||x - (1 + \varepsilon)^{-1}x||_C \leqq \varepsilon(1 + \varepsilon)^{-1}||x||_H \max_{0\leqq t\leqq 1} R^{\frac{1}{2}}(t, t) \leqq \delta\varepsilon$ by (3.7). Therefore

$$\begin{aligned} P[Y_{\alpha^i} \notin (r^{\frac{1}{2}}K)_{2\delta\varepsilon}] &\leqq P[||Y_{\alpha^i} - (2\log[\alpha^i])^{-\frac{1}{2}}\sum_{j=1}^k \xi_{[\alpha^i]}^{(j)}e_j||_C > \varepsilon] \\ &\quad + P[(2\log[\alpha^i])^{-\frac{1}{2}}||\sum_{j=1}^k \xi_{[\alpha^i]}^{(j)}e_j||_H > (1 + \varepsilon)r^{\frac{1}{2}}] \\ &= O(\alpha^{-\rho i}) + O(i^{k-1}\alpha^{-ir(1+\varepsilon)^2}). \end{aligned} \tag{3.11}$$

The last relation above follows from (3.6) and the following estimate:

$$\begin{aligned} &P[||\sum_{j=1}^k \xi_{[\alpha^i]}^{(j)}e_j||_H > (1 + \varepsilon)(2r\log[\alpha^i])^{\frac{1}{2}}] \\ &\quad = P[\sum_{j=1}^k (\xi_1^{(j)})^2 > (1 + \varepsilon)^2(2r\log[\alpha^i])] = O(\alpha^{-ir(1+\varepsilon)^2}(\log\alpha^i)^{k-1}), \end{aligned}$$

since $\sum_{j=1}^k (\xi_1^{(j)})^2$ has a χ^2-distribution with k degrees of freedom. From (3.11), it follows that

$$\sum_{n=1}^\infty \alpha^{rn}\sum_{i=n}^\infty P[Y_{\alpha^i} \notin (r^{\frac{1}{2}}K)_{2\delta\varepsilon}] < \infty. \tag{3.12}$$

Since $\delta > 1$ and

$$\begin{aligned} ||Y_m - Y_{\alpha^i}||_C &\leqq (\max_{0\leqq t\leqq 1} |X([\alpha^i]t)|)|(2m^\lambda\log m)^{-\frac{1}{2}} - (2[\alpha^i]^\lambda\log[\alpha^i])^{-\frac{1}{2}}| \\ &\quad + (2m^\lambda\log m)^{-\frac{1}{2}}\max_{0\leqq t\leqq 1} |X(mt) - X([\alpha^i]t)|, \end{aligned}$$

we obtain that for all large i,

$$
\begin{aligned}
(3.13)\qquad & P[\max_{\alpha^i \leqq m < \alpha^{i+1}} ||Y_m - Y_{\alpha^i}||_C > 2\delta\varepsilon] \\
& \leqq P[\max_{0\leqq t\leqq 1} |X([\alpha^i]t)| > \varepsilon(\alpha^{\lambda/2} - 1)^{-1}(2[\alpha^i]^\lambda \log \alpha^i)^{\frac{1}{2}}] \\
& \quad + P[\max_{0\leqq s,t \leqq \alpha^{i+1}, |t-s| \leqq (\alpha-1)\alpha^i} |X(t) - X(s)| > \varepsilon(2\alpha^{i\lambda} \log \alpha^i)^{\frac{1}{2}}] .
\end{aligned}
$$

Since the process $n^{-\lambda/2}X(nt)$, $0 \leqq t \leqq 1$, has the same distribution as $X(t)$, $0 \leqq t \leqq 1$, it follows that

$$
\begin{aligned}
(3.14)\qquad & P[\max_{0\leqq t\leqq 1} |X([\alpha^i]t)| > \varepsilon(\alpha^{\lambda/2} - 1)^{-1}(2[\alpha^i]^\lambda \log \alpha^i)^{\frac{1}{2}}] \\
& = P[\max_{0\leqq t\leqq 1} |X(t)| > \varepsilon(\alpha^{\lambda/2} - 1)^{-1}(2 \log \alpha^i)^{\frac{1}{2}}] .
\end{aligned}
$$

By (3.3), we can apply Fernique's inequality (cf. [3], [13]) to obtain that for all large i,

$$
\begin{aligned}
(3.15)\qquad & P[\max_{0\leqq t\leqq 1} |X(t)| > \varepsilon(\alpha^{\lambda/2} - 1)^{-1}(2 \log \alpha^i)^{\frac{1}{2}}] \\
& \leqq 16 \int_{\rho^{\frac{1}{2}}(2\log\alpha^i)^{\frac{1}{2}}}^{\infty} \exp(-u^2/2)\, du , \qquad \text{in view of (3.9a)} \\
& = O(\alpha^{-\rho i}) .
\end{aligned}
$$

We note that

$$
\begin{aligned}
& P[\max_{0\leqq s,t\leqq \alpha^{i+1}, |t-s|\leqq(\alpha-1)\alpha^i} |X(t) - X(s)| > \varepsilon(2\alpha^{i\lambda} \log \alpha^i)^{\frac{1}{2}}] \\
& = P[\max_{0\leqq s,t\leqq 1, |t-s|\leqq(\alpha-1)/\alpha} |X(\alpha^{i+1}t) - X(\alpha^{i+1}s)| > \varepsilon(2\alpha^{i\lambda} \log \alpha^i)^{\frac{1}{2}}] \\
& = P[\max_{0\leqq s,t\leqq 1, |t-s|\leqq(\alpha-1)/\alpha} |X(t) - X(s)| > \varepsilon(2\alpha^{-\lambda} \log \alpha^i)^{\frac{1}{2}}] \\
(3.16)\qquad & \leqq \sum_{\nu=2}^{\alpha/(\alpha-1)} P[\max_{(\nu-2)(\alpha-1)/\alpha\leqq s,t\leqq \nu(\alpha-1)/\alpha} |X(t) - X(s)| \\
& \quad > \varepsilon(2\alpha^{-\lambda} \log \alpha^i)^{\frac{1}{2}}] \\
& \leqq \sum_{\nu=2}^{\alpha/(\alpha-1)} P\left[\max_{(\nu-2)(\alpha-1)/\alpha\leqq t\leqq \nu(\alpha-1)/\alpha} |X(t) - X((\nu-2)(\alpha-1)/\alpha)| \right. \\
& \quad \left. > \left(\frac{\varepsilon}{2}\right)(2\alpha^{-\lambda} \log \alpha^i)^{\frac{1}{2}} \right] \\
& \leqq 4q^2\alpha/(\alpha - 1) \int_{\rho^{\frac{1}{2}}(2\log\alpha^i)^{\frac{1}{2}}}^{\infty} \exp(-u^2/2)\, du ,
\end{aligned}
$$

where q is given by (3.9b). The last relation above follows from (3.9b) and Fernique's inequality since (3.3) holds and for $(\nu-2)(\alpha-1)/\alpha \leqq t \leqq \nu(\alpha-1)/\alpha$, $E(X(t) - X((\nu-2)(\alpha-1)/\alpha))^2 \leqq \psi^2(2(\alpha-1)/\alpha)$. It then follows from (3.13), (3.14), (3.15) and (3.16) that

$$
(3.17)\qquad \sum_{n=1}^{\infty} \alpha^{rn} \sum_{i=n}^{\infty} P[\max_{\alpha^i\leqq m<\alpha^{i+1}} ||Y_m - Y_{\alpha^i}||_C > 2\delta\varepsilon] < \infty .
$$

From (3.10), (3.12) and (3.17), we obtain that $\sum_{n=1}^{\infty} \alpha^{rn} P[L(\varepsilon) \geqq \alpha^n] < \infty$. Therefore (3.8) holds. Since K is compact, $(r^{\frac{1}{2}}K)_{4\delta\varepsilon}$ can be covered by a finite union of $5\delta\varepsilon$-spheres in $C[0, 1]$. As ε is arbitrary, we obtain from (3.8) that the sequence (Y_n) is r-quickly relatively compact in $C[0, 1]$ and the set of its r-quick limit points is contained in $\bigcap_{\varepsilon>0} (r^{\frac{1}{2}}K)_{4\delta\varepsilon} = K$, since K is closed.

Given any $h \in r^{\frac{1}{2}}K$, we shall show that h is an r-quick limit point of Y_n. Define $L^*(\varepsilon) = \sup\{n \geqq 1 : ||Y_n - h||_C \leqq \varepsilon(3 + ||h||_C)\}$. We shall show that $E(L^*(\varepsilon))^r = \infty$.

Letting $g = (1-\varepsilon)h$, we have $||g-h||_C = \varepsilon||h||_C$ and so we need only show that

$$(3.18)\qquad E\tilde{L}^r(\varepsilon) = \infty\,, \qquad \text{where} \quad \tilde{L}(\varepsilon) = \sup\{n \geqq 1 : ||Y_n - g||_C \leqq 3\varepsilon\}\,.$$

Since g has the expansion $g(t) = \sum_{j=1}^{\infty} g_j e_j(t)$ with the above series converging uniformly for $t \in [0, 1]$, we can choose k_0 such that $||g - \sum_{j=1}^{k} g_j e_j||_C < \varepsilon$ for all $k \geqq k_0$. By Lemma 1, we can choose $k = k(\varepsilon, \rho, k_0) \geqq k_0$ such that (3.6) holds. Hence to prove (3.18), it suffices to show

$$(3.19)\qquad \sum_{n=1}^{\infty} n^{r-1} P[||(2\log n)^{-\frac{1}{2}} \sum_{j=1}^{k} \xi_n^{(j)} e_j - \sum_{j=1}^{k} g_j e_j||_C \leqq \varepsilon] = \infty\,.$$

For $j = 1, \cdots, k$, we define

$$B_n^{(j)} = [||\{\xi_n^{(j)} - (2\log n)^{\frac{1}{2}} g_j\} e_j||_C \leqq (\varepsilon/k)(2\log n)^{\frac{1}{2}}]\,,$$
$$C_n^{(j)} = [|\xi_n^{(j)} - (2\log n)^{\frac{1}{2}} g_j| \leqq (\varepsilon/k\Gamma)(2\log n)^{\frac{1}{2}}]\,,$$

where $\Gamma = \max_{0\leqq t\leqq 1} R^{\frac{1}{2}}(t, t) \geqq ||e_j||_C$. It is easy to see that $B_n^{(j)} \supset C_n^{(j)}$ and $\bigcap_{j=1}^{k} B_n^{(j)} \subset [||(2\log n)^{-\frac{1}{2}} \sum_{j=1}^{k} \xi_n^{(j)} e_j - \sum_{j=1}^{k} g_j e_j||_C \leqq \varepsilon]$. Hence to prove (3.19), it suffices to show

$$(3.20)\qquad \sum_{n=1}^{\infty} n^{r-1} P[|\xi_1^{(j)} - (2\log n)^{\frac{1}{2}} g_j| \leqq (\varepsilon/k\Gamma)(2\log n)^{\frac{1}{2}} \quad \text{for } j = 1, \cdots, k] = \infty\,.$$

Let Φ denote the distribution function of the standard normal distribution. Since $\xi_1^{(1)}, \cdots, \xi_1^{(k)}$ are independent, we have for $n \geqq n_0$,

$$(3.21)\qquad \begin{aligned} &P[|\xi_1^{(j)} - (2\log n)^{\frac{1}{2}} g_j| \leqq (\varepsilon/k\Gamma)(2\log n)^{\frac{1}{2}} \text{ for } j = 1, \cdots, k] \\ &\quad \geqq \prod_{j=1}^{k} \{\Phi((2\log n)^{\frac{1}{2}}(|g_j| + (\varepsilon/k\Gamma))) - \Phi((2\log n)^{\frac{1}{2}}|g_j|)\} \\ &\quad \geqq C(\log n)^{-k/2} \exp(-(\textstyle\sum_{j=1}^{k} g_j^2)\log n) \end{aligned}$$

where C is a positive constant. Since g belongs to $(1-\varepsilon)r^{\frac{1}{2}}K$, $\sum_{j=1}^{\infty} g_j^2 < r$ and so (3.20) follows from (3.21).

COROLLARY 1. *Let $W(t)$, $t \geqq 0$, be the standard Wiener process. Define K_0 as in* (1.8) *and set $Y_n(t) = W(nt)/(2n\log n)^{\frac{1}{2}}$, $0 \leqq t \leqq 1$. Then for every $r > 0$,*

$$(3.22)\qquad E(\sup\{n : Y_n \notin (r^{\frac{1}{2}}K_0)_\varepsilon\})^r < \infty \qquad \textit{for all} \quad \varepsilon > 0\,,$$

and so the sequence $(Y_n)_{n\geqq 2}$ is r-quickly relatively compact in $C[0, 1]$. The set of its r-quick limit points is $r^{\frac{1}{2}}K_0$.

PROOF. As is well known, $R(s, t) = \min(s, t)$ is the covariance kernel for the process $W(t)$ and K_0 is the unit ball in the reproducing kernel Hilbert space corresponding to $W(t)$, $0 \leqq t \leqq 1$. The desired conclusion is therefore a special case of Theorem 4.

4. An r-quick version of the strong invariance principle and the proof of Strassen's conjecture. To prove Theorem 2 on Strassen's conjecture as stated in Section 1, we shall first obtain in Theorem 5 below an r-quick version of the strong invariance principle and then apply Corollary 1 of the preceding section.

LEMMA 2. *Suppose* $X_1, X_2, \cdots$ *are* i.i.d. *random variables with mean* 0 *and* $EX_1^2 < \infty$. *Let* $b: [0, \infty) \to [0, \infty)$ *be ultimately nondecreasing such that* $\lim_{t\to\infty} t^{-\frac{1}{2}} b(t) = \infty$. *Let* $r > 0$. *Suppose*

$$\sum n^{r-1} P[|S_n| > b(n)] < \infty . \tag{4.1}$$

Then for every $\varepsilon > 1$,

$$\sum n^{r-1} P[\max_{j \leq m} |S_j| > \varepsilon b(\varepsilon m) \text{ for some } m \geqq n] < \infty . \tag{4.2}$$

PROOF. Without loss of generality, we can assume that $b(t)$ is nondecreasing for $t \geqq 1$. Take $\alpha > 1$ such that $\alpha^2 < \varepsilon$. We note that

$$\begin{aligned}
&\sum_{n \geqq \alpha} n^{r-1} P[\max_{j \leqq m} |S_j| > \varepsilon b(\varepsilon m) \text{ for some } m \geqq n] \\
&\quad \leqq \sum_{i=1}^{\infty} \sum_{\alpha^i \leqq n \leqq \alpha^{i+1}} n^{r-1} P[\max_{j \leqq m} |S_j| > \varepsilon b(\varepsilon m) \text{ for some } m \geqq \alpha^i] \\
&\quad \leqq c \sum_{i=1}^{\infty} \alpha^{ri} P[\max_{j \leqq m} |S_j| > \varepsilon b(\varepsilon m) \text{ for some } m \geqq \alpha^i] \\
&\quad \leqq c \sum_{i=1}^{\infty} \alpha^{ri} \sum_{n=i}^{\infty} P[\max_{j \leqq m} |S_j| > \varepsilon b(\varepsilon m) \text{ for some } \alpha^{n+1} \geqq m \geqq \alpha^n] \\
&\quad \leqq c_1 \sum_{n=1}^{\infty} \alpha^{r(n+1)} P[\max_{j \leqq \alpha^{n+1}} |S_j| > \varepsilon b(\varepsilon \alpha^n)] \\
&\quad \leqq c_2 \sum_{n=1}^{\infty} \sum_{\alpha^{n+1} \leqq m < \alpha^{n+2}} m^{r-1} P[\max_{j \leqq \alpha^{n+1}} |S_j| > \varepsilon b(\varepsilon \alpha^{-2} \alpha^{n+2})] \\
&\quad \leqq c_3 \sum_{m=1}^{\infty} m^{r-1} P[\max_{j \leqq m} |S_j| > \varepsilon b(\varepsilon \alpha^{-2} m)]
\end{aligned}$$

where c, c_1, c_2, c_3 are constants. Since $\varepsilon > \alpha^2$, we need only show

$$\sum_{n=1}^{\infty} n^{r-1} P[\max_{j \leqq n} |S_j| > \varepsilon b(n)] < \infty . \tag{4.3}$$

To prove (4.3), we make use of the Lévy inequality (cf. [12, page 248]) to obtain

$$\begin{aligned}
P[\max_{j \leqq n} |S_j| > \varepsilon b(n)] &\leqq 2P[|S_n| > \varepsilon b(n) - (2nEX_1^2)^{\frac{1}{2}}] \\
&\leqq 2P[|S_n| > b(n)]
\end{aligned}$$

for all large n since $n^{\frac{1}{2}} = o(b(n))$. Therefore (4.3) follows from (4.1).

THEOREM 5. *Suppose* $X_1, X_2, \cdots$ *are* i.i.d. *real-valued random variables with* $EX_1 = 0$, $EX_1^2 = 1$ *and* $E|X_1|^{2(r+1)}(\log^+ |X_1| + 1)^{-(r+1)} < \infty$. *Define* ζ_n *as in* (1.11). *Then there is a standard Wiener process* $W(t)$, $t \geqq 0$, *such that defining* $Y_n(t) = W(nt)/(2n \log n)^{\frac{1}{2}}$, $0 \leqq t \leqq 1$, *we have*

$$E(\sup\{n \geqq 1 : \|\zeta_n - Y_n\|_C > \varepsilon\})^r < \infty \qquad \textit{for every } \varepsilon > 0 . \tag{4.4}$$

PROOF. The proof of Theorem 5 makes use of Theorem 3 of [5] which we have already referred to in Section 2. Let $\sigma_c^2 = \operatorname{Var} X_1 I_{[|X_1| \leqq c]}$, $X_i' = \{X_i I_{[|X_i| \leqq c]} - EX_1 I_{[|X_1| \leqq c]}\}/\sigma_c$, $X_i'' = X_i - X_i'$, where c is chosen so large that $r^{\frac{1}{2}} E^{\frac{1}{2}}(X_1'')^2 < \varepsilon/3$. Therefore letting $S_n'' = X_1'' + \cdots + X_n''$, we obtain by Theorem 3 of [5] that $\sum n^{r-1} P[|S_n''| > (\varepsilon/3)(2n \log n)^{\frac{1}{2}}] < \infty$. Hence by Lemma 2,

$$\sum n^{r-1} P[\max_{j \leqq m} |S_j''| > (\varepsilon/2)(2m \log m)^{\frac{1}{2}} \text{ for some } m \geqq n] < \infty . \tag{4.5}$$

Therefore

$$E(\sup\{n : (2n \log n)^{-\frac{1}{2}} \max_{j \leqq n} |S_j''| > \varepsilon/2\})^r < \infty . \tag{4.6}$$

Let $S_n' = X_1' + \cdots + X_n'$ and let ζ_n' be obtained by linearly interpolating $(2n \log n)^{-\frac{1}{2}} S_i'$ at $t = i/n\,(1 \leqq i \leqq n)$. Then by the Skorohod embedding theorem, there is (without loss of generality) a standard Wiener process $W(t)$, $t \geqq 0$, together with a sequence of i.i.d. nonnegative random variables $T_1, T_2, \cdots$ such that $ET_1 = 1$ and with probability 1,

$$S_n' = W(\textstyle\sum_{i=1}^n T_i)\,, \qquad n = 1, 2, \cdots .$$

Since X_1' is bounded, $ET_1^p < \infty$ for all $p > 0$. Take any $1 > \delta > \frac{1}{2}$. We note that

$$\begin{aligned}
&P[\|\zeta_m' - Y_m\|_C > \varepsilon/2 \text{ for some } m \geqq n] \\
&\quad \leqq P[\max_{1 \leqq i \leqq m} \max_{|t-i| \leqq m^\delta} |W(t) - W(i)| > (\varepsilon/4)(2m \log m)^{\frac{1}{2}} \\
&\qquad\qquad \text{for some } m \geqq n \text{ and} \\
&\qquad\qquad \max_{1 \leqq i \leqq m} |\textstyle\sum_{j=1}^i T_j - i| \leqq m^\delta \text{ for all } m \geqq n] \\
&\qquad + P[\max_{1 \leqq i \leqq m} |\textstyle\sum_{j=1}^i T_j - i| > m^\delta \text{ for some } m \geqq n] \\
&\quad = P_n^{(1)} + P_n^{(2)}\,, \qquad \text{say.}
\end{aligned} \tag{4.7}$$

Since $ET_1^p < \infty$ for all $p > 0$, $\sum n^{r-1} P[|\sum_{j=1}^n T_j - 1| > \varepsilon n^\delta] < \infty$ for all $\varepsilon > 0$. Therefore by Lemma 2,

$$\textstyle\sum_{n=1}^\infty n^{r-1} P_n^{(2)} < \infty\,. \tag{4.8}$$

Noting that

$$\begin{aligned}
P_n^{(1)} &\leqq \textstyle\sum_{m=n}^\infty \{3m^\delta P[\max_{0 \leqq t \leqq 4m^\delta} |W(t)| > (\varepsilon/8)(2m \log m)^{\frac{1}{2}}] \\
&\qquad + \textstyle\sum_{3m^\delta < i \leqq m} P[\max_{|t-i| \leqq m^\delta} |W(t) - W(i - m^\delta)| > (\varepsilon/8)(2m \log m)^{\frac{1}{2}}]\} \\
&\leqq c_1 \textstyle\sum_{m \geqq n} m \exp(-c_2 m^{1-\delta} \log m)\,,
\end{aligned}$$

where c_1, c_2 are positive constants, we obtain that

$$\textstyle\sum_{n=1}^\infty n^{r-1} P_n^{(1)} < \infty\,. \tag{4.9}$$

From (4.6), (4.7), (4.8) and (4.9), the desired conclusion follows.

Proof of Theorem 2. We now proceed to prove Theorem 2 on Strassen's conjecture as stated in Section 1. With the same notations as in Theorem 5, we obtain (1.14) from Corollary 1 and Theorem 5, so the sequence $(\zeta_n)_{n \geqq 2}$ is r-quickly relatively compact in $C[0, 1]$.

Suppose x is an r-quick limit point of the sequence $(Y_n)_{n \geqq 2}$. Take any $\varepsilon > 0$ and let $L = \sup\{n : \|Y_n - \zeta_n\|_C > \varepsilon\}$. We note that if $n > L$ and $\|x - Y_n\|_C < \varepsilon$, then $\|x - \zeta_n\|_C < 2\varepsilon$. Therefore

$$L + 1 + \sup\{n : \|x - \zeta_n\|_C < 2\varepsilon\} \geqq \sup\{n : \|x - Y_n\|_C < \varepsilon\}\,.$$

Hence

$$\begin{aligned}
&E(L + 1)^r + E(\sup\{n : \|x - \zeta_n\|_C < 2\varepsilon\})^r \\
&\qquad \geqq 2^{-r} E(\sup\{n : \|x - Y_n\|_C < \varepsilon\})^r = \infty\,.
\end{aligned}$$

Since $EL^r < \infty$ by Theorem 5, it then follows that x is an r-quick limit point of the sequence $(\zeta_n)_{n\geqq 2}$. Likewise if x is an r-quick limit point of $(\zeta_n)_{n\geqq 2}$, then it is also an r-quick limit point of $(Y_n)_{n\geqq 2}$. Hence by Corollary 1, $r^{\frac{1}{2}}K_0$ is the set of r-quick limit points of $(\zeta_n)_{n\geqq 2}$.

5. Some applications of Strassen's conjecture. In [15], Strassen gave some beautiful applications of his functional form of the law of the iterated logarithm. In this section, we shall give similar applications of Theorem 2. First we study in the following lemmas some properties of r-quick limit sets which will be used in the sequel.

LEMMA 3. *Let (M, d) be a metric space and let (ζ_n) be a sequence of random variables taking values in M. Then the set K of r-quick limit points of (ζ_n) is a closed subset of M. Consequently if M is complete and (ζ_n) is r-quickly relatively compact in M, then K is a compact subset of M.*

PROOF. Let x_i be a sequence of points in K such that $\lim_{i\to\infty} x_i = x$. We shall show that $x \in K$. Assume the contrary. Then there exists an open neighborhood V of x such that $E(\sup\{n : \zeta_n \in V\})^r < \infty$. Since $x_i \in V$ for all large i, this contradicts that every x_i is an r-quick limit point of (ζ_n).

Now assume that M is complete and (ζ_n) is r-quickly relatively compact in M. Then given any $\varepsilon > 0$, there is a finite union $U(\varepsilon)$ of closed ε-spheres in M such that

$$E(\sup\{n : \zeta_n \notin U(\varepsilon)\})^r < \infty . \tag{5.1}$$

It follows easily from (5.1) that $K \subset U(\varepsilon)$. Hence K is totally bounded. Since K is a closed subset of M and is therefore complete, it is a compact subset of M.

LEMMA 4. *Let (M_1, d_1), (M_2, d_2) be metric spaces. Let $\varphi : M_1 \to M_2$ be a continuous function. Suppose (ζ_n) is a sequence of random variables taking values in M_1. Let $\mathscr{L}(\zeta_n)$ (respectively $\mathscr{L}(\varphi(\zeta_n))$) denote the set of r-quick limit points of (ζ_n) (respectively of $(\varphi(\zeta_n))$). Then*

$$\varphi[\mathscr{L}(\zeta_n)] \subset \mathscr{L}(\varphi(\zeta_n)) . \tag{5.2}$$

Assume further that the following condition holds:

(5.3) $\mathscr{L}(\zeta_n)$ *is compact and for every* $\varepsilon > 0$, $E(\sup\{n : \zeta_n \notin \mathscr{L}_\varepsilon(\zeta_n)\})^r < \infty$, *where* $\mathscr{L}_\varepsilon(\zeta_n)$ *denotes the open ε-neighborhood of* $\mathscr{L}(\zeta_n)$ *in* M_1.

Then the sequence (ζ_n) (respectively $(\varphi(\zeta_n))$) is r-quickly relatively compact in M_1 (respectively M_2), and equality holds in (5.2), i.e.,

$$\varphi[\mathscr{L}(\zeta_n)] = \mathscr{L}(\varphi(\zeta_n)) . \tag{5.4}$$

PROOF. To prove (5.2), let x be an r-quick limit point of (ζ_n) in M_1. Let V be an open neighborhood of $\varphi(x)$. Then there is an open neighborhood U of x

such that $\varphi[U] \subset V$. Obviously

$$E(\sup\{n : \varphi(\zeta_n) \in V\})^r \geqq E(\sup\{n : \zeta_n \in U\})^r = \infty .$$

Therefore $\varphi(x)$ is an r-quick limit point of $(\varphi(\zeta_n))$ in M_2.

We now assume (5.3). Given $\delta > 0$, let N_δ denote the open δ-neighborhood of $\varphi[\mathscr{L}(\zeta_n)]$ in M_2. By the continuity of φ, there exists an open set M containing $\mathscr{L}(\zeta_n)$ such that $\varphi[M] \subset N_\delta$. Let $\varepsilon = \inf\{d_1(a, b) : a \in \mathscr{L}(\zeta_n), b \in M_1 - M\}$. Since $\mathscr{L}(\zeta_n)$ is a compact set disjoint from the closed set $M_1 - M$, $\varepsilon > 0$. Noting that $\mathscr{L}_\varepsilon(\zeta_n) \subset M$ and so $\varphi[\mathscr{L}_\varepsilon(\zeta_n)] \subset N_\delta$, we obtain that $\sup\{n : \varphi(\zeta_n) \notin N_\delta\} \leqq \sup\{n : \zeta_n \notin \mathscr{L}_\varepsilon(\zeta_n)\}$. Hence it follows from (5.3) that

$$E(\sup\{n : \varphi(\zeta_n) \notin N_\delta\})^r < \infty \qquad \text{for every} \quad \delta > 0 . \tag{5.5}$$

By the continuity of φ and the compactness of $\mathscr{L}(\zeta_n)$, $\varphi[\mathscr{L}(\zeta_n)]$ is a compact subset of M_2. Therefore (5.5) implies that $(\varphi(\zeta_n))$ is r-quickly relatively compact in M_2 and

$$\mathscr{L}(\varphi(\zeta_n)) \subset \bigcap_{\delta>0} N_\delta . \tag{5.6}$$

The compactness of $\varphi[\mathscr{L}(\zeta_n)]$ implies that it is a closed subset of M_2, so $\bigcap_{\delta>0} N_\delta = \varphi[\mathscr{L}(\zeta_n)]$. Hence from (5.2) and (5.6), (5.4) follows.

LEMMA 5. *Let R denote the real line and (θ_n) be a sequence of real-valued random variables such that (θ_n) is r-quickly relatively compact in R. Let K be the set of all r-quick limit points of (θ_n) and let $y = \sup\{x : x \in K\}$. Then y is finite and*

$$\limsup_{n\to\infty} \theta_n = y \quad (r\text{-quickly}) . \tag{5.7}$$

PROOF. Since (θ_n) is r-quickly relatively compact in R, it follows from Lemma 3 that K is compact. Hence y is finite and $y \in K$. Therefore for any $c < y$, noting that (c, ∞) is an open neighborhood of y, we have $ET_c^{\,r} = \infty$, where $T_c = \sup\{n : \theta_n \geqq c\}$ as defined in (1.1).

Now let $c > y$ and set $2\varepsilon = c - y$. The r-quick relative compactness of (θ_n) implies that there exists a finite union $U = \bigcup_{i=1}^m (a_i, b_i)$ of open intervals (a_i, b_i) each of length ε such that

$$E(\sup\{n : \theta_n \notin U\})^r < \infty . \tag{5.8}$$

If $a_i > y$, then each $x \in [a_i, b_i]$ is not an r-quick limit point of (θ_n) and therefore there is an open neighborhood V_x such that $E(\sup\{n : \theta_n \in V_x\})^r < \infty$; so by the Heine-Borel theorem, there is an open set $G \supset [a_i, b_i]$ such that $E(\sup\{n : \theta_n \in G\})^r < \infty$. Hence without loss of generality, we can assume that $a_i \leqq y$ and therefore $b_i \leqq y + \varepsilon (< c)$ for all $i = 1, \cdots, m$, so (5.8) implies that $ET_c^{\,r} < \infty$. Therefore we have established (5.7).

In what follows, we shall let $X_1, X_2, \cdots$ be i.i.d. real-valued random variables such that $EX_1 = 0$, $EX_1^2 = 1$ and $E|X_1|^{2(r+1)}(\log^+|X_1| + 1)^{-(r+1)} < \infty$. Let $S_n = X_1 + \cdots + X_n$ and define ζ_n as in (1.11) and $r^{\frac{1}{2}}K_0$ as in (1.12). Then Theorem 2 is applicable and ζ_n satisfies condition (5.3), as it is well known that $r^{\frac{1}{2}}K_0$ is a compact subset of $C[0, 1]$. We now give some applications of Theorem 2 below.

EXAMPLE 1. Define $\varphi: C[0, 1] \to R$ by $\varphi(x) = x(1)$. Clearly φ is continuous, and so by Theorem 2 and Lemmas 4 and 5,

$$\limsup_{n\to\infty} (2n \log n)^{-\frac{1}{2}} S_n = \sup_{x \in r^{\frac{1}{2}} K_0} x(1) \quad (r\text{-quickly})\,. \tag{5.9}$$

Since the supremum in (5.9) is attained with $x(t) = r^{\frac{1}{2}} t$, we have established (1.16), thus completing the proof of Theorem 3.

EXAMPLE 2. Define $\varphi: C[0, 1] \to R$ by $\varphi(x) = ||x||_C$. Then φ is continuous and $\sup_{x \in r^{\frac{1}{2}} K_0} \varphi(x) = r^{\frac{1}{2}}$. Therefore by Theorem 2 and Lemmas 4 and 5,

$$\limsup_{n\to\infty} (2n \log n)^{-\frac{1}{2}} \max_{1\le j\le n} |S_j| = r^{\frac{1}{2}} \quad (r\text{-quickly})\,. \tag{5.10}$$

EXAMPLE 3. Let f be a continuous real-valued function on $[0, 1]$ and set $F(t) = \int_t^1 f(s)\,ds$, $0 \leqq t \leqq 1$. Define $\varphi: C[0, 1] \to R$ by $\varphi(x) = \int_0^1 x(t) f(t)\,dt$. Then φ is continuous and $\sup_{x \in r^{\frac{1}{2}} K_0} \varphi(x) = (r \int_0^1 F^2(t)\,dt)^{\frac{1}{2}}$ (cf. [15, page 219]). Therefore by Theorem 2 and Lemmas 4 and 5,

$$\limsup_{n\to\infty} \int_0^1 f(t)\zeta_n(t)\,dt = (r \textstyle\int_0^1 F^2(t)\,dt)^{\frac{1}{2}} \quad (r\text{-quickly})\,. \tag{5.11}$$

We note that

$$\begin{aligned} &|n^{-1} \textstyle\sum_{i=1}^n f(i/n)(2n \log n)^{-\frac{1}{2}} S_i - \int_0^1 f(t)\zeta_n(t)\,dt| \\ &= |\textstyle\sum_{i=1}^n \int_{(i-1)/n}^{i/n} (f(i/n) - f(t))\zeta_n(t)\,dt + (2n)^{-1} \sum_{i=1}^n f(i/n)(2n \log n)^{-\frac{1}{2}} X_i| \\ &\leqq \alpha_n (2n \log n)^{-\frac{1}{2}} \max_{1\le j\le n} |S_j| + ||f||_C (2n \log n)^{-\frac{1}{2}} \max_{1\le j\le n} |X_j|\,, \end{aligned} \tag{5.12}$$

where $\alpha_n = \max\{|f(i/n) - f(t)| : (i-1)/n \leqq t \leqq i/n,\ i = 1, \cdots, n\}$. We now show that

$$\limsup_{n\to\infty} (n \log n)^{-\frac{1}{2}} \max_{1\le j\le n} |X_j| = 0 \quad (r\text{-quickly})\,. \tag{5.13}$$

To prove (5.13), since $E|X_1|^{2(r+1)}(\log^+ |X_1| + 1)^{-(r+1)} < \infty$, it follows that

$$\sum n^r P[|X_1| > \varepsilon(n \log n)^{\frac{1}{2}}] < \infty \qquad \text{for all} \quad \varepsilon > 0\,. \tag{5.14}$$

Obviously (5.14) implies that

$$\sum n^{r-1} P[\max_{j\le n} |X_j| > \varepsilon(n \log n)^{\frac{1}{2}}] < \infty \qquad \text{for all} \quad \varepsilon > 0\,. \tag{5.15}$$

By an argument similar to the proof of Lemma 2, we obtain that

$$\begin{aligned} &\textstyle\sum_{n=2}^\infty n^{r-1} P[\max_{j\le m} |X_j| > \varepsilon(m \log m)^{\frac{1}{2}} \text{ for some } m \geqq n] \\ &\qquad \leqq c \textstyle\sum_{m=1}^\infty m^{r-1} P[\max_{j\le m} |X_j| > (\varepsilon/2)(m \log m)^{\frac{1}{2}}]\,, \end{aligned} \tag{5.16}$$

where c is a positive constant. Using (5.15) and (5.16), we establish (5.13).

From (5.10), (5.11), (5.12) and (5.13), it follows that

$$\limsup_{n\to\infty} (2n^3 \log n)^{-\frac{1}{2}} \textstyle\sum_{i=1}^n f(i/n) S_i = (r \int_0^1 F^2(t)\,dt)^{\frac{1}{2}} \quad (r\text{-quickly})\,. \tag{5.17}$$

EXAMPLE 4. Let $p \geqq 1$ be a real constant. Define $\varphi: C[0, 1] \to R$ by $\varphi(x) = \int_0^1 |x(t)|^p\,dt$. Clearly φ is continuous. Therefore making use of Theorem 2 and Lemmas 4 and 5, we obtain by a similar argument as in [15, pages 220–221] that

$$\begin{aligned} &\limsup_{n\to\infty} n^{-1-(p/2)} (2 \log n)^{-p/2} \textstyle\sum_{i=1}^n |S_i|^p \\ &\qquad = 2(r/p)^{p/2} (p+2)^{(p-2)/2} \{\textstyle\int_0^1 (1 - t^p)^{-\frac{1}{2}}\,dt\}^{-p} \quad (r\text{-quickly})\,. \end{aligned} \tag{5.18}$$

6. r-quick convergence and statistical applications. While Theorems 2 and 3 deal with r-quick versions of the law of the iterated logarithm, we shall examine in this section the r-quick version of the strong law of large numbers. Our results in this section are mainly expository in nature, reformulating earlier results in the literature in the framework of r-quick convergence. We shall also outline some statistical applications which have appeared elsewhere to indicate the usefulness of the concept of r-quick convergence.

First we note that for a sequence (θ_n) of real-valued random variables, the statement $\lim_{n\to\infty} \theta_n = 0$ a.s. can be expressed as $P[T_\varepsilon^* < \infty] = 1$ for all $\varepsilon > 0$, where

$$T_\varepsilon^* = \sup\{n \geqq 1 : |\theta_n| \geqq \varepsilon\} . \tag{6.1}$$

Therefore in analogy with Definition 1, we define r-quick convergence as follows.

DEFINITION 3. Let (M, d) be a metric space and let ζ, ζ_n $(n = 1, 2, \cdots)$ be random variables taking values in M. Then

$$\lim_{n\to\infty} \zeta_n = \zeta \quad (r\text{-quickly}) \tag{6.2}$$

if and only if

$$E(\sup\{n \geqq 1 : d(\zeta, \zeta_n) \geqq \varepsilon\})^r < \infty \qquad \text{for all } \varepsilon > 0 . \tag{6.3}$$

Suppose $X_1, X_2, \cdots$ are i.i.d. real-valued random variables and $0 < \beta < 2$. Let $S_n = X_1 + \cdots + X_n$. The Marcinkiewicz–Zygmund strong law of large numbers states that

$$n^{-1/\beta} S_n \to 0 \quad \text{a.s.} \Leftrightarrow E|X_1|^\beta < \infty \quad \text{and in the case} \quad \beta \geqq 1 , \quad EX_1 = 0 . \tag{6.4}$$

An r-quick version of the above strong law is the following: For any $p > 1/\alpha$ and $\alpha > \frac{1}{2}$,

$$\begin{aligned} & E|X_1|^p < \infty \quad \text{and for the case} \quad \alpha \leqq 1 , \quad EX_1 = 0 \\ & \Leftrightarrow \sum n^{p\alpha-2} P[|S_n| \geqq \varepsilon n^\alpha] < \infty \qquad \text{for all } \varepsilon > 0 \\ & \Leftrightarrow \sum n^{p\alpha-2} P[\sup_{k \geqq n} |S_k/k^\alpha| \geqq \varepsilon] < \infty \qquad \text{for all } \varepsilon > 0 \\ & \Leftrightarrow n^{-\alpha} S_n \to 0 \quad (p\alpha - 1 \text{ quickly}) \end{aligned} \tag{6.5}$$

(cf. [1], [2]). The results (2.1) and (2.2) can therefore be regarded as an extension of (6.5) in the limiting case $\alpha = \frac{1}{2}$.

In [7], the r-quick version (6.5) of the strong law has been extended to i.i.d. random variables $X_1, X_2, \cdots$ taking values in a separable Banach space B. It is proved that for $\alpha \geqq 1$ and $p > 1/\alpha$,

$$\begin{aligned} & E||X_1||^p < \infty \quad \text{and for the case} \quad \alpha = 1 , \\ & \qquad EX_1 = 0 \quad \text{in the Bochner sense} \\ & \Leftrightarrow \sum n^{p\alpha-2} P[||S_n|| \geqq \varepsilon n^\alpha] < \infty \qquad \text{for all } \varepsilon > 0 \\ & \Leftrightarrow n^{-\alpha} S_n \to 0 \quad (p\alpha - 1 \text{ quickly}) . \end{aligned} \tag{6.6}$$

A counterexample is given in [7] to show that (6.6) may fail to hold for $\frac{1}{2} < \alpha < 1$ and $p\alpha > 1$. However, if the Banach space B satisfies additional assumptions (for example, if B is a Hilbert space), then (6.6) still holds for $\frac{1}{2} < \alpha < 1$ and $p\alpha > 1$. Applications to the Cramér–von Mises statistic and to likelihood function statistics are given in [7].

In [9], by making use of the notion of r-quick convergence, we obtain asymptotic approximations for the expected sample size in certain invariant sequential tests. For example, for the sequential t-test which tests sequentially the null hypothesis H_0 that the observations $Z_1, Z_2, \cdots$ are i.i.d. $N(\zeta, \sigma^2)$ with $\zeta/\sigma = \gamma_0$ versus the alternative H_1 that $Z_1, Z_2, \cdots$ are i.i.d. $N(\zeta, \sigma^2)$ with $\zeta/\sigma = \gamma_1$, where $\gamma_0 \neq \gamma_1$ are given and ζ, σ are unknown parameters, we stop at stage

$$N = \inf\{n \geqq 1 : l_n \notin (-a, b)\},$$

where l_n is the logarithm of the likelihood ratio of the maximal invariant at stage n (cf. [9]). Although l_n has a fairly complicated structure, we have the following approximation: There exists a constant c for which

$$|l_n - n\psi(T_n)| \leqq c, \qquad n = 1, 2, \cdots$$

where

$$T_n = (n^{-1} \textstyle\sum_1^n Z_i)/(n^{-1} \sum_1^n Z_i^2)^{\frac{1}{2}} \qquad \text{and}$$

$$\psi(y) = \beta(\gamma_1 y) - \beta(\gamma_0 y) - \tfrac{1}{2}\gamma_1^2 + \tfrac{1}{2}\gamma_0^2,$$

$$\beta(u) = \tfrac{1}{2}u\alpha(u) + \log\alpha(u), \qquad \alpha(u) = \tfrac{1}{2}\{u + (u^2 + 4)^{\frac{1}{2}}\}.$$

(See [9].) Now assume that $Z_1, Z_2, \cdots$ are i.i.d. (not necessarily normal) with $0 < E|Z_1|^{2(r+1)} < \infty$ for some $r > 0$. Then by (6.5),

$$\lim_{n\to\infty} T_n = \lambda \quad (r\text{-quickly}), \tag{6.7}$$

where $\lambda = (EZ_1)/(EZ_1^2)^{\frac{1}{2}}$. Let us first consider the case where $\psi(\lambda) > 0$. It is easy to see that as $\min(a, b) \to \infty$, $\psi(\lambda)N/b \to 1$ a.s. Choose $\varepsilon > 0$ such that $\psi(t) \geqq \eta\,(> 0)$ for $|t - \lambda| < \varepsilon$. Then defining $L = \sup\{n : |T_n - \lambda| \geqq \varepsilon\}$, we obtain (cf. [9]) that $N \leqq L + 2 + \eta^{-1}(b + c)$. Since $EL^r < \infty$ by (6.7), it then follows by the dominated convergence theorem that as $\min(a, b) \to \infty$,

$$EN^r \sim (b/\psi(\lambda))^r. \tag{6.8}$$

Similarly if $\psi(\lambda) < 0$, then $EN^r \sim (a/|\psi(\lambda)|)^r$.

In [8], we make use of r-quick convergence to prove uniform integrability theorems in renewal theory. Applications of r-quick convergence to linear rank statistics are given in [10], where it is proved that the remainder term in the Chernoff–Savage representation of linear rank statistics, when suitably normalized, in fact converges to 0 r-quickly under certain assumptions. Since r-quick convergence implies almost sure convergence, we obtain as an immediate corollary an invariance principle and a law of the iterated logarithm for linear rank statistics. Some applications of these results to study the expected sample size of certain sequential rank tests are also given in [10].

REFERENCES

[1] BAUM, L. E. and KATZ, M. (1965). Convergence rates in the law of large numbers. *Trans. Amer. Math. Soc.* **120** 108–123.

[2] CHOW, Y. S. and LAI, T. L. (1975). Some one-sided theorems on the tail distribution of sample sums with applications to the last time and largest excess of boundary crossings. *Trans. Amer. Math. Soc.* **208** 51–72.

[3] FERNIQUE, X. (1964). Continuité des processus gaussiens. *C. R. Acad. Sci. Paris Sér. A–B* **258** 6058–6060.

[4] FERNIQUE, X. (1970). Integrabilité des vecteurs gaussiens. *C. R. Acad. Sci. Paris Sér. A* **270** 1698–1699.

[5] LAI, T. L. (1974). Limit theorems for delayed sums. *Ann. Probability* **2** 432–440.

[6] LAI, T. L. (1974). Reproducing kernel Hilbert spaces and the law of the iterated logarithm for Gaussian processes. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **29** 7–19.

[7] LAI, T. L. (1974). Convergence rates in the strong law of large numbers for random variables taking values in Banach spaces. *Bull. Inst. Math., Acad. Sinica* **2** 67–85.

[8] LAI, T. L. (1975). On uniform integrability in renewal theory. *Bull. Inst. Math., Acad. Sinica* **3** 99–105.

[9] LAI, T. L. (1975). Termination, moments and exponential boundedness of the stopping rule for certain invariant sequential probability ratio tests. *Ann. Statist.* **3** 581–598.

[10] LAI, T. L. (1975). On Chernoff-Savage statistics and sequential rank tests. *Ann. Statist.* **3** 825–845.

[11] LAMPERTI, J. (1962). Semi-stable stochastic processes. *Trans. Amer. Math. Soc.* **104** 62–78.

[12] LOÈVE, M. (1963). *Probability Theory*, 3rd ed. Van Nostrand, Princeton.

[13] OODAIRA, H. (1972). On Strassen's version of the law of the iterated logarithm for Gaussian processes. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **21** 289–299.

[14] SLIVKA, J. (1969). On the law of the iterated logarithm. *Proc. Nat. Acad. Sci. U.S.A.* **63** 289–291.

[15] STRASSEN, V. (1964). An invariance principle for the law of the iterated logarithm. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **3** 211–226.

[16] STRASSEN, V. (1967). Almost sure behavior of sums of independent random variables and martingales. *Proc. Sixth Berkeley Symp. Math. Statist. Prob.* **2** 315–343.

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