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THE RANGE OF A RANDOM WALK IN TWO-DIMENSIONAL TIME[1]

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Let $[X_{ij}: i > 0, j > 0]$ be a double sequence of i.i.d. random variables taking values in the d-dimensional lattice E_d. Also let $S_{mn} = \sum_{i=1}^{m} \sum_{j=1}^{n} X_{ij}$. Then the range of random walk $[S_{mn}: m > 0, n > 0]$ up to time (m, n), denoted by R_{mn}, is the cardinality of the set $[S_{pq}: 0 < p \leqq m, 0 < q \leqq n]$, i.e., the number of distinct points visited by the random walk up to time (m, n). In this paper a strong law for R_{mn}, when $d \geqq 3$, has been established. Namely, it has been proved that $\lim R_{mn}/ER_{mn} = 1$ a.s. as either (m, n) or m (n) tends to infinity.

1. Introduction. Let $[X_{ij}: i > 0, j > 0]$ be a double sequence of independently, identically distributed random variables (i.i.d.) which takes values in d-dimensional integer lattice E_d. The double sequence $[S_{mn}: m > 0, n > 0]$ defined by $S_{mn} = \sum_{i=1}^{m} \sum_{j=1}^{n} X_{ij}$ is called the random walk in two-dimensional time generated by X_{11} or a two-parameter random walk or simply a random walk when there is no danger of confusion. In this paper we will study the asymptotic behavior of the range of two-parameter random walk. To be more specific, let the range of random walk up to time (m, n), denoted by R_{mn}, be the cardinality of the set $[S_{ij}: 0 < i \leqq m, 0 < j \leqq n]$, i.e., the number of distinct lattice points visited by the random walk up to time (m, n). Then one would like to know how R_{mn} behaves as (m, n) tends to infinity.

Although the range of one-parameter random walk has been studied extensively starting with Dvoretzky and Erdös [1] and then by Jain and Orey [2], Jain and Pruitt [3, 4, 5, 6, 7], no papers have been published, as far as we know, investigating the range of two-parameter random walks.

In this work after giving some notations and preliminary estimates in Section 2, we will prove in Section 3, that a strong law holds for R_{mn} when $d \geqq 3$. This means that R_{mn}/ER_{mn} is asymptotically one almost surely as either (m, n) or m (n) tends to infinity.

2. Notations and preliminaries. From the random walk in two-dimensional time one can induce one-parameter random walks, which will be of considerable interest, as follows:

Let $[X_{ij}: (i, j) \in I^+ \times I^+]$, ($I^+$ = the set of positive integers), be the corresponding double sequence of i.i.d. random variables, defined on the probability space

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$(\Omega, \mathscr{F}, P)$, and

(2.1) $$X_i^n = X_{i1} + X_{i2} + \cdots + X_{in}, \qquad (i, n) \in I^+ \times I^+ .$$

Then fixing $n \in I^+$, the process $[S_m^n : m \in I^+]$ defined by

(2.2) $$S_m^n = \sum_{i=1}^m X_i^n, \qquad m \in I^+ .$$

will give us a one-parameter random walk.

DEFINITION 2.1. The two-parameter random walk generated by X_{11} is called genuinely d-dimensional if the group generated by the support of X_{11}, $[x \in E_d : P[X_{11} = x] > 0]$, is d-dimensional. Note that in this case the support of X_{11} is not contained in a hyperplane through the origin. Also the associated one-parameter random walk $[S_m^1 : m \in I^+]$ is genuinely d-dimensional.

For the definitions and terminologies used for one-parameter random walk, we refer the reader to [8].

The two-parameter random walk may take place on a proper subgroup of E_d. In this case, the subgroup is isomorphic to some E_k, $k \leqq d$; if $k < d$, then the transformation should be made (see [8], page 66) and the problem considered in k-dimensions. We will assume throughout this paper that this reduction has been made, if necessary, and the random walk is genuinely d-dimensional.

For an arbitrary set A in $I^+ \times I^+$, $\sum_{(i,j) \in A} X_{ij}$ will be denoted by S_A. For convenience we let $P[S_0^n = 0] = 1$, $n \in I^+$, and we will use u_m^n for $P[S_m^n = 0]$, U^n for $\sum_{m=0}^\infty P[S_m^n = 0]$ and r^n for $P[S_m^n \neq 0$ for all $m \in I^+]$, $n \in I^+$.

The following theorem (see [8], page 72–73) will give us a uniform bound for $P[S_A = x]$, $x \in E_d$, where A is a finite subset of $I^+ \times I^+$ with cardinality $|A|$.

THEOREM 2.1. *For a genuinely d-dimensional random walk generated by X_{11} there exists a constant c, independent of x, such that for every finite set A in $I^+ \times I^+$, $P[S_A = x] \leqq c|A|^{-\frac{1}{2}d}$, provided that the symmetrized random walk generated by $X_{11} - X_{11}'$ is also genuinely d-dimensional, where X_{11} and X_{11}' are independently identically distributed.*

Finally, let R_m^n be the range of one-parameter random walk $[S_k^n : k \in I^+]$ up to time m, i.e., the cardinality of the set $[S_k^n : 1 \leqq k \leqq m]$. Then the following theorem, with a beautiful proof in [8], page 38, is true.

THEOREM 2.2. *Let $n \in I^+$ be fixed. Then* $\lim_{m\to\infty} R_m^n/m = r^n$ a.s.

GENERAL REMARKS.

(i) Throughout this paper c is a "universal constant" in the sense that it may depend only on the distribution of X_{11} and we will allow it to change in each step in computations.

(ii) $[a]$ will denote the integer part of the real number a, i.e., the greatest integer smaller than or equal to a.

(iii) Let $[a_{mn} : (m, n) \in I^+ \times I^+]$ be a double sequence. Then we say a_{mn} approches a, as $(m, n) \to \infty$, if given $\varepsilon > 0$, there exists $M(\varepsilon)$ a positive integer depending on ε, such that; if $m, n > M(\varepsilon)$, then $|a_{mn} - a| < \varepsilon$.

3. The strong law for R_{mn} when $d \geqq 3$. Throughout this section, first, we will assume that the genuine dimensions of the random walk and its associated symmetrized random walk are the same in order to be able to use Theorem 2.1. Then at the end we will remove this assumption.

The strong law for R_{mn} is obtained by approximating R_{mn} by another double sequence of random variables, which we call Q_{mn}, defined by

$$Q_{mn} = \sum_{q=1}^{n} R_m^q . \tag{3.1}$$

Clearly $R_{mn} \leqq Q_{mn} \leqq mn$ and in fact we will eventually show that,

$$R_{mn} \sim mn \sim Q_{mn} \quad \text{a.s.}, \tag{3.2}$$

as $(m, n) \to \infty$.

THEOREM 3.1. *Let $d \geqq 3$, then EQ_{mn}/mn converges to one as $(m, n) \to \infty$.*

PROOF. Fix q and let F_p^q be the event that the random walk $[S_i^q : i \in I^+]$ visits a new point relative to previous times on the time line $y = q$, i.e.,

$$F_p^q = [S_i^q \neq S_p^q : i = 1, 2, \cdots, p-1] , \qquad F_1^q = \Omega . \tag{3.3}$$

Then we have

$$R_m^q = \sum_{p=1}^{m} I(F_p^q) , \tag{3.4}$$

where $I(F_p^q)$ is the indicator function of F_p^q. Now let

$$r_p^q = P[S_i^q \neq 0 : i = 1, 2, \cdots, p] . \tag{3.5}$$

Then following the work of Dvoretzky and Erdös ([1], page 353–356) one can easily show that r_p^q approaches $r^q = 1/U^q$ as p tends to infinity and in fact, using the estimate for u_m^n, we have

$$\begin{aligned} r_p^q - r^q &\leqq P[S_i^q = 0 : \text{for some } \; i \geqq p+1] \\ &\leqq \textstyle\sum_{i=p+1}^{\infty} u_i^q \leqq c \sum_{i=p+1}^{\infty} i^{-\frac{1}{2}d} q^{-\frac{1}{2}d} = O(p^{1-\frac{1}{2}d} q^{-\frac{1}{2}d}) . \end{aligned} \tag{3.6}$$

But

$$r^q = 1/U^q = 1/(1 + \textstyle\sum_{p=1}^{\infty} u_p^q) \geqq 1/(1 + c q^{-\frac{1}{2}d}) \geqq 1 - c q^{-\frac{1}{2}d} . \tag{3.7}$$

Thus,

$$ER_m^q = m r^q + O(q^{-\frac{3}{2}} m^{\frac{1}{2}}) , \qquad EQ_{mn} = \textstyle\sum_{q=1}^{n} ER_m^q \geqq m(n - c) . \tag{3.8}$$

For $EI(F_p^q) = r_{p-1}^q$, $(r_0^q = 1)$.

Dividing EQ_{mn}, in (3.8), by mn and letting $(m, n) \to \infty$ one has

$$\lim_{(m,n)\to\infty} EQ_{mn}/mn \geqq 1 . \tag{3.9}$$

But

$$EQ_{mn}/mn \leqq 1 . \tag{3.10}$$

Therefore we have the desired result.

THEOREM 3.2. *Let $d \geqq 3$, then Q_{mn}/mn converges to one* a.s. *as $(m, n) \to \infty$.*

PROOF. The main part of the proof is to get an estimate for the variance of Q_{mn} of the form Var $Q_{mn} = O(m^{\frac{3}{2}}n^{\frac{1}{2}})$, once we have such an estimate, then we can use Chebyshev's inequality to get

(3.11) $$P[|Q_{mn} - EQ_{mn}| > \varepsilon EQ_{mn}] \leqq \operatorname{Var} Q_{mn}/\varepsilon^2(EQ_{mn})^2 = O(m^{-\frac{1}{2}}n^{-\frac{3}{2}}) .$$

Now (3.11) together with the standard Borel–Cantelli lemma give us

(3.12) $$\lim_{(p,q)\to\infty} Q_{m_p n_q}/EQ_{m_p n_q} = 1 \quad \text{a.s.,}$$

where $m_p = [p^\nu]$, $n_q = [q^\nu]$ with $\nu > 2$. But for $m_{p-1} < m \leqq m_p$ and $n_{q-1} < \leqq n_q$,

(3.13) $$\frac{Q_{m_{p-1}n_{q-1}}}{m_{p-1}n_{q-1}} \cdot \frac{m_{p-1}n_{q-1}}{m_p n_q} \leqq \frac{Q_{mn}}{mn} \leqq \frac{Q_{m_p n_q}}{m_p n_q} \cdot \frac{m_p n_q}{m_{p-1}n_{q-1}} .$$

Therefore (3.12) and the previous theorem give us the result.

To get an estimate for the variance of Q_{mn}, first use (3.4) to obtain

$$\begin{aligned}\operatorname{Var} R_m{}^q &= \textstyle\sum_{j=1}^m \sum_{k=1}^m [EI(F_j{}^q \cap F_k{}^q) - EI(F_j{}^q)EI(F_k{}^q)] \\ &= 2 \textstyle\sum \sum_{1\leqq j<k\leqq n} [EI(F_j{}^q \cap F_k{}^q) - EI(F_j{}^q)E(F_k{}^q)] \\ &\quad + \textstyle\sum_{j=1}^m EI(F_j{}^q)[1 - E(F_j{}^q)] .\end{aligned}$$

Now follow the argument given in [8], pages 35–38, to get

(3.14) $$\operatorname{Var} R_m{}^q \leqq 2ER_m{}^q(ER^q_{m-[m/2]} - ER_m{}^q + ER^q_{[m/2]}) + \textstyle\sum_{k=0}^{m-1} r_k{}^q(1 - r_k{}^q) .$$

Since $1 - r_p{}^q \leqq \sum_{i=1}^p P[S_i{}^q = 0] = O(q^{-\frac{3}{2}})$, (3.8) and (3.14) will easily give us,

(3.15) $$\operatorname{Var} R_m{}^q = O(q^{-\frac{3}{2}}m^{\frac{3}{2}}) .$$

But,

$$\begin{aligned}\operatorname{Var} Q_{mn} &= \operatorname{Var} (\textstyle\sum_{q=1}^n R_m{}^q) \\ &= \textstyle\sum_{q=1}^n \operatorname{Var} R_m{}^q + \sum\sum_{p\neq q} \operatorname{Cov} (R_m{}^p, R_m{}^q) \\ &\leqq \textstyle\sum_{q=1}^n \operatorname{Var} R_m{}^q + \sum\sum_{p\neq q} (\operatorname{Var} R_m{}^p)^{\frac{1}{2}}(\operatorname{Var} R_m{}^q)^{\frac{1}{2}} \qquad (3.16) \\ &= (\textstyle\sum_{q=1}^n (\operatorname{Var} R_m{}^q)^{\frac{1}{2}})^2 = (\sum_{q=1}^n O(q^{-\frac{3}{4}}m^{\frac{3}{4}}))^2 \\ &= (O(m^{\frac{3}{4}}n^{\frac{1}{4}}))^2 = O(m^{\frac{3}{2}}n^{\frac{1}{2}}) .\end{aligned}$$

□

COROLLARY 3.1. *Let* $d \geqq 3$, *then* Q_{mn}/EQ_{mn} *converges to one almost surely as* $(m, n) \to \infty$.

PROOF. The proof is an immediate consequence of the preceding two theorems.

To reach our goal, it, suffices to show that $(Q_{mn} - R_{mn})/mn$ converges to zero almost surely as $(m, n) \to \infty$. To prove this we will need two lemmas. Let us also use $\langle m, n\rangle$ to denote $[(i, j): 1 \leqq i \leqq m, 1 \leqq j \leqq n]$.

LEMMA 3.1. *Let* $d \geqq 3$ *and*

$$T^1_{mn} = \textstyle\sum_{i=1}^m \sum_{j=1}^n I(\bigcup_{p=i+1}^m \bigcup_{q=j+1}^n I[S_{pq} = S_{ij}]) .$$

Then T^1_{mn}/mn *converges to zero almost surely as* $(m, n) \to \infty$.

PROOF. As usual first we will get an upper estimate for ET^1_{mn} and this estimate will be good enough to get us through. We will proceed as follows:

$$
\begin{aligned}
EI(\bigcup_{p=i+1}^{m} \bigcup_{q=j+1}^{n} [S_{pq} = S_{ij}]) &= P(\bigcup_{p=i+1}^{m} \bigcup_{q=j+1}^{n} [S_{pq} = S_{ij}]) \\
&\leqq \sum_{p=i+1}^{m} \sum_{q=j+1}^{n} P[S_{pq} = S_{ij}] \\
&= \sum_{p=i+1}^{m} \sum_{q=j+1}^{n} P[S_A = 0] \\
&\leqq \sum_{p=i+1}^{m} \sum_{q=j+1}^{n} c(pq - ij)^{-\frac{3}{2}} \\
&\leqq c \int_i^m \int_j^n (uv - ij)^{-\frac{3}{2}}\, du\, dv \\
&\leqq c(ij)^{-\frac{1}{2}} \int_1^\infty 2/y(y-1)^{\frac{1}{2}}\, dy = c(ij)^{-\frac{1}{2}} ,
\end{aligned} \tag{3.17}
$$

where $A = \langle p, q \rangle - \langle i, j \rangle$. Therefore,

$$ET^1_{mn} \leqq c \sum_{i=1}^{m} \sum_{j=1}^{n} (ij)^{-\frac{1}{2}} \leqq c(mn)^{\frac{1}{2}} . \tag{3.18}$$

Now by Chebyshev's inequality,

$$P[T^1_{mn}/mn \geqq \varepsilon] = P[T^1_{mn} \geqq \varepsilon mn] \leqq ET^1_{mn}/\varepsilon mn \leqq c(mn)^{-\frac{1}{2}} . \tag{3.19}$$

This together with the Borel–Cantelli lemma shows that for $m_p = [p^\alpha]$, $n_q = [q^\alpha]$ such that $\alpha > 2$,

$$\lim_{(p,q)\to\infty} T^1_{m_p n_q}/m_p n_q = 0 \quad \text{a.s.} \tag{3.20}$$

But for $m_{p-1} < m \leqq m_p$, $n_{q-1} < n \leqq n_q$,

$$0 \leqq \frac{T^1_{mn}}{mn} \leqq \frac{T^1_{m_p n_q}}{m_p n_q} \cdot \frac{m_p n_q}{m_{p-1} n_{q-1}} \leqq c \frac{T^1_{m_p n_q}}{m_p n_q} . \tag{3.21}$$

Now (3.20) will conclude the proof.

REMARK 3.1. Notice that the preceding lemma is still valid if we fix n (m) and let $m \to \infty$ ($n \to \infty$).

LEMMA 3.2. *Let* $d \geqq 3$ *and*

$$T^2_{mn} = \sum_{i=1}^{m} \sum_{j=1}^{n} I(\bigcup_{p=1}^{i-1} \bigcup_{q=j+1}^{n} [S_{pq} = S_{ij}]) . \tag{3.22}$$

Then T^2_{mn}/mn *converges to zero almost surely as* $(m, n) \to \infty$.

PROOF. The idea of the proof will be similar to the one we gave for Lemma 3.1, except that we do not have the type of independence we had before. Therefore the proof will be a little lengthier than the previous one. For $1 \leqq p < i, j < q \leqq n$, let $A = \langle p, q \rangle - \langle p, j \rangle$ and $B = \langle i, j \rangle - \langle p, j \rangle$. Then we have

$$
\begin{aligned}
P[S_{pq} = S_{ij}] = P[S_A = S_B] &= \sum_x P[S_A = x, S_B = x] \\
&= \sum_x P[S_A = x] \cdot P[S_B = x] \leqq c(|A| \vee |B|)^{-d/2} \\
&\leqq 2^{d/2} c(|A| + |B|)^{-d/2} ,
\end{aligned} \tag{3.23}
$$

where the first inequality follows from Theorem 2.1. Thus,

$$P[S_{pq} = S_{ij}] \leqq c[p(q-j) + j(i-p)]^{-\frac{3}{2}} . \tag{3.24}$$

Now we consider two cases:

Case I: $n - j < j$ or $j > n/2$. For this case we have

$$
\begin{aligned}
&EI(\textstyle\bigcup_{p=1}^{i-1}\bigcup_{q=j+1}^{n}[S_{pq} = S_{ij}]) \\
&\qquad \leqq c \textstyle\sum_{p=1}^{i-1}\sum_{q=j+1}^{n}[p(q-j) + j(i-p)]^{-\frac{3}{2}} \\
&\qquad = c \textstyle\sum_{p=1}^{i-1}\sum_{q=2j-n}^{j-1}(ij - pq)^{-\frac{3}{2}} \\
&\qquad \leqq c \textstyle\sum_{p=1}^{i-1}\sum_{q=1}^{j-1}(ij - pq)^{-\frac{3}{2}} \leqq c(ij)^{-\frac{1}{2}}\,,
\end{aligned} \tag{3.25}
$$

where the last inequality follows very much as in the proof of Lemma 3.1.

Case II: $n - j \geqq j$. For this case, similarly to (3.25), we get

$$
\begin{aligned}
&EI(\textstyle\bigcup_{p=1}^{i-1}\bigcup_{q=1}^{n}[S_{pq} = S_{ij}]) \\
&\qquad \leqq c \textstyle\sum_{p=1}^{i-1}\sum_{q=1}^{n-j}[pq + j(i-p)]^{-\frac{3}{2}} \\
&\qquad = c \textstyle\sum_{p=1}^{i-1}\sum_{q=1}^{j-1}(ij - pq)^{-\frac{3}{2}} + c\sum_{p=j}^{i-1}\sum_{q=0}^{n-2j}(ij + pq)^{-\frac{3}{2}}\,.
\end{aligned} \tag{3.26}
$$

Now consider only the last sum,

$$
\begin{aligned}
&\textstyle\sum_{p=1}^{i-1}\sum_{q=0}^{n-2j}(ij + pq)^{-\frac{3}{2}} \\
&\qquad = \textstyle\sum_{p=1}^{i-1}(ij)^{-\frac{3}{2}} + \sum_{p=1}^{i-1}\sum_{q=1}^{n-2j}(ij + pq)^{-\frac{3}{2}} \\
&\qquad \leqq (ij)^{-\frac{1}{2}} + \textstyle\int_0^{i-1}\int_0^{n-2j}(ij + uv)^{-\frac{3}{2}}\,du\,dv \\
&\qquad \leqq (ij)^{-\frac{1}{2}} + (ij)^{-\frac{1}{2}}\textstyle\int_0^{(n-2j)/j}\int_0^{(i-1)/i}(1 + xy)^{-\frac{3}{2}}\,dx\,dy \\
&\qquad \leqq (ij)^{-\frac{1}{2}} + (ij)^{-\frac{1}{2}}\textstyle\int_0^{(n-2j)/j}(\int_0^1(1 + xy)^{-\frac{3}{2}}\,dy)\,dx \\
&\qquad \leqq (ij)^{-\frac{1}{2}} + 2(ij)^{-\frac{1}{2}}\log\left(\frac{n-j}{j}\right) \leqq c\,\frac{1 + \log(n/j)}{(ij)^{\frac{1}{2}}}\,.
\end{aligned} \tag{3.27}
$$

Therefore (3.25), (3.26) and (3.27) imply that

$$
EI(\textstyle\bigcup_{p=1}^{i-1}\bigcup_{q=j+1}^{n}[S_{pq} = S_{ij}]) \leqq c(ij)^{-\frac{1}{2}}(1 + \log(n/j))\,. \tag{3.28}
$$

Consequently,

$$
\begin{aligned}
ET_{mn}^2 &\leqq c\textstyle\sum_{i=1}^{m}\sum_{j=1}^{n}(ij)^{-\frac{1}{2}}(1 + \log(n/j)) \\
&\leqq cm^{\frac{1}{2}}\textstyle\sum_{j=1}^{n}j^{-\frac{1}{2}}(1 + \log(n/j)) \leqq cm^{\frac{1}{2}}\int_0^n x^{\frac{1}{2}}(1 + \log(n/x))\,dx \\
&\leqq c(mn)^{\frac{1}{2}}\textstyle\int_1^\infty u^{\frac{3}{2}}(1 + \log u)\,du \leqq c(mn)^{\frac{1}{2}}\,.
\end{aligned} \tag{3.29}
$$

Now again an argument similar to the one given at the end of Lemma 3.1 gives us the result.

REMARK 3.2. Notice that the preceding lemma is still valid if we fix n (m) and let $m \to \infty$ ($n \to \infty$).

THEOREM 3.3. *Let $d \geqq 3$, then R_{mn}/ER_{mn} converges to one almost as $(m, n) \to \infty$.*

PROOF. Let the difference between Q_{mn} and R_{mn} be T_{mn}. As we already mentioned, by Corollary 3.1: if T_{mn}/mn converges to zero almost surely, then R_{mn}/mn converges to one a.s. too. But $R_{mn}/mn \leqq 1$, hence the dominated convergence theorem can be used in order to show that ER_{mn}/mn converges to one too and thus the result follows.

To prove that T_{mn}/mn converges to zero a.s., we will proceed as follows:

Let F_i^j be the event which we defined in (3.3); then it is easy to see that

$$\begin{aligned} T_{mn} &= \sum_{i=1}^m \sum_{j=1}^n I(\bigcup_{p=1}^m \bigcup_{q=j+1}^n [F_i^j;\, S_{pq} = S_{ij}]) \\ &\leqq \sum_{i=1}^m \sum_{j=1}^n [I(\bigcup_{p=1}^{i-1} \bigcup_{q=j+1}^n [F_i^j;\, S_{pq} = S_{ij}]) \\ &\quad + I(\bigcup_{q=j+1}^n [F_i^j;\, S_{iq} = S_{ij}]) \\ &\quad + I(\bigcup_{p=i+1}^m \bigcup_{q=j+1}^n [F_i^j; S_{pq} = S_{ij}])] \\ &\leqq T_{mn}^2 + T_{mn}^1 + \sum_{i=1}^m \sum_{j=1}^n I(\bigcup_{q=j+1}^n [S_{iq} = S_{ij}]) \\ &= T_{mn}^1 + T_{mn}^2 + \sum_{i=1}^m \sum_{j=1}^n (1 - I(\bigcap_{q=j+1}^n [S_{iq} \neq S_{ij}])) \\ &= T_{mn}^1 + T_{mn}^2 + mn - Q_{mn}^* , \end{aligned} \tag{3.30}$$

where Q_{mn}^* is the "dual" of Q_{mn} in the sense that we should interchange the role of m and n. Hence by symmetry Q_{mn}^*/mn also converges to one almost surely as $(m, n) \to \infty$. Now in the following inequality

$$0 \leqq T_{mn}/mn \leqq T_{mn}^1/mn + T_{mn}^2/mn + 1 - Q_{mn}^*/mn , \tag{3.31}$$

which is a consequence of (3.30), let $(m, n) \to \infty$ and use the last two lemmas to get the result.

REMARK 3.3. By Chebyshev's inequality and using the estimate we had for the variance of R_m^n in (3.15), one can easily see that R_m^n converges to m almost surely as $n \to \infty$. Now this and Theorem 2.2 imply that,

$$\begin{aligned} &\lim_{n\to\infty} Q_{mn}/mn = 1 \quad \text{a.s.} ; \\ &\lim_{m\to\infty} Q_{mn}/mn = (r^1 + r^2 + \cdots + r^n)/n . \end{aligned} \tag{3.32}$$

Hence by Remarks 3.1 and 3.2 we have

$$\lim_{m\to\infty} R_{mn}/mn = \lim_{m\to\infty} Q_{mn}/mn = (r^1 + r^2 + \cdots + r^n)/n . \tag{3.33}$$

This together with the dominated convergence theorem imply that the strong law still holds for R_{mn} when n is fixed and $m \to \infty$, and by symmetry it also holds when m is fixed and $n \to \infty$.

Finally, to remove the assumption that we made at the beginning of this section, note that the d corresponding to the upper estimates for the $P[S_A = x]$ is in fact the genuine dimension of the symmetrized random walk regardless of the genuine dimension of random walk itself (see the proof.). Also in the case that X_{11} is genuinely d-dimensional but not $X_{11} - X_{11}'$, it is clear that the support of X_{11} must be the translate $x_0 + H$ of a hyperplane H through the origin (a proper subgroup). Thus S_{mn} is contained in $(mn)x_0 + H$. Now since $x_0 \notin H$, $S_{pq} \neq S_{mn}$ as long as $pq \neq mn$. Therefore we can conclude the following results:

(i) The difference between the genuine dimension of a random walk and its associated symmetrized random walk is at most one. This tells us that Theorem 3.3 holds true for $d \geqq 4$ without any assumption on the dimension of X_{11}. Also for any finite set A in $I^+ \times I^+$ we have.

$$P[S_{mn} = x] \leqq c(mn)^{-1} , \qquad d \geqq 3 . \tag{3.34}$$

(ii) Now suppose the genuine dimension of X_{11} is 3 and it is strictly bigger than the dimension of its symmetrized random walk. Then it is clear, from the above argument, that $Q_{mn} = mn$ a.s. and $T^1_{mn} = 0$ a.s. Therefore it only remains to take care of T^2_{mn}. But in this case, using (3.34), we have,

$$(3.35) \qquad \begin{aligned} ET^2_{mn} &= \sum_{i=1}^m \sum_{j=1}^n P[\bigcup_{pq=ij;1\leq p\leq i-1} [S_{pq} = S_{ij}]] \\ &\leqq \sum_{i=1}^m \sum_{j=1}^n \sum_{1\leq p\leq i-1;pq=ij} c(j(i-p))^{-1} \\ &= c \sum_{i=1}^m \sum_{j=1}^n j^{-1} \log(i) = O(m \log(m) \log(n)) . \end{aligned}$$

Hence for $m \leqq n$ and every $\delta \in (\frac{1}{2}, 1)$ we have,

$$(3.36) \qquad ET^2_{mn} = O((mn)^\delta) ,$$

which is sufficient to get us through. For it follows that T^2_{mn}/mn converges to zero a.s. as $(m,n) \to \infty$ in the upper half part of $I^+ \times I^+$. Therefore R_{mn}/mn tends to one a.s. as $(m, n) \to \infty$ in that region. Now by using Q^*_{mn} and its associated T^*_{mn} we can conclude that R_{mn}/mn also approaches to one a.s. as $(m, n) \to \infty$ in the remaining region $(m > n)$.

The case when m (n) is fixed and $n \to \infty$ $(m \to \infty)$ can be handled similarly. The limit behavior of R_{mn}/mn, in both cases, is one.

Finally we invite the reader to consult the author's thesis for the corresponding results on the range of random walk when the dimension of the random walk is either one or two. Since the results are not complete to our satisfaction, we have not made any attempt to make them available for general publication.

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REFERENCES

[1] Dvoretzky, A. and Erdös, P. (1951). Some problems on random walk in space. *Proc. Second Berkeley Symp. Math. Stat. Prob.* 353-367, Univ. of California Press.

[2] Jain, N. C. and Orey, S. (1968). On the range of random walk. *Israel J. Math.* **6** 373-380.

[3] Jain, N. C. and Pruitt, W. E. (1970). The range of recurrent random walk in the plane. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **16** 279-292.

[4] Jain, N. C. and Pruitt, W. E. (1971). The range of transient random walk. *J. Analyse Math.* **24** 369-393.

[5] Jain, N. C. and Pruitt, W. E. (1972). The range of random walk. *Proc. Sixth Berkeley Symp. Math. Stat. Prob.* **3** 31-50, Univ. of California Press.

[6] Jain, N. C. and Pruitt, W. E. (1972). The law of iterated logarithm for the range of random walk. *Ann. Math. Statist.* **43** 1692-1697.

[7] Jain, N. C. and Pruitt, W. E. (1972). Further limit theorems for the range of random walk. To appear in *J. Analyse Math.*

[8] Spitzer F. (1964). *Principles of Random Walk.* Van Nostrand, Princeton.

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