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PURELY ATOMIC STRUCTURES SUPPORTING UNDOMINATED AND NONUNIFORMLY INTEGRABLE MARTINGALES[1]

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Let $(F_n)_{n=1,2,\cdots}$ be a sequence of sigma-fields on a set Ω, each F_n purely atomic with respect to a measure P. Let C denote a nested sequence of sets C_n, where C_n is a P-atom of F_n for each n. Define $S(C) = \Sigma_n(P(C_n - C_{n+1})/P(C_n))$. Then every L^1-bounded martingale relative to $(F_n)_{n=1,2,\cdots}$ and P is uniformly integrable if and only if S is finite-valued, and every such martingale is dominated if and only if S is uniformly bounded.

1. Introduction. Let Ω be a set and $(F_n)_{n=1,2,\cdots}$ a nested sequence of sigma-fields on Ω. Suppose F_n is generated by a partition of Ω, $(I(j, n))_{j=1,2,\cdots,m_n}$ where m_n may be infinite. Let P be a measure on $F = \bigvee_n F_n$ such that for all n and j, $P(I(j, n)) > 0$. Call the pair $((F_n)_{n=1,2,\ldots}, P)$ a *purely atomic structure* on Ω.

A *chain* is a nested sequence of partition sets $I(j_1, 1) \supset I(j_2, 2) \supset \cdots$. We denote this chain by $\bigwedge_i I(j_i, i)$. A chain may also be denoted by the letter C, and the atom of F_n in C will be denoted C_n.

We now define a function on the space of all chains:

$$S(C) = \Sigma_n(P(C_n - C_{n+1})/P(C_n)) .$$

S may, of course, be infinite.

If $(M_n)_{n=1,2,\cdots}$ is a martingale relative to $(F_n)_{n=1,2,\cdots}$ and P, we say that M is *supported by* $((F_n)_{n=1,2,\ldots}, P)$. If $E(\sup_n M_n) < \infty$, M is *dominated*.

In this paper, we establish the following result: every L^1-bounded martingale on a purely atomic structure is uniformly integrable if and only if S is finite-valued; and every L^1-bounded martingale on the structure is dominated if and only if S is uniformly bounded.

2. Characterization of purely atomic structures supporting nonuniformly integrable L^1-bounded martingales.

PROPOSITION. *Suppose C is a chain. $S(C) = \infty$ if and only if $P(\bigcap_n C_n) = 0$.*

PROOF.

$$\begin{aligned} P(C_i) &= P(C_1)\,(P(C_2)/P(C_1)) \cdots (P(C_i)/P(C_{i-1})) \\ &= P(C_1) \textstyle\prod_{n=1,\cdots,i-1} (1 - (P(C_n - C_{n+1})/P(C_n))) . \end{aligned}$$

The product converges to zero as $i \to \infty$ if and only if $S(C) = \infty$.

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THEOREM 1. *A purely atomic structure* $((F_n)_{n=1,2,\dots}, P)$ *supports nonuniformly integrable* L^1*-bounded martingales if and only if there exists a chain* C *with* $S(C) = \infty$.

PROOF. (1) Suppose C is a chain with $S(C) = \infty$. Let $M_n = 1_{C_n} P(C_n)^{-1}$. Then M is a martingale supported by $((F_n)_{n=1,2,\dots}, P)$. By the proposition, $M_n \to 0$ a.s. Since $E(M_n) = 1$ for all n, $M_n \nrightarrow 0$ in L^1, so M is not uniformly integrable.

(2) If M is a nonuniformly integrable L^1-bounded martingale supported by $((F_n)_{n=1,2,\dots}, P)$, f is the a.s. limit of M_n, and $X_n = M_n - E(f \mid F_n)$, then X is a nonuniformly integrable martingale supported by $((F_n)_{n=1,2,\dots}, P)$ and $X_n \to 0$ a.s.

Since X is not identically zero, there exist integers n and j_n such that $|X_n| = c \neq 0$ on $I(j_n, n)$. Since $X_n = E(f \mid F_n)$, there is some integer j_{n+1} such that $I(j_{n+1}, n+1) \subset I(j_n, n)$ and $|X_{n+1}| \geqq c$ on $I(j_{n+1}, n+1)$. Continuing in this manner we obtain a chain $\bigwedge_i I(j_i, i)$ such that for $i \geqq n$, $|X_i| \geqq c$ on $I(j_i, i)$. Since $X_m \to 0$ a.s., $P(\bigcap_i I(j_i, i)) = 0$, and so $S(\bigwedge_i I(j_i, i)) = \infty$.

3. Characterization of purely atomic structures supporting undominated L^1-bounded martingales. For each partition set $I(j, n)$, let

$$T(I(j, n)) = \sup_{(C: C_n = I(j,n))} S(C) .$$

LEMMA. *Suppose for each chain* C *of* $((F_n)_{n=1,2,\dots}, P)$, $S(C) < \infty$ *but* $\sup_C S(C) = \infty$. *Then either*:

(1) *there exist an* n *and* $(j_i)_{i=1,2,\dots}$ *such that* $T(I(j_k, n)) > k$;

or

(2) *there exists a chain* C *such that for all* n, $T(C_n) = \infty$.

PROOF. Suppose the purely atomic structure has the required property and condition (1) is not satisfied. Then, for each n, there exists at least one and at most a finite number of partition sets $I(k, n)$ with $T(I(k, n)) = \infty$. Also, if $T(I(k, n)) = \infty$ there exists at least one partition set $I(j, n+1) \subset I(k, n)$ with $T(I(j, n+1)) = \infty$. Condition (2) follows.

THEOREM 2. *A purely atomic structure* $((F_n)_{n=1,2,\dots}, P)$ *supports undominated* L^1*-bounded martingales if and only if there exists* $c < \infty$ *such that* $S(C) \leqq c$ *for every chain* C.

PROOF. (1) Suppose S is unbounded. By Theorem 1, we may assume that S is finite-valued. The lemma reduces our considerations to two cases:

CASE 1. There exist n and $(j_i)_{i=1,2,\dots}$ with $T(I(j_k, n)) > k$.

Choose chains $(C^k)_{k=1,2,\dots}$ with $C_n{}^k = I(j_{k+n}, n)$ and $S(C^k) > k + n$. Now $P(\bigcap_n C_n{}^k) > 0$, so we may define

$$\begin{aligned} f &= (k^2 P(\textstyle\bigcap_n C_n{}^k))^{-1} && \text{on} \quad \textstyle\bigcap_n C_n{}^k, \quad k = 1, 2, \cdots \\ &= 0 && \text{elsewhere.} \end{aligned}$$

Since f is P-integrable, we may define the martingale $(M_n = E(f \mid F_n))_{n=1,2,\dots}$. Set $A(k, m) = C_m{}^k - C_{m+1}^k$. For $m \geqq n$,

$$1_{A(k,m)} M_m = (k^2 P(C_m{}^k))^{-1} ,$$

and so

$$\begin{aligned} E(\sup_n M_n) &\geqq E(\textstyle\sum_k \sum_m 1_{A(k,m)} \sup_n M_n) \\ &\geqq \textstyle\sum_k E(\sum_{m=n}^{\infty} 1_{A(k,m)} M_m) \\ &= \textstyle\sum_k \sum_{m=n}^{\infty} P(A(k, m))(k^2 P(C_m{}^k))^{-1} \\ &\geqq \textstyle\sum_k (1/k) \\ &= \infty , \end{aligned}$$

since

$$\begin{aligned} k + n < S(C^k) &= \textstyle\sum_{m=1}^{n-1} P(A(k, m))/P(C_m{}^k) + \sum_{m=n}^{\infty} P(A(k, m))/P(C_m{}^k) \\ &\leqq n + \textstyle\sum_{m=n} P(A(k, m))/P(C_m{}^k) . \end{aligned}$$

Thus, M is an undominated L^1-bounded martingale on $((F_n)_{n=1,2,\cdots}, P)$.

CASE 2. There exists a chain C such that for all n, $T(C_n) = \infty$.

Suppose $S(C) = c$. Since $T(C_1) = \infty$, there exists a chain C^1 with $C_1^1 = C_1$ and $S(C^1) > c + 1$. Let $m_1 = \inf_i (i: C_i^1 \neq C_i)$. Since $T(C_{m_1}) = \infty$, there exists a chain C^2 with $C_{m_1}^2 = C_{m_1}$ and $S(C^2) > c + 2$. Let $m_2 = \inf_i (i: C_i^2 \neq C_i)$. Proceeding in this way, we obtain an increasing sequence of integers $(m_j)_{j=1,2,\cdots}$ and a sequence of chains $(C^j)_{j=1,2,\cdots}$, such that $C_{m_{j-1}}^j = C_{m_{j-1}}$, $C_{m_j}^j \neq C_{m_j}$, and $S(C^j) > c + j$.

Set

$$\begin{aligned} f &= (j^2 P(\textstyle\bigcap_n C_n{}^j))^{-1} \qquad \text{on} \quad \bigcap_n C_n{}^j , \quad j = 1, 2, \cdots \\ &= 0 \qquad \text{elsewhere.} \end{aligned}$$

As in Case 1, $(M_n = E(f \mid F_n))_{n=1,2,\cdots}$ defines an undominated L^1-bounded martingale on $((F_n)_{n=1,2,\cdots}, P)$.

(2) Suppose for each chain C, $S(C) \leqq c$. By Theorem 1, we need only show: if f is a P-integrable function, $f \geqq 0$, then $(M_n = E(f \mid F_n))_{n=1,2,\cdots}$ is a dominated martingale.

By the proposition, there is an at most countable number of chains $(C^i)_{i=1,2,\cdots}$, with $\bigcup_i (\bigcap_n C_n{}^i) = \Omega$, and for all i $P(\bigcap_n C_n{}^i) > 0$. Thus $f = \sum_i f_i 1_{\cap_n C_n{}^i}$.

For each i, define the martingale M^i by

$$M_n{}^i = E(f_i 1_{\cap_m C_m{}^i} \mid F_n) .$$

Then on $A(i, n)$, $\sup_m M_m{}^i = (P(C_n{}^i))^{-1} f_i P(\bigcap_m C_m{}^i)$, so

$$\begin{aligned} E(\sup_m M_m{}^i) &= \textstyle\sum_n (P(A(i, n))/P(C_n{}^i)) f_i P(\bigcap_m C_m{}^i) \\ &= S(C^i) f_i P(\textstyle\bigcap_m C_m{}^i) \\ &\leqq c f_i P(\textstyle\bigcap_m C_m{}^i) . \end{aligned}$$

But $\sup_m M_m \leqq \sum_i \sup_m M_m{}^i$, so

$$\begin{aligned} E(\sup_m M_m) &\leqq \textstyle\sum_i E(\sup_m M_m{}^i) \\ &\leqq c \textstyle\sum_i f_i P(\bigcap_m C_m{}^i) \\ &= cE(f) , \end{aligned}$$

and we see that M is dominated.

4. Notes.

(1) If there exists a chain C with $S(C) = \infty$, it is possible to construct an undominated uniformly integrable martingale.

(2) Suppose $(X_n)_{n=1,2,\cdots}$ is a stochastic process with countable state space I. If $F_n = \sigma(X_m, m \leqq n)$ and P is the measure induced by X on $\bigvee_n F_n$, then $((F_n)_{n=1,2,\ldots}, P)$ is a purely atomic structure. We can represent chains by elements of I^∞: if $(i_1, i_2, \cdots, i_n, \cdots) = C$, then $C_n = (X_1 = i_1, \cdots, X_n = i_n)$. The summands appearing in the definition of S correspond to the conditional probability $P(X_{n+1} \neq i_{n+1} \mid X_1 = i_1, \cdots, X_n = i_n)$.

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