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THE FOUNDATIONS OF STOCHASTIC GEOMETRY[1]

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We show how to build models of random collections of geometrical objects: lines, circles, line segments, etc. This basic problem in stochastic geometry is solved using the theory of point processes on abstract spaces.

1. Introduction. This paper provides the foundations for the study of random collections of congruent geometrical objects. Stochastic geometry, in the sense of Kendall (1974), is the representation of such a collection as a point process on a suitable space. We deal with the problem of identifying the 'natural' structures on this space and so define the point processes using the theory of Ripley (1976a).

Davidson (1968, 1970) considered general stationary line processes. His approach (for oriented lines) was to take a particular parameterization and use the induced topology. For unoriented lines other parameterizations are equally natural but give rise to different topologies (Ripley (1976b)).

Throughout we assume that G is a locally compact second countable Hausdorff (LCD) topological group acting continuously on a Hausdorff topological space X (i.e., there is a continuous map $(g, x) \to gx$ from $G \times X$ to X satisfying $g(hx) = (gh)x$ and $ex = x$).

2. The representation. Suppose 0 is a nonempty subset of X; 0 is a typical geometrical object. Let $\mathscr{O}$ be the class of subsets of X congruent to 0 under G. For each $A \in \mathscr{O}$ let $f(A) = \{g : g0 = A\}$ and let $f(0) = H$. Then H is a subgroup of G. Let q be the (open) quotient map $G \to G/H$.

PROPOSITION 1. *$q \circ f$ is a bijection from $\mathscr{O}$ to G/H.*

PROOF. Elementary.

We will identify $\mathscr{O}$ with G/H. The simplest nontrivial example is the group G of rigid motions of the plane X with the usual topologies (Nachbin (1965)). Various choices of 0 show that $\mathscr{O}$ is not determined by H, and that not all subgroups can occur.

We give G/H the quotient topology, which is Hausdorff if and only if H is closed, in which case G/H is a LCD space. Let τ_1 be the topology induced on $\mathscr{O}$.

PROPOSITION 2. *If 0 is closed then H is closed.*

PROOF. $H = \{g : g0 = 0\} = \{g : g0 \subset 0, g^{-1}0 \subset 0\} = \bigcap_{x \in 0} \{g : gx \in 0, g^{-1}x \in 0\}$.

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The example of an open disc in the plane shows the converse to be false.

The topology τ_1 depends on the topology of the group (which is often rather artificial) and does not seem completely natural. If $\mathscr{O}$ can be represented by some Hausdorff Baire space on which G acts continuously (as in Davidson's example of line processes), Lemma 2 of Bourbaki ((1963), Appendix 1) shows that this space is isomorphic to $(\mathscr{O}, \tau_1)$. This is a useful method of identifying τ_1. We use $A \uparrow E$ ('A hits E') as shorthand for $A \cap E \neq \emptyset$.

PROPOSITION 3. *If U is open then $\{A: A \in \mathscr{O}, A \uparrow U\}$ is open in τ_1.*

PROOF. $f^{-1}(\{A: A \uparrow U\}) = \{g: g0 \uparrow U\} = \bigcup_{x \in 0}\{g: gx \in U\}$ which is open, and q is an open map.

3. Another approach. We suppose throughout this section that 0 is closed. Then $\mathscr{O}$ is contained in $\mathscr{F}$, the class of closed subsets of X, so $\mathscr{O}$ can be given any of the topologies proposed for $\mathscr{F}$. The compact-finite topology with subbasic open sets $\{F: F \uparrow U\}$ and $\{F: F \cap K = \emptyset\}$ for $U \in \mathscr{G}$, the open sets, and $K \in \mathscr{K}$, the compact sets, considered by Choquet (1953–1954), Mrowka (1958), Fell (1962) and Matheron (1972, 1975) seems to be the most suitable because of the following theorem. We define G to act on $\mathscr{F}$ by $gF = \{gx: x \in F\}$.

THEOREM 1. *G acts continuously on $\mathscr{F}$.*

PROOF. The map $(g, F) \to gF$ obviously satisfies all the conditions except continuity. Suppose $F_0 \in \mathscr{F}$, $g_0 \in G$ and $V = \{F: F \uparrow G_i, i = 1, \cdots, n, F \cap K = \emptyset\}$ is an open neighbourhood of $g_0 F_0$ with $G_i \in \mathscr{G}$ and $K \in \mathscr{K}$. For each i pick $x_i \in (g_0^{-1}G_i) \cap F_0$. We can find open neighbourhoods U_i of g_0 and V_i of x_i with $U_i V_i \subset G_i$. If $F \uparrow V_i$ and $g \in U_i$ then $gF \cap G_i \supset g(F \cap V_i) \neq \emptyset$, so $gF \cap V_i$. Let (W_n) be a decreasing base of compact symmetric neighbourhoods of the identity in G. Then $(W_n K)$ is a sequence of compact sets decreasing to K, so for some m $W_m K \cap g_0 F_0 = \emptyset$. Let $U_0 = W_m g_0$ and $K_1 = g_0^{-1} W_m K$. Let $U' = \bigcap_0^n U_i$ and $V' = \{F: F \uparrow V_i, i = 1, \cdots, n, F \cap K_1 = \emptyset\}$. Then $U' \times V'$ is an open neighbourhood of (g_0, F_0) and $U'(V') \subset V$. Thus $(g, F) \to gF$ is continuous at (g_0, F_0).

PROPOSITION 4. *If 0 is closed and K compact then $\{A: A \in \mathscr{O}, A \uparrow K\}$ is closed in τ_1.*

PROOF. By Theorem 1 $\{g: g0 \uparrow K\}$ is closed and q-saturated, so $\{A: A \uparrow K\}$ is open in τ_1.

Propositions 3 and 4 show that the injection map $(\mathscr{O}, \tau_1) \to \mathscr{F}$ is continuous so the topology τ_2 which this induces on $\mathscr{O}$ is coarser than τ_1. Now suppose X is a LCD space. Then $\mathscr{F}$ is a LCD space (Matheron (1975)) and τ_1 and τ_2 have the same Borel σ-field $\mathscr{A}$ (Kuratowski (1968) Section 39 or Dellacherie and Meyer (1975) III 21). Other topologies on $\mathscr{F}$, notably Michael's finite topology (Michael (1951)), have the same Borel σ-field on $\mathscr{F}$ (Ripley (1976a)) and hence on $\mathscr{O}$. Thus $\mathscr{A}$ is the natural σ-field to choose on $\mathscr{O}$. The topologies τ_1 and τ_2 may differ (but see Proposition 5(ii)).

EXAMPLE. Let G be the group of rational translations on R with the discrete topology. Then G is a LCD topological group acting continuously on R. Let $0 = \{0\}$. Then H is the subgroup of the identity, so τ_1 is the discrete topology, whereas τ_2 is induced from the rationals by $x \to \{x\}$.

4. Geometrical point processes. We require our random collections of objects to be locally finite in the sense that only a finite number hit each member of a class $\mathscr{D}$ of 'bounded' sets in X. We may assume that $\mathscr{D}$ is an ideal in $\mathscr{P}(X)$ (the class of subsets of X) covering X, and we will assume $\mathscr{D}' = \mathscr{D} \cap \mathscr{G}$ is cofinal in D under inclusion. Let $\mathscr{B}$ be the ideal in $\mathscr{P}(\mathscr{O})$ generated by $\{\{A : A \uparrow E\} : E \in \mathscr{D}\}$. Then $\{\{A : A \uparrow F\} : F \in \mathscr{D}'\}$ is cofinal in $\mathscr{B}$, so by Proposition 3 $\mathscr{C} = \mathscr{A} \cap \mathscr{B}$ is cofinal in $\mathscr{B}$. Thus if H is closed then $(\mathscr{O}, \mathscr{A}, \mathscr{B})$ is a *bounded space* (Ripley (1976a)). If 0 is closed then Proposition 4 allows us to assume only that $\mathscr{D} \cap \mathscr{K}_o$ is cofinal in $\mathscr{D}$.

PROPOSITION 5. *Suppose $\mathscr{D}$ is the class of relatively compact subsets. Then*:

(i) *$\mathscr{B}$ contains all the compact sets of $(\mathscr{O}, \tau_1)$.*

(ii) *Suppose X is a topological homogeneous space G/J, J is compact, and $0 = HK$ where K is compact in X. Then $\mathscr{B}$ is the class of relatively compact sets in τ_1, $\mathscr{O}$ is closed in $\mathscr{F}_0 = \mathscr{F} \setminus \{\emptyset\}$ and $\tau_1 = \tau_2$.*

PROOF. (i) Suppose K is compact in $\mathscr{O}$. There is a compact set $E \subset G$ with $q(E) = K$ (Bourbaki (1966) I Section 10.4, Proposition 10). Fix $x \in 0$, and let $F = Ex$ which is compact. Then $K = q(\{g : g \in E\}) \subset q(\{g : (gHx) \uparrow F\}) \subset \{A : A \in \mathscr{O}, A \uparrow F\} \in \mathscr{B}$. (ii) Suppose E is a compact set in X. Let $L = \{g : gK \uparrow E\}$ and let p be the quotient map $G \to G/J$. Then p is proper, so $p^{-1}(K)$ and $p^{-1}(E)$ are compact and $L = \{hg^{-1} : p(g) \in K, p(h) \in E\}$ is compact. Thus $\{A : A \in \mathscr{O}, A \uparrow E\} = q(\{g : (gHK) \uparrow E\}) = q(L)$ is compact. Now suppose $g_n 0 \to F \in \mathscr{F}_0$. Then $(g_n 0)$ hits a compact set E infinitely often and so for a subsequence we may assume that $g_n \in L$. Then (g_n) has a cluster point g, and $(g_n 0)$ has a cluster point $g0$ by Theorem 1. Thus $g_n 0 \to g0 \in \mathscr{O}$. Now $\mathscr{O}$ is a Baire subspace (since X and $\mathscr{F}$ are LCD spaces) and so by Lemma 2 of Bourbaki ((1963), Appendix 1) $(\mathscr{O}, \tau_2)$ is a topological homogeneous space, hence $\tau_1 = \tau_2$.

The conditions of (ii) are usually satisfied in Euclidean geometry. The example of half-infinite rays in the plane shows that the condition on 0 is necessary (here $\tau_1 = \tau_2$ but the other conclusions are false). Part (i) shows that the section on more general random sets in Ripley (1976a) is applicable.

We assume from now on that $(\mathscr{O}, \mathscr{A}, \mathscr{B})$ is a bounded space (i.e., $\mathscr{C}$ is cofinal in $\mathscr{B}$ and $\mathscr{A}$ contains all singletons). Let N be the class of completely additive functions $n : \mathscr{C} \to Z_+$, the nonnegative integers, and let $\mathscr{N}$ be the smallest σ-field on N making the evaluation maps e_A measurable for all $A \in \mathscr{C}$. Then (Ripley (1976a)) a point process on $(\mathscr{O}, \mathscr{A}, \mathscr{B})$ is a measurable map from a probability space to $(N, \mathscr{N})$. We define G to act on N by $gn(A) = n(g^{-1}A)$.

PROPOSITION 6. *Suppose $\mathscr{O}$ has a countable cover from $\mathscr{B}$. Then the map $(g, n) \to gn$ is measurable on $G \times N$, G having the Borel σ-field.*

PROOF. Suppose E is a Borel subset of G. Then $(g, n) \to \int 1_{E\times A}(g, u)\, dn(u) = 1_E(g)n(A)$ is measurable for all A in the σ-field $\mathscr{A}$ which is generated by $\mathscr{C}$. By the usual approximation we have $(g, n) \to \int f(g, u)\, dn(u)$ measurable for all bounded measurable functions f on $X \times G$. We may take $f(g, u) = 1_A(gu)$ for $A \in \mathscr{C}$, so $(g, n) \to gn(A)$ is measurable.

Proposition 6 generalizes a result of Mecke ((1967), Anhang B). Ambartzumian (1972) uses the space $(\mathscr{O}, \mathscr{A}, \mathscr{B})$ with $G = X = R^n$ and $0 = \{0\}$. He considers a point process to be a map Z from a probability space to N such that $(g, \omega) \to gZ(\omega)(A)$ is measurable for all open spheres A. By Ripley ((1976a), Theorem 4(i)) $(g, \omega) \to gZ(\omega)$ is measurable, so this is a point process in our sense and conversely by Proposition 6. Thus every probability on $\mathscr{N}$ is induced by a point process in Ambartzumian's sense, answering his final question (Ambartzumian (1972)), and his method for Palm probabilities is available.

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