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AN OPTIMAL STOPPING PROBLEM FOR SUMS OF DICHOTOMOUS RANDOM VARIABLES[1]

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Let Y_t be a stochastic process starting at y which changes by i.i.d. dichotomous increments X_t with mean 0 and variance 1. The cost of proceeding one step is one and the payoff is zero unless n steps are taken and the final value $\hat{Y}$ of Y_t is negative in which case the payoff is $\hat{Y}^2$. The optimal procedure consists of stopping as soon as $Y_t \geq \tilde{y}_m$ where m is the number of steps left to be taken. The limit of $\tilde{y}_m$ as $m \to \infty$ is desired as a function of $p = P(X_t < 0)$. This limit $\tilde{y}$ is evaluated for p rational and proved to be continuous in p. One can use $\tilde{y}$ to relate the solution of optimal stopping problems involving a Wiener process to those involving certain discrete-time discrete-process stopping problems. Thus $\tilde{y}$ is useful in calculating simple numerical approximations to solutions of various stopping problems.

1. Introduction. The limiting behavior of the solution of the following problem can be used to relate the solution of a class of continuous time stopping problems involving a Wiener process to certain discrete time, discrete process, stopping problems. This relationship can be used to estimate the error of a relatively simple computational approximation to the solutions of stopping problems. In the last section we elaborate on this paragraph and relate it to a problem in sequential analysis.

A stochastic process $\{Y_t, t = -n, -n+1, \cdots, 0\}$ starting at $Y_{-n} = y$ is observed for at most n successive times. At each time Y_t changes by either a or b so that the change has mean 0 and variance 1. To observe a new value of Y_t involves a cost of one unit. The observer receives a reward only if he has observed all n steps and the final value Y_0 is negative in which case he receives the square of the final value.

Clearly it pays to continue observing if Y_t is highly negative and to stop if Y_t is highly positive. In Section 3 we shall prove that an optimal procedure consists of continuing as long as Y_t stays below $\tilde{y}_{-t}$ where $-t$ is the remaining number of steps to be observed and that $\tilde{y}_{-t}$ converges to $\tilde{y}$ as $-t \to \infty$. In Section 4 an expression for $\tilde{y}$ is obtained in terms of a contour integral for the case of rational a/b and in Section 5 it is shown that $\tilde{y}$ is continuous in a/b. In Section 6 some related results are presented.

2. Notation. The process $\{Y_t, t = -n, -n+1, \cdots\}$ starts at $Y_{-n} = y$ and

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can be expressed as

$$Y_{t+1} = Y_t + X_t$$

where $\{X_t, t = 0, \pm 1, \pm 2, \cdots\}$ are independently and identically distributed according to

$$P\{X_t = a\} = p$$
$$P\{X_t = b\} = 1 - p$$

where $0 < p < 1$ and a and b are such that $EX_t = 0$ and $EX_t^2 = 1$ which is equivalent to

$$pa + (1 - p)b = 0$$
$$pa^2 + (1 - p)b^2 = 1 .$$

We shall take

$$a = -((1 - p)/p)^{\frac{1}{2}} \qquad \text{and} \qquad b = (p/(1 - p))^{\frac{1}{2}} .$$

Note that $a/b = -(1 - p)/p$ is rational if and only if p is.

The stopping problem consists of finding a stopping rule which stops after N steps in order to maximize the expected gain designated by $v(y, n)$. Here N is possibly random. Then the gain consists of the payoff Y_0^2 if $Y_0 \leqq 0$ and $N = n$ steps are observed minus a cost of N if N steps are observed. In the event that $N < n$, there is no payoff. We shall designate the optimal expected gain by $\tilde{v}((y, n)$. Associated with an arbitrary stopping rule we are concerned with N, and $T = -n + N$, the value of the subscript upon termination, and $\hat{Y} = Y_T$, the value of Y_t upon termination. The event

$$F_n = \{N = n, \hat{Y} < 0\}$$

and its complement E_n are of importance. Note that N, T, $\hat{Y}$, and the probability measure depend implicitly upon the initial point $(y, -n)$ as well as the stopping rule.

3. Monotonicity and continuity properties and bounds. In this section we show that an optimal strategy consists of stopping when $Y_t \geqq \tilde{y}_{-t}$ where $\tilde{y}_n$ is negative and decreases monotonically as $n \to \infty$ to some number $\tilde{y} \geqq -b/2$. Also $\tilde{v}(y, n)$ converges monotonically to $\tilde{v}(y)$ as $n \to \infty$ where $\tilde{v}(y)$ is continuous and satisfies a simple functional equatation.

Comparing the expected gains of taking an observation and stopping, and applying backward induction it is easy to see that

$$\tilde{v}(y, n) = \max\{0, p\tilde{v}(y + a, n - 1) + (1 - p)\tilde{v}(y + b, n - 1) - 1\} , \quad n > 0 \tag{3.1}$$

with $\tilde{v}(y, 0) = y^2$ for $y \leqq 0$ and $\tilde{v}(y, 0) = 0$ for $y \geqq 0$. It is also apparent that an optimal policy consists of stopping after observing $Y_t = y$, if $\tilde{v}(y, -t) = 0$. This describes the optimal policy in terms of a *stopping set* of points (y, t) at which it pays to stop or the complementary *continuation set* on which $\tilde{v}(y, -t) > 0$. This set does not depend on the initial value n specified in the problem and thus our solution is simultaneously applicable for all initial points $(y, -n)$, $n > 0$.

LEMMA 3.1. *$\tilde{v}(y, n)$ is monotonic decreasing in n.*

PROOF. We observe that

$$p(y + a)^2 + (1 - p)(y + b)^2 - 1 = y^2 .$$

Hence, if $y + b \leqq 0$,

$$p\tilde{v}(y + a, 0) + (1 - p)\tilde{v}(y + b, 0) - 1 = y^2 ,$$

but if $y + b > 0 \geqq y + a$, the left-hand side of the above equality becomes $p(y + a)^2 - 1 < y^2$. It easily follows from (3.1) that

$$\begin{aligned} \tilde{v}(y, 1) &= y^2 && \text{for } \; y \leqq -b \\ 0 < \tilde{v}(y, 1) &< y^2 && \text{for } \; -b < y < -a - p^{-\frac{1}{2}} < 0 \\ \tilde{v}(y, 1) &= 0 && \text{for } \; y \geqq -a - p^{-\frac{1}{2}} , \end{aligned}$$

and hence $\tilde{v}(y, 1) \leqq \tilde{v}(y, 0)$.

For $n > 0$, $\tilde{v}(y, n + 1)$ can be considered the optimal expected gain of an n step stopping problem with terminal payoff function $\tilde{v}(y, 1) \leqq \tilde{v}(y, 0)$. This problem is less favorable than our initial problem and hence $\tilde{v}(y, n + 1) \leqq \tilde{v}(y, n)$. □

This proof incidentally demonstrates that for $t = -1$, the stopping points are $\{(y, -1) : y \geqq \tilde{y}_1 = -a - p^{-\frac{1}{2}}\}$ where $\tilde{y}_1 < 0$. Hence applying Lemma 3.1, $\tilde{v}(y, n) = 0$ for $y \geqq \tilde{y}_1$.

LEMMA 3.2. *$\tilde{v}(y, n)$ is monotonic decreasing in y.*

PROOF. For a given initial point $(y, -n)$, the optimal procedure can be described in terms of the $X_{-n}, X_{-n+1}, \cdots$, which lead to stopping. Apply this same rule for the initial point $(y - \varepsilon, -n)$ with $\varepsilon > 0$. (Here this rule is possibly suboptimal.) Then

$$\tilde{v}(y - \varepsilon, n) - \tilde{v}(y, n) \geqq \int_{F_n} [(\hat{Y} - \varepsilon)^2 - \hat{Y}^2] \, dP \geqq 0$$

where we recall that $F_n = \{N = n, \hat{Y} < 0\}$ and P is the probability distribution induced by the optimal procedure with initial point $(y, -n)$. □

Theorem 3.1 follows immediately from Lemmas 3.1 and 3.2.

THEOREM 3.1. *The optimal stopping set can be described as*

$$\{(y, -n) : y \geqq \tilde{y}_n, n \geqq 1\}$$

where $\{\tilde{y}_n\}$ is a monotonic decreasing negative sequence. Furthermore $\tilde{v}(y, n) = 0$ for $y \geqq \tilde{y}_n$.

We may now define

$$\tilde{y} = \lim_{n\to\infty} \tilde{y}_n \tag{3.2}$$

and

$$\tilde{v}(y) = \lim_{n\to\infty} \tilde{v}(y, n) . \tag{3.3}$$

LEMMA 3.3. *$y^2 \geqq \tilde{v}(y, n) \geqq y^2 - b^2/4$ for $y \leqq 0$, and $\tilde{y}_n \geqq -b/2$.*

PROOF. Since $\{W_i = Y^2_{-n+i} - i, i = 0, 1, 2, \cdots\}$ is a martingale, the optimal stopping theorem ([5], page 300) yields

$$y^2 = E(\hat{Y}^2 - N) \tag{3.4}$$

for any stopping procedure. Then

$$v(y, n) = E(\hat{Y}^2 - N) - \int_{E_n} \hat{Y}^2 \, dP \tag{3.5}$$

where $E_n = \{N < n \text{ or } N = n, \hat{Y} \geqq 0\}$.

Let $y < 0$ and consider the special possibly suboptimal procedure which consists of continuing as long as $Y_t < c < 0$. Then

$$v(y, n) \geqq y^2 - \max\,[c^2, (c + b)^2] .$$

Taking $c = -b/2$ we have $\tilde{v}(y, n) \geqq v(y, n) \geqq y^2 - b^2/4$. Thus $\tilde{v}(y, n) > 0$ for $y < -b/2$ and $\tilde{y}_n \geqq -b/2$. □

Some useful continuity properties are included in

THEOREM 3.2. *$\tilde{v}(y, n)$ and $\tilde{v}(y)$ are continuous and for $y > 0$, their derivative numbers are bounded between $2(y - b)$ and $\min\,[0, 2(y - \tilde{y})]$.*

PROOF. First we note that $\{Y_t, -n \leqq t \leqq 0\}$ is a martingale and hence the optimal stopping theorem yields

$$y = E\hat{Y} = \int_{E_n} \hat{Y} \, dP + \int_{F_n} \hat{Y} \, dP \tag{3.6}$$

for any stopping procedure. Let us restrict ourselves to procedures which stop if $Y_t > 0$ and do not stop before $t = 0$ if $Y_t < \tilde{y}_n$. These include all optimal procedures. Then if $y < 0$,

$$y - b \leqq \int_{F_n} \hat{Y} \, dP \leqq y - \tilde{y}_n .$$

Applying the argument of Lemma 3.2, we have, for $\varepsilon > 0$, $\tilde{v}(y - \varepsilon, n) - \tilde{v}(y, n) \geqq -2\varepsilon \int_{F_n} \hat{Y} \, dP + \varepsilon^2 \int_{F_n} dP \geqq -2\varepsilon(y - \tilde{y}_n)$. The same argument yields

$$\begin{aligned} \tilde{v}(y + \varepsilon, n) - \tilde{v}(y, n) &\geqq \int_{F_n} [(\hat{Y} + \varepsilon)^2 - \hat{Y}^2] \, dP - \int_{F_n\{\hat{Y}+\varepsilon>0\}} (\hat{Y} + \varepsilon)^2 \, dP \\ &\geqq 2\varepsilon(y - b) - \varepsilon^2 . \end{aligned}$$

These bounds and the previous results imply our Theorem 3.2. □

Let $n \to \infty$ in Equation 3.1. With the help of the previous results it is easy to see that

THEOREM 3.3.

$$0 \geqq \tilde{y} \geqq -b/2$$
$$y^2 \geqq \tilde{v}(y) \geqq y^2 - b^2/4\,, \qquad for \quad y \leqq 0$$

and

$$\tilde{v}(y) = p\tilde{v}(y+a) + (1-p)\tilde{v}(y+b) - 1 \qquad for \quad y \leqq \tilde{y} \tag{3.7}$$
$$\tilde{v}(y) = 0 \qquad for \quad y \geqq \tilde{y}\,.$$

Note that the two equations of (3.7) imply $p\tilde{v}(\tilde{y}+a) + (1-p)\tilde{v}(\tilde{y}+b) - 1 = 0$.

4. The case of rational p. Suppose $p = r/m$ where r and m are relatively prime integers with $0 < r < m$. Let $s = m - r > 0$. Then $a = -sh$ and $b = rh$ where $h = (rs)^{-\frac{1}{2}}$. In this case the possible values of X_t are commensurate, i.e., integral multiples of h, and the possible values of Y_t are restricted to a lattice of values $y + ih$, $i = 0, \pm 1, \pm 2, \cdots$. For convenience we represent this set as $\{c + ih : i = 0, \pm 1, \pm 2, \cdots\}$ where c is selected so that $\tilde{y} - h < c \leqq \tilde{y}$.

For each value of c, equation (3.7) becomes a classical mth order linear difference equation and $\tilde{v}(y)$ is a solution of

$$v(y) = \frac{r}{m} v(y - sh) + \frac{s}{m} v(y + rh) - 1 \qquad y \leqq \tilde{y} \tag{4.1}$$

$$v(y) = 0 \qquad y \geqq \tilde{y} \tag{4.2}$$

A particular solution of (4.1) is given by $v(y) = y^2$. The general solution of (4.1) can be expressed in terms of the roots x_i of the algebraic equation

$$mx^s = r + sx^m \tag{4.3}$$

which is easily seen to have a double root at $x = 1$ and no other double root. Then the general solution of (4.1) is

$$v(y) = y^2 + d_0 + d_0' y + \textstyle\sum_{i=1}^{m-2} d_i x_i^{(y-c)/h}\,. \tag{4.4}$$

Then the bound on $\tilde{v}(y)$ in Theorem 3.3 implies that for $\tilde{v}(y)$, $d_0' = 0$ and $d_i = 0$ for all i for which $|x_i| < 1$. The fact that $\tilde{v}(y) = 0$ for $y \geqq \tilde{y}$ implies

$$\tilde{v}(c + ih) = 0 \qquad \text{for} \quad i = 1, 2, \cdots, r \tag{4.5}$$

which imposes r condition on the remaining d_i. (The fact that $\tilde{v}(y) = 0$ for $y > c + rh$ is not useful since equation (4.1) applies only for $y \leqq \tilde{y}$ or equivalently $y + rh \leqq \tilde{y} + rh < c + (r+1)h$.) Thus to determine the coefficients d_i it is desirable to have $r - 1$ roots x_i for which $|x_i| > 1$. In the following lemma we establish this property and use it to derive an expression for $\tilde{y}$ in terms of a contour integral.

LEMMA 4.1 *The equation* $mx^s = r + sx^m$ *has one double root at* $x = 1$, $r - 1$ *distinct roots for which* $|x_i| > 1$, *and* $s - 1$ *distinct roots for which* $|x_i| < 1$.

PROOF. 3ince the x_i^{-1} satisfy the same equation with r and s interchanged, it suffices to show that there are $s-1$ roots inside the unit circle. Let

$$A(x) = x^{-s}B(x) = \frac{r}{m}x^{-s} + \frac{s}{m}x^{r} - 1\,. \tag{4.6}$$

Consider the path C which follows the unit circle counter clockwise in the complex plane except for a short vertical line near $x = 1$ from $(1-\varepsilon) - i\eta$ to $(1-\varepsilon) + i\eta$. The number of times $A(x)$ circles the origin as x goes around C is s less than the number of x_i such that $|x_i| < 1$. Our proof will be complete if we show that this is -1, i.e., if there is one clockwise circuit.

Near $x = 1$, $A(x) \approx rs(x-1)^2/2$ and along the line segment, $A(x)$ moves from a point in the second quadrant clockwise about 0 to a point in the third quadrant. Along the circular part of C, $A(x)$ is a weighted average of points on the unit circle -1, and is confined to the half plane where the real part is negative. The lemma follows. □

Let $x_0 = 1, x_1, x_2, \cdots, x_{r-1}$ be the $r-1$ distinct roots of (4.3) outside the unit circle and $x_r, x_{r+1}, \cdots, x_{m-2}$ be the distinct roots inside the unit circle. Let

$$D = \begin{bmatrix} 1 & x_1 & \cdots & x_{r-1} \\ 1 & x_1^2 & \cdots & x_{r-1}^2 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_1^r & \cdots & x_{r-1}^r \end{bmatrix} \tag{4.7}$$

which is nonsingular, and let

$$\mathbf{d}' = (d_0, d_1, \cdots, d_{r-1})$$

and let $\mathbf{i}^j$ be the column vector whose ith element is i^j, $1 \leqq i \leqq r$, with $\mathbf{1} = \mathbf{i}^0$. Then condition (4.5) translates to

$$D\mathbf{d} = -(c^2\mathbf{1} + 2ch\mathbf{i} + h^2\mathbf{i}^2)\,. \tag{4.8}$$

Hence,

$$\tilde{v}(c) = \mathbf{1}'\mathbf{d} + c^2 = c^2(1 - \mathbf{1}'D^{-1}\mathbf{1}) - 2ch\mathbf{1}'D^{-1}\mathbf{i} - h^2\mathbf{1}'D^{-1}\mathbf{i}^2\,. \tag{4.9}$$

The following theorem "evaluates" $\tilde{v}(c)$ and $\tilde{y}$.

THEOREM 4.1. *For some positive constant k (independent of c)*

$$\tilde{v}(c) = -2kh\{c + h[\tfrac{1}{2} + \textstyle\sum_{|x_i|>1}(1-x_i)^{-1}]\} \qquad \text{for} \quad \tilde{y} - h < c \leqq \tilde{y} \tag{4.10}$$

and

$$\tilde{y} = -[\tfrac{1}{2} + \textstyle\sum_{|x_i|>1}(1-x_i)^{-1}]h\,. \tag{4.11}$$

PROOF. Let $E(x) = \mathbf{1}'D^{-1}\mathbf{x} - 1$ where $\mathbf{x}' = (x, x^2, \cdots, x^r)$. Since $D^{-1}\mathbf{x}_{i-1}$ is the ith column of the identity matrix $E(x_{i-1}) = 0$ for $i = 1, 2, \cdots, r$. Since $E(0) \neq 0$, E is a nondegenerate rth degree polynomial in x and it follows that

$$E(x) = k_1 \textstyle\prod_{i=0}^{r-1}(x - x_i)$$

where $k_1 \neq 0$. Moreover,

$$E(1) = \mathbf{1}'D^{-1}\mathbf{1} - 1 = 0\,,$$
$$E'(1) = \mathbf{1}'D^{-1}\mathbf{i} = k_1 \prod_{i=1}^{r-1}(1 - x_i) \neq 0\,,$$
$$E''(1) = \mathbf{1}'D^{-1}(\mathbf{i}^2 - \mathbf{i}) = 2E'(1) \sum_{i=1}^{r-1}(1 - x_i)^{-1}$$

and applying (4.9)

$$\tilde{v}(c) = -2chE'(1) - h^2[E''(1) + E'(1)]\,.$$

Equation (4.10) follows with $k = E'(1)$. Since $\tilde{v}(c) \geqq 0$, and is decreasing in c for $c \leqq \tilde{y}$, it follows that $E'(1)>0$. Equation (4.11) is an immediate consequence of the fact that $\tilde{v}(\tilde{y}) = 0$. □

To obtain an expression for $\tilde{y}$ in terms of a contour integral, we establish two lemmas. The second expresses $\sum_{|x_i|<1}(1 - x_i)^{-1}$ in terms of a contour integral and the first obtains a simple expression for $\sum_{x_i \neq 1}(1 - x_i)^{-1}$.

LEMMA 4.2. *The expansion of* $B'(x)/B(x)$ *about* $x = 1$ *is of the form*

$$B'(x)/B(x) = 2(x - 1)^{-1} + \frac{r + 2s - 3}{3} + \frac{(x - 1)}{18}[s^2 + rs + r^2 - 12s - 6r + 15] + \cdots$$

and

$$\sum_{x_i \neq 1}(1 - x_i)^{-1} = \frac{r + 2s - 3}{3}\,.$$

PROOF. Since $B(x) = (s/m)\prod_{x_i \neq 1}(x - x_i)\cdot(x - 1)^2$, it is easy to see that $B'(x)/B(x) = d[\log B(x)]/dx = 2(x - 1)^{-1} + \sum_{x_i \neq 1}(x - x_i)^{-1}$, and the constant in the expansion of $B'(x)/B(x)$ about $x = 1$ is $\sum_{x_i \neq 1}(1 - x_i)^{-1}$. Expanding $B'(x)$ and $B(x)$ about $x = 1$ and dividing yields the expansion above. □

LEMMA 4.3.

$$\sum_{|x_i|<1}(1 - x_i)^{-1} = \frac{1}{2\pi}\int_0^{2\pi}\left[\frac{ms(e^{ir\theta} - 1)e^{i(s-1)\theta}}{r + se^{im\theta} - me^{is\theta}} - \frac{2}{e^{i\theta} - 1} - \frac{r + 2s - 3}{3}\right]\frac{e^{i\theta}\,d\theta}{1 - e^{i\theta}}\,.$$

PROOF. By virtue of the expansion of Lemma 4.2,

$$\left[\frac{B'(x)}{B(x)} - 2(x - 1)^{-1} - \frac{r + 2s - 3}{3}\right]\frac{1}{(1 - x)}$$

which has poles of magnitude $(1 - x_i)^{-1}$ at each $x_i \neq 1$, is regular at $x = 1$. Our result follows by taking the contour integral about the unit circle. □

Combining Theorem 4.1 with the last two lemmas we can write

$$\tilde{y} = \left[S + \frac{1}{2} - \frac{r + 2s}{3}\right]h \tag{4.12}$$

where S is the right-hand side of the statement of Lemma 4.3.

It should be remarked that the cases where $p = 1/m$ and $p = (m - 1)/m$ are simple to handle without using the contour integral. When $p = 1/m$, $r = 1$ and there are no roots of (4.3) outside the unit circle. Then $\tilde{y} = -h/2 = -b/2 = -(m-1)^{-\frac{1}{2}}/2$. When $p = (m-1)/m$, $r = m-1$, $s = 1$ and except for the double root at $x = 1$, all the roots are outside the unit circle. Then $\sum_{|x_i|>1} (1 - x_i)^{-1} = (r + 2s - 3)/3 = (m - 2)/3$ and $\tilde{y} = -h(2m - 1)/6$, where $h = (m - 1)^{-\frac{1}{2}} = -a$. Since $b = (m - 1)h$, $\tilde{y} \approx -b/3$ in that case.

Incidentally, in the case $p = 1/m$, application of (4.4) and (4.8) yields

$$\tilde{v}(y) = y^2 - (c + b)^2 \qquad \text{for} \quad y \leqq \tilde{y} = -b/2$$

where c is the remainder, $-3b/2 < c \leqq -b/2$, after a suitable multiple of b is added to y. The term $(c + b)^2$ can also be expressed as

$$\inf_j (y + jb)^2 ,$$

i.e., the square of the distance from y to the closest multiple of b, and varies from 0 to $b^2/4$.

5. Continuity in p. In Section 4, the functional equation (3.7) reduced to a difference equation which was used to derive expressions for $\tilde{y}$ and $\tilde{v}(y)$ for the case where p is rational. In the irrational case that technique is not directly applicable. However, we shall show that $\tilde{y}$ and $\tilde{v}(y)$ are continuous in p and thus their values in the irrational case can be approximated by replacing p by a nearby rational.

We shall find it expedient to introduce another stopping problem which is equivalent to the original problem. That consists of minimizing

$$u(y, n) = \int_{E_n} \hat{Y}^2 \, dP$$

and has minimizing value $\tilde{u}(y, n)$. To compare the cases for two values p and p_0 we shall use the subscript 0 to represent the case of p_0. Thus $\tilde{u}_0(y, n)$, $\tilde{y}_{0n}$, $\tilde{y}_0$, $\hat{Y}_0$, etc. all correspond to the case of p_0.

Lemma 5.1 implies the equivalence of the original problem and the minimization problem and the fact that as $n \to \infty$, $\tilde{u}(y, n) \to \tilde{u}(y) = y^2 - \tilde{v}(y)$.

LEMMA 5.1. *For any stopping rule*, $v(y, n) = y^2 - u(y, n)$.

PROOF. This result is an immediate consequence of equations (3.4) and (3.5). □

It should be remarked that there are some conveniences to be gained from the equivalence of these two problems. The original problem made certain monotonicity properties easy to obtain. The new problem permits us to deal with $\hat{Y}$ on E_n where $\hat{Y}$ is bounded between $-b/2$ and b.

Our overall plan consists of using a bound on $P(E_n)$ to bound $\tilde{v}(\tilde{y}) - \tilde{v}(\tilde{y}, n)$. This with the bound on the derivative numbers of $\tilde{v}(y, n)$ (of Theorem 3.2) leads to a bound on $\tilde{y} - \tilde{y}_n$. Finally a bound on $\tilde{v}(y, n) - \tilde{v}_0(y, n)$ leads to a bound on $\tilde{y}_n - \tilde{y}_{0n}$ which combines with two applications of the previous result to bound $\tilde{y} - \tilde{y}_0$.

LEMMA 5.2. *If y is in a bounded interval and p is in a closed subinterval of* $(0, 1)$, *there is a constant K such that $P(F_n) \leqq Kn^{-\frac{1}{2}}$ when the optimal procedure is applied.*

PROOF. Let $y < 0$ since the case $y \geqq 0$ is trivial. Note that $P(F_n) \leqq P\{(\max_{1 \leqq m < n} \sum_{i=1}^m X_i) < -y\} = P\{N \geqq n\}$ where N is the first integer m for which $S_m = \sum_{i=1}^m X_i \geqq -y$. $M_X(\lambda) = E[\exp(\lambda X_i)]$ be the moment generating function (m.g.f.) of X_i and let $\lambda > 0$. The sequence $\{\exp[\lambda S_m - m \log M_X(\lambda)], m = 1, 2, \cdots\}$ is a martingale with mean 1. Because $EX_i = 0$, $u = -\log M_X(\lambda) < 0$. By optional stopping,

$$1 = E(e^{\lambda S_N + Nu}) \leqq e^{\lambda(-y+b)} M_N(u)$$

where M_N is the m.g.f. of N. But

$$M_N(u) \leqq 1 - P\{N \geqq n\} + e^{nu} P\{N \geqq n\}$$

and also

$$P\{N \geqq n\} \leqq \frac{1 - e^{-\lambda(-y+b)}}{1 - e^{nu}}\,. \tag{5.1}$$

Now let $\lambda = n^{-\frac{1}{2}}$. Then $M_X(\lambda) = 1 + \frac{1}{2}\lambda^2 + 0(\lambda^3)$ and it is easy to see that the right-hand side of (5.1) is asymptotically equivalent to $(-y + b)(1 - e^{-\frac{1}{2}})^{-1} n^{-\frac{1}{2}}$. A more detailed calculation yields our desired result. □

The above lemma is basically a result on the distribution of the maximum of sums of i.i.d. random variables and is a consequence of a general result of Spitzer [7]. The bound obtained using that result involves the solution of a Wiener–Hopf equation.

LEMMA 5.3. *If y is in a bounded interval and p is in a closed subinterval of* $(0, 1)$ *there is a constant K such that*

$$\tilde{v}(y, n) \geqq \tilde{v}(y) \geqq \tilde{v}(y, n) - b^2 K n^{-\frac{1}{2}}$$

and

$$\tilde{y}_n - \tilde{y} \leqq b K^{\frac{1}{2}} n^{-\frac{1}{4}}\,.$$

PROOF. For the stopping problem with initial point $(y, -n_1)$, with $n_1 > n$, apply the optimal procedure for initial point $(y, -n)$. More precisely stop if $Y_t \geqq \tilde{y}_{n-n_1-t}$ for $-n_1 \leqq t < n - n_1$. For $t \geqq n - n_1$ stop if $Y_t \geqq -b/2$ or when $t = 0$. This suboptimal procedure leads to $u(y, n_1)$ where

$$\tilde{u}(y, n_1) \leqq u(y, n_1) \leqq \tilde{u}(y, n) + b^2 P(F_n)\,.$$

Let $n_1 \to \infty$ and apply Lemmas 5.1 and 5.2 and the first part of the result follows. Now let $y = \tilde{y} \geqq -b/2$ and hence

$$\tilde{v}(\tilde{y}, n) - \tilde{v}(\tilde{y}_n, n) = \tilde{v}(\tilde{y}, n) \leqq b^2 P(F_n)\,.$$

But, the proof of Theorem 3.2 implies that $\tilde{v}(\tilde{y}, n) - \tilde{v}(\tilde{y}_n, n) \geqq (\tilde{y}_n - \tilde{y})^2$. □

LEMMA 5.4. *If* $y \leqq 0$

$$\tilde{v}_0(y, n) \geqq \tilde{v}(y, n) - (\varepsilon_{1n}b^2 + 2b\varepsilon_{2n} + \varepsilon_{2n}^2)$$

where $\varepsilon_{1n} = n|p - p_0|$ *and* $\varepsilon_{2n} = n \max(|a - a_0|, |b - b_0|)$.

PROOF. We shall apply the optimal stopping rule for (y, n, p) to (y, n, p_0) where that rule will be interpreted in terms of the "history" of positive and negative values of X_t which lead to stopping. It is convenient to think of X_t and X_{0t} as being formed by generating a random variable Z_t uniform on (0, 1) and letting $X_t = a$ if $Z_t \leqq p$ and b otherwise. The same Z_t can serve to generate X_t and also X_{0t} corresponding to p_0. Thus we see that in n steps, the "histories" of positive and negative steps for X_t and X_{0t} will differ only if some Z_t is between p and p_0, an event with probability no larger than ε_{1n}. To help define our stopping rule, let $X_t^* = a$ when $X_{0t} = a_0$ (i.e., when $Z_t \leqq p_0$) and let $X_t^* = b$ when $X_{0t} = b_0$. Our rule consists of stopping the Y_{0t} process when $Y_t^* \geqq \tilde{y}_{-t}$.

If the histories don't differ, then $\hat{Y}^* = \hat{Y}$ and $|\hat{Y} - \hat{Y}_0| \leqq \varepsilon_{2n}$. Let H_n^+ be the event of common histories and H_n^- be its complement. Then

$$u_0(y, n) = \int_{H_n^+ E_n E_{0n}} \hat{Y}_0^2 \, dP + \int_{H_n^+ F_n E_{0n}} \hat{Y}_0^2 \, dP + \int_{H_n^- E_{0n}} \hat{Y}_0^2 \, dP$$

and

$$\tilde{u}(y, n) = \int_{E_n} \hat{Y}^2 \, dP \geqq \int_{H_n^+ E_n E_{0n}} \hat{Y}^2 \, dP \, .$$

On $H_n^+ E_n E_{0n}$, $\hat{Y}_0^2 \leqq \hat{Y}^2 + 2b\varepsilon_{2n} + \varepsilon_{2n}^2$. On $H_n^+ F_n E_{0n}$, $N = N_0 = n$ and $\hat{Y} < 0$ but $\hat{Y}_0 \geqq 0$ and hence $\hat{Y}_0^2 \leqq \varepsilon_{2n}^2$. Finally on E_{0n}, $|\hat{Y}_0 - \hat{Y}^*| \leqq \varepsilon_{2n}$ and $|\hat{Y}^*| \leqq b$ and hence $\hat{Y}_0^2 \leqq (b + \varepsilon_{2n})^2$. Hence

$$\tilde{u}_0(y, n) \leqq u_0(y, n) \leqq \tilde{u}(y, n) + 2b\varepsilon_{2n} + \varepsilon_{2n}^2 + \varepsilon_{1n}b^2 \, . \qquad \square$$

LEMMA 5.5. *If* $\tilde{y}_n > \tilde{y}_{0n}$, *then*

$$\tilde{y}_n - \tilde{y}_{0n} \leqq (\varepsilon_{1n}b^2 + 2b\varepsilon_{2n} + \varepsilon_{2n}^2)^{\frac{1}{2}} \, .$$

PROOF.

$$0 = \tilde{v}_0(\tilde{y}_{0n}, n) \geqq \tilde{v}(\tilde{y}_{0n}, n) - (\varepsilon_{1n}b^2 + 2b\varepsilon_{2n} + \varepsilon_{2n}^2) \, .$$

Using the proof of Theorem 3.2

$$\tilde{v}(\tilde{y}_{0n}, n) = \tilde{v}(\tilde{y}_{0n}, n) - \tilde{v}(\tilde{y}_n, n) \geqq (\tilde{y}_n - \tilde{y}_{0n})^2 \, . \qquad \square$$

THEOREM 5.1. $\tilde{y}$ *is continuous in* p.

PROOF. From Lemma 5.3 we have bounds for $\tilde{y}_n - \tilde{y}$ and $\tilde{y}_{0n} - \tilde{y}_0$. From Lemma 5.5 we have a bound on $\tilde{y}_n - \tilde{y}_{0n}$ if $\tilde{y}_n > \tilde{y}_{0n}$ and a similar bound if $\tilde{y}_n \leqq \tilde{y}_{0n}$. These combine to give a bound on $|\tilde{y} - \tilde{y}_0|$. Given an interval containing p and p_0, then a, b, a_0 and b_0 are bounded, and a constant K (from Lemma 5.3) is determined. For n sufficiently large the bounds for $\tilde{y}_n - \tilde{y}$ and $\tilde{y}_{0n} - \tilde{y}_0$ derived from Lemma 5.3 can be made arbitrarily small. Given that value of n, ε_{1n} and ε_{2n} can be made sufficiently small by taking $p - p_0$ small enough so that the bound on $|\tilde{y}_n - \tilde{y}_{0n}|$ is arbitrarily small. Hence $|\tilde{y} - \tilde{y}_0|$ can be made arbitrarily small. $\square$

Theorem 5.1 permits us to approximate $\bar{y}_0$ for irrational p_0 by computing $\bar{y}$ for nearby rational p. Moreover the derivation carries implicitly in it a method of estimating the error of this approximation.

In Figure 1, we present the values of $\bar{y}$ determined from (4.12) by numerical integration for a finite sequence of rational values of p. This figure suggests the conjecture that $\bar{y}$ is not differentiable in p.

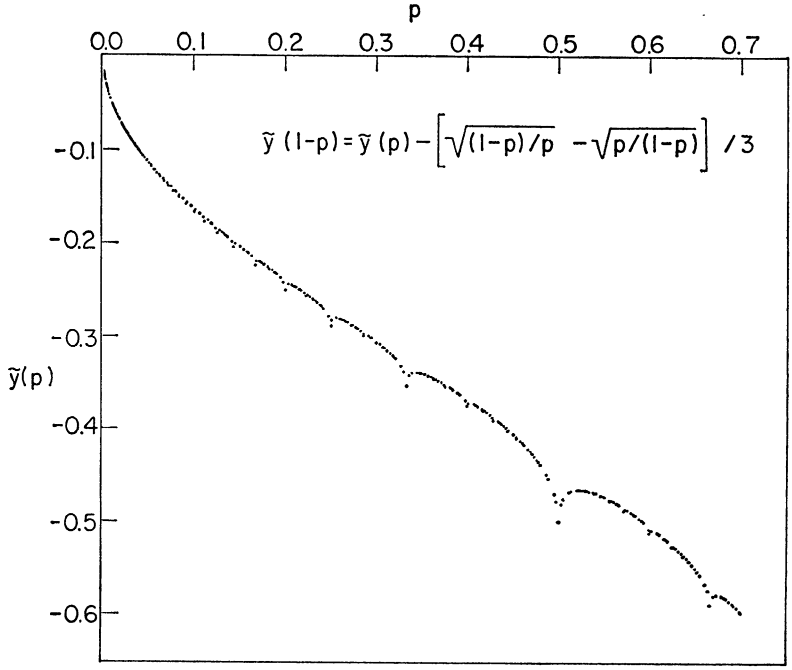

FIG. 1.

6. Related results. An alternative proof was developed for the continuity of $\bar{y}$ as a function of p. The proof in Section 5 has the advantage that it provides a means of computing bounds on $|\bar{y} - \bar{y}_0|$. The alternative proof involves several results of intrinsic interest in themselves and we shall present these here.

Where Section 5 basically involves a bound on the probability distribution of the time of first passage above a constant of a sum of i.i.d. (dichotomous) random variables, the alternative uses

$$W_c(y) \doteq E(\hat{Y}_c^2) \tag{6.1}$$

where $\hat{Y}_c$ is the value of Y_t at the first time where $Y_t > c$. We also define

$$w_c^*(y) = E(\hat{Y}_c^{*2}) \tag{6.2}$$

where $\hat{Y}_c^*$ is the value of Y_t at the first time $Y_t \geqq c$. We omit the subscript c when $c = \bar{y}$. Note that these definitions (6.1) and (6.2) are independent of the t coordinate of the initial point (y, t).

In this section we shall prove that $\bar{y}_n > \bar{y}$ for all n and use this to prove that

$$w(y) = \tilde{u}(y) = w^*(y) .$$

A corollary is that $c = \bar{y}$ minimizes $w_c(y)$ and $w_c^*(y)$. The result that $w_c(y) = w_c^*(y)$ for $c = \bar{y}$ is not trivial. It is not true for general c; and its proof, which invokes the fact that $\bar{y}_n > \bar{y}$, seems to be tied to the smoothness imposed by the fact that $c = \bar{y}$ is the solution of an optimization problem.

LEMMA 6.1. $\bar{y}_n > \bar{y}$.

PROOF. Direct computation in the proof of Lemma 3.1 shows that $\tilde{v}(y, 1) < \tilde{v}(y, 0)$, for $-b < y < 0$. If $(y, -n - 1)$ is an initial point such that there exists a possible path Y_t which leads to $(y_1, -1)$ with $-b < y_1 < 0$ without stopping under the optimal procedure, then,

$$\tilde{v}(y, n + 1) < \tilde{v}(y, n) \tag{6.3}$$

since $\tilde{v}(y, n + 1)$ can be considered as the optimal expected gain of an n-step problem with terminal payoff $\tilde{v}(y, 1)$. In particular if $y = \bar{y}$, ($\bar{y} \geqq -b/2$ by Theorem 3.3) and n_1 is designated, such a path is easily constructed for some y_1 between $-b$ and $\bar{y}_1$ and for some $n > n_1$.

Hence, for arbitrary $n_1 > 0$, there is an $n > n_1$, so that

$$\tilde{v}(\bar{y}, n_1) \geqq \tilde{v}(\bar{y}, n) > \tilde{v}(\bar{y}, n + 1) \geqq 0$$

using monotonicity, (6.3), and the nonnegativity of $\tilde{v}$, and thus

$$\bar{y} < \bar{y}_{n_1} . \qquad \square$$

LEMMA 6.2. $w(y) = \tilde{u}(y)$.

PROOF. Let $\hat{Y}$, T and N correspond to the suboptimal rule of stopping when $Y_t > \bar{y}$, and let $\tilde{Y}$, $\tilde{N}$ and $\tilde{T}$ correspond to the optimal rule. Then by Lemma 6.1, $N \leqq \tilde{N}$ and

$$\tilde{u}(y, n) = \int_{\{\tilde{N}=N<n/2\}} \hat{Y}^2 \, dP + \int_C \tilde{u}(\hat{Y}, -T) \, dP \tag{6.4}$$

where C is the complement of $\{\tilde{N} = N < n/2\}$. Thus on C either $N \geqq n/2$ or $N < n/2$ and $N < \tilde{N}$. As $n \to \infty$, $P\{N \geqq n/2\} \to 0$. On the set where $N < n/2$ and $N < \tilde{N}$, $\bar{y} < \hat{Y} < \bar{y}_{[n/2]} \to \bar{y}$ and $|\tilde{u}(\hat{Y}, -T) - \hat{Y}^2| = \tilde{v}(\hat{Y}, -T) < 3b|\hat{Y} - \bar{y}_{[n/2]}| \to 0$ where the last inequality derives from Theorem 3.2 and the fact that $\bar{y} > -b/2$. Hence

$$\tilde{u}(y, n) - \int_{\{N<n/2\}} \hat{Y}^2 \, dP \to 0 .$$

The first term converges to $\tilde{u}(y)$. The second term converges to $w(y)$. $\square$

THEOREM 6.1. $w(y) = \tilde{u}(y) = w^*(y)$.

PROOF. Let $\hat{Y}^*$, T^*, and N^* correspond to the rule of stopping when $Y_t \geqq \bar{y}$. While the proof of Lemma 6.2 required Lemma 6.1 to infer $N \leqq \tilde{N}$ the fact that $N^* \leqq \tilde{N}$ follows from the definition. The remainder of the proof of Lemma 6.2 applies to $w^*(y)$ directly. $\square$

Note that if c is a possible value of Y_t, the distribution of $\hat{Y}_c$ and $\hat{Y}_c^*$ are quite different and $w_c(y)$ is not in general equal to $w_c^*(y)$.

COROLLARY 1. $w_c(y) \geqq w(y)$ *and* $w_c^*(y) \geqq w^*(y)$.

PROOF. The suboptimal strategy of stopping when $Y_t > c$ leads to

$$u(y, n) = \int_{\{N<n\}} \hat{Y}^2 \, dP + \int_{\{N=n, \hat{Y} \geqq 0\}} \hat{Y}^2 \, dP \geqq \tilde{u}(y, n) .$$

As $n \to \infty$ the second term in the sum approaches 0 and the first approaches $w_c(y)$. At the same time $\tilde{u}(y, n) \to \tilde{u}(y) = w(y)$. The same proof applies for $w_c^*(y)$. □

7. Background. In a series of papers [1, 2, 3, 4], the sequential problem of testing whether the mean of a normal distribution with known variance is positive or negative was approximated by the continuous time problem of deciding the sign of the drift of a Wiener process. The latter problem reduces to a stopping problem involving a zero drift standard Wiener process $Z(t)$. If stopping takes place at $(\hat{Z}, T)$, there is a payoff $g(\hat{Z}, T)$. The continuous time problem has the advantage that its solution is related to a problem in analysis, a free boundary problem involving that heat equation. Moreover a numerical solution of the continuous time problem can be approximated by applying backward induction on a truncated version of the original discrete time problem.

This apparent circularity seems more embarrassing than is the case. First, the continuous time problem allows us to derive valuable characteristics of the solution including asymptotic approximations. Moreover there is an excellent and simple approximate relation between the solutions of the discrete and continuous time problems which allows us to use a single backward induction to approximate the solution of the continuous time problem and the solution of an entire class of discrete time problems.

More specifically, a discrete time version of the above stopping problem is obtained when one is permitted to stop only on a discrete set of possible values of t, say $\{t_0 + n\delta, n = 1, 2, \cdots\}$. Then, between successive possible stopping times Z changes by a normal deviate with mean 0 and variance δ. It is shown in [4] that the difference between the optimal boundaries $\tilde{z}(t)$ and $\tilde{z}_\delta(t)$ of the two problems is approximately given by

$$\tilde{z}_\delta = \tilde{z} \pm .5824\delta^{\frac{1}{2}}$$

(the sign is determined so as to make the continuation region smaller). The number .5824 comes from the limiting value of $\tilde{y}$ in the solution of an *associated* problem. That problem is the same as the one originally posed in this paper except that the X_t are normally distributed with mean 0 and variance 1.

The programming of the backward induction for the numerical calculation is easier if the Wiener process is approximated, not by the sum of independent normal random variables, but by the sum of dichotomous variables which take on the values $\pm\delta^{\frac{1}{2}}$ with probability $\frac{1}{2}$. In this case the above approximation is replaced by

$$\tilde{z}_\delta = \tilde{z} \pm .5\delta^{\frac{1}{2}}$$

where .5 is limiting value $\bar{y}$ in the problem of this paper when $p = \frac{1}{2}$.

The last result would be adequate to approximate the solution of the continuous time stopping problems. However, in a recent case [6], the continuous time problem was used to approximate a discrete time dichotomous problem derived from an application to clinical trials described in more detail below. There, the parameter of concern was the probability of success which could be far from $\frac{1}{2}$. Thus we can use the special $p = \frac{1}{2}$ problem to correct the numerical approximation to the solution of the continuous time problem. But now we need $\bar{y}$ of the problem for general p for the approximation $\tilde{z}_\delta = \tilde{z} \pm \bar{y}\delta^{\frac{1}{2}}$ to relate the solution of the continuous time problem to the dichotomous variable, discrete time, clinical trials problem.

The application to clinical trials arises as follows: An experimental treatment which is characterized by an unknown probability of success (denoted by p) is to be compared with a standard treatment which is characterized by a known probability of success (denoted by p_0). A finite horizon of patients, each of whom must be treated with one of the available treatments, is anticipated. Sampling is initiated with the experimental treatment and is continued with this treatment until a decision is made in favor of one of the treatments. The remaining patients are then treated with the favored treatment. Costs are incurred for each patient unsuccessfully treated and also for each patient treated prior to the time of decision.

A normal version of this problem, obtained by replacing the dichotomous random variables by normal random variables leads to the consideration of the following Gaussian version of the problem: Observe a Wiener process $X(t)$ with (unknown) drift μ and (known) variance σ^2 per unit time for $t \leqq N$. One incurs a constant cost of sampling per unit time and upon terminating sampling at time t, one must choose either to accept the payoff $X(t)$ or to continue and receive $X(N)$. If μ has a normal prior distribution, the Bayes solution (after normalization) is the stopping problem in which one observes $Y(s)$, a zero drift standard Wiener process in the $-s$ scale for $s \geqq 1$, and where the stopping cost at (y, s) is given by $d(y, s) = \gamma/s - y$ for $y > 0$ and $\gamma/s - y/s$ for $y \leqq 0$. In this problem, Y represents the current estimate of μ, s^{-1} is its precision and γ is a normalized cost parameter. The values of $\bar{y}(p_0)$ are then used in the manner described to relate the solution of this continuous time stopping problem to the solution of the clinical trials problem.

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REFERENCES

[1] BREAKWELL, J. and CHERNOFF, H. (1962). Sequential tests for the mean of a normal distribution II. *Ann. Math. Statist.* **35** 162–173.

[2] CHERNOFF, H. (1961). Sequential tests for the mean of a normal distribution. *Proc. Fourth Berkeley Symp. Math. Statist. Prob.* **1** 79–91, Univ. of California Press.

[3] CHERNOFF, H. (1965). Sequential tests for the mean of a normal distribution III. *Ann. Math. Statist.* **36** 28–54.

[4] CHERNOFF, H. (1965). Sequential tests for the mean of a normal distribution IV. *Ann. Math. Statist.* **36** 55–68.

[5] DOOB, J. L. (1953). *Stochastic Processes.* Wiley, New York.

[6] Petkau, A. J. (1975). Sequential medical trials for comparing an experimental with a standard treatment. Technical Report No. 1, Department of Mathematics, M.I.T.

[7] Spitzer, F. (1960). A Tauberian theorem and its probability interpretation. *Trans. Amer. Math. Soc.* **94** 150–169.

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