The Annals of Probability
1977, Vol. 5, No. 2, 235–242

THE HAUSDORFF DIMENSION OF THE RANGE OF THE N-PARAMETER WIENER PROCESS

BY LANH TAT TRAN

Indiana University

Let $W^{(N,d)}$ be the N-parameter Wiener process with values in R^d. It is shown that almost all sample functions of $W^{(N,d)}$ have dimensional number $2N$ and zero $2N$-measure when $d \geqq 2N$. Our results extend earlier ones of Taylor for $N = 1$.

1. Introduction and notation. Let $W^{(N)}$ be the N-parameter Wiener process; that is, a real valued Gaussian process with zero means and covariance $\prod_{i=1}^{N} (s_i \wedge t_i)$ where $s = \langle s_i \rangle$, $t = \langle t_i \rangle$, $s_i \geqq 0$, $t_i \geqq 0$, $i = 1, 2, \cdots, N$. Then $W^{(N,d)}$ is to be the process with values in R^d such that each component is an N-parameter Wiener process and the components are independent. In the one-parameter case, $N = 1$, Taylor (1953) showed that almost all d-dimensional Brownian sample functions have Hausdorff dimensional number 2 and zero 2-measure when $d \geqq 2$. The purpose of this paper is to extend Taylor's results to $W^{(N,d)}$.

The notation of Orey and Pruitt (1973) will be used. The parameter space is the set $t \in R_+{}^N$ with all components nonnegative. We sometimes write t as $\langle t_1, \cdots, t_N \rangle$ or simply $\langle t_i \rangle$. When all $t_i = a$, $\langle t_i \rangle$ is written as $\langle a \rangle$. For $s = \langle s_i \rangle$ and $t = \langle t_i \rangle$ with $s_i \leqq t_i$, $\times_{i=1}^{N} [s_i, t_i]$ is denoted by $\Delta(s, t)$, or $\Delta(t)$ in case $s = \langle 0 \rangle$. Sets of this form in the parameter space are sometimes referred to as intervals. Let $W = W^{(N,d)}$ for simplicity. Denote the ith component of W by W^i where $1 \leqq i \leqq d$. Let $s, t \in R_+{}^N$. Then the variance of $W^i(t) - W^i(s)$ can be verified to be $|S(s, t)|$ where $S(s, t)$ is the symmetric difference between $\Delta(s)$ and $\Delta(t)$ and $|\ |$ denotes the N-dimensional Lebesgue measure. We shall write $\delta(s, t)$ for $|S(s, t)|$. Occasionally, c will be used to denote constants whose values are unimportant and may be different from line to line.

We will often use the scaling property of W and state that W has continuous sample functions and stationary independent increments. For an account of these properties and further information on W, consult Kitagawa (1951) Chentsov (1956), Yeh (1960, 1963a, b), Delporte (1966), C. Park (1969), W. J. Park (1970), Zimmerman (1972), Orey and Pruitt (1973).

Orey and Pruitt (1973) showed that almost all sample functions of W have positive d-dimensional volume when $d < 2N$. In this case, the sample functions have dimensional number d almost surely. We assume $d \geqq 2N$ in the rest of this paper. In Section 2, the concept of capacity is used to show that almost all sample functions of W have dimensional number $2N$. The result of this section

Received January 23, 1975; revised September 22, 1975, July 9, 1976.
AMS 1970 *subject classifications.* Primary 60G15; Secondary 60G17.
Key words and phrases. Wiener process, capacity, Hausdorff dimension.

is a simple generalization of Taylor's result for Brownian motion. Section 3, which is the more difficult part of our paper, is devoted to showing that the $2N$-measure of the sample functions is almost always zero. Taylor (1953, Theorem 2) showed that for $N = 1$, the 2-measure of almost all sample functions of Brownian motion is zero. However, due to the particular nature of Taylor's proof which requires the strong Markov property, his method of argument does not provide an immediate generalization to general N. Orey and Pruitt (1973) showed that almost all sample functions of W have zero d-dimensional volume when $d \geqq 2N$. This does not imply that the $2N$-measure of the sample functions is almost always zero for $d > 2N$. Theorem 3.1 of Section 3 implies that almost all sample functions of $W^{(N,d)}$ have zero d-dimensional volume when $d \geqq 2N$ and so is stronger then the result of Orey and Pruitt stated above.

2. Dimension of the range.

THEOREM 2.1. *Let $d \geqq 2N$. Then $\{W_t : t \in R_+^N\}$ has dimensional number $2N$ almost surely.*

PROOF. This will be shown in two parts.

(i) If $\alpha < 2N$, then $\{W_t : t \in \Delta(\langle\frac{1}{2}\rangle, \langle 1\rangle)\}$ has positive α-measure a.s.

Let $I = \Delta(\langle\frac{1}{2}\rangle, \langle 1\rangle)$. Consider

$$\int_I \int_I E|W(t) - W(s)|^{-\alpha}\, ds\, dt\,. \tag{2.1}$$

The expectation inside (2.1) can be written as

$$\sigma^{-\alpha}(W^1(s) - W^1(t))E\{\textstyle\sum_{i=1}^d [(W^i(s) - W^i(t))/\sigma(W^i(s) - W^i(t))]^2\}^{-\alpha/2}$$

which is equal to

$$\sigma^{-\alpha}(W^1(s) - W^1(t))E[\chi^2(d)]^{-\alpha/2}$$

where $\chi^2(d)$ denotes the chi-square random variable with d degrees of freedom. Observe that $E[\chi^2(d)]^{-\alpha/2}$ is finite for $\alpha < d$. Since $s, t \in \Delta(\langle\frac{1}{2}\rangle, \langle 1\rangle)$,

$$\sigma^2(W^1(s) - W^1(t)) > c[|t_1 - s_1| + \cdots + |t_N - s_N|]\,.$$

Therefore, (2.1) is bounded by

$$c \int_I \int_I [|t_1 - s_1| + \cdots + |t_N - s_N|]^{-\alpha/2}\, ds\, dt\,. \tag{2.2}$$

By a change of variables and integration, (2.2) can be seen to be finite if

$$\int_{I'} (t_1 + \cdots + t_N)^{-\alpha/2}\, dt \quad \text{is finite for} \quad I' = \Delta(\langle 0\rangle, \langle\tfrac{1}{2}\rangle)\,. \tag{2.3}$$

But (2.3) is clearly finite for $\alpha < 2N$ by a simple integration. The expectation sign in (2.1) can now be taken outside to show that

$$\int_I \int_I |W(t) - W(s)|^{-\alpha}\, ds\, dt < \infty \quad \text{a.s.}$$

Now, Theorem B of Taylor (1955) implies that the α-capacity of $\{W_t : t \in \Delta(\langle\frac{1}{2}\rangle, \langle 1\rangle)\}$ is positive. The proof of this part is completed by the equivalence of Hausdorff dimension and capacity dimension for compact sets in Euclidean space. For an account of this information, consult Kametani (1946) and Taylor (1955).

(ii) If $\alpha > 2N$, then a.s. $\{W_t : t \in R_+{}^N\}$ has zero α-measure.

Let $\eta > 0$. By Theorem 2.4 of Orey and Pruitt (1973), for almost all ω, there is an $\varepsilon(\omega)$ such that $|W(t) - W(s)| < \delta(s, t)^{\eta+\frac{1}{2}}$ whenever $\delta(s, t) < \varepsilon(\omega)$ and $s, t \in \Delta(\langle\frac{1}{2}\rangle, \langle 1\rangle)$. With the availability of this property, the argument of Theorem 1 of Taylor can be easily extended to the general case; and so more details will not be presented. It will also follow from Theorem 3.1 of Section 3 that a.s. $\{W_t : t \in R_+{}^N\}$ has zero α-measure if $\alpha > 2N$.

3. $2N$-measure of $\{W_t : t \in R_+{}^N\}$ is a.s. zero when $d \geqq 2N$. Let $\{m_n\}$ be a sequence of positive integers with $m_{n+1} = 2m_n$. For each m_n, partition $\Delta(\langle\frac{1}{2}\rangle, \langle 1\rangle)$ into $m_n{}^N$ cubicles with sides parallel to coordinate axes and equal to $2^{-1}m_n{}^{-1}$. Let $G(m_n)$ be the collection of these cubicles. Order the cubicles of $\bigcup_{k=1}^\infty G(m_k)$ in such a way that the cubicles of $G(m_k)$ precede the cubicles of $G(m_{k+1})$ for all $k \geqq 1$. Denote the ordered collection of cubicles by $\{C_n\}$. Observe that for two cubicles C_i, C_j with $j > i \geqq 1$ then either C_i contains C_j or they have disjoint interior.

LEMMA 3.1. *Let C_k be a cubicle of $\{C_n\}$ with sides equal to a_k. Let t^k be the least vertex of cubicle C_k,* i.e., *closest to $\langle 0\rangle$. Let μ be any positive number. Define*

$$D_k = [\omega : \sup_{t\in C_k} |W(t, \omega) - W(t^k, \omega)| < \mu a_k{}^{\frac{1}{2}}] . \tag{3.1}$$

Then, there exists a positive number β and an integer k_0 such that $P(D_k) > \beta$ for all k with $k > k_0$.

PROOF. This lemma has been shown by Taylor (1953) for $N = 1$. Assume $N \geqq 2$ and let $t \in C_k$. Define $p_i(t) = \langle t_1{}^k, \cdots, t_{i-1}^k, t_i, t_{i+1}^k, \cdots, t_N{}^k\rangle$. Write $t_i = t_i{}^k + e_i$ for some e_i, $1 \leqq i \leqq N$. Observe that $0 \leqq e_i \leqq a_k$. The variance of each component of $W(t) - W(t^k)$ is $\delta(t^k, t)$ and $\delta(t^k, t) = \prod_{i=1}^N (t_i{}^k + e_i) - \prod_{i=1}^N t_i{}^k$. Now, $\delta(t^k, t)$ can be expanded as the sum of $2^N - 1$ terms. It is then easy to see that $S(t^k, t)$ can be written as the union of $2^N - 1$ nonoverlapping intervals such that the Lebesgue measure of each interval corresponds to one term of the sum above. Precisely, we can write $S(t^k, t)$ as the union of $\bigcup_{i=1}^N S(t^k, p_i(t))$ and $2^N - 1 - N$ other intervals so that each interval of the second group has at least two sides smaller or equal to a_k. Any interval with only one side smaller or equal to a_k has been accounted for in $\bigcup_{i=1}^N S(t^k, p_i(t))$. Therefore, we now can write $W(t) - W(t^k)$ as

$$\textstyle\sum_{i=1}^N [W(p_i(t)) - W(t^k)] + Y_t ,$$

where Y_t is the sum of the increments of W over $2^N - 1 - N$ nonoverlapping intervals.

Define

$$F_i = [\sup_{t\in C_k} |W(p_i(t)) - W(t^k)| < \mu(2N)^{-1}a_k{}^{\frac{1}{2}}]$$
$$F = [\sup_{t\in C_k} |Y_t| < \mu 2^{-1}a_k{}^{\frac{1}{2}}] .$$

Clearly

$$P(D_k) \geqq P(F \bigcap_{i=1}^N F_i)\,.$$

Since W has independent increments, F and the F_i's are independent. We obtain

$$P(D_k) \geqq P(F) \prod_{i=1}^N P(F_i)\,.$$

Y_t is the sum of the increments of W over $2^N - 1 - N$ intervals. Each of these intervals has at least two sides smaller or equal to a_k. Theorem 2.1 of Orey and Pruitt (1973) can now be used to conclude that $P(F)$ goes to 1 as k goes to infinity. Furthermore,

$$\begin{aligned} P(F_i) &\geqq P[\sup_{t \in C_k} |(W(p_i(t)) - W(t^k)) \textstyle\prod_{j \neq i} (t_j{}^k)^{-\frac{1}{2}}| < \mu(2N)^{-\frac{1}{2}} a_k{}^{\frac{1}{2}}] \\ &\geqq P[\sup_{0 \leqq t \leqq a_k} |W^{(1,d)}(t)| < \mu(2N)^{-\frac{1}{2}} a_k{}^{\frac{1}{2}}]\,, \end{aligned}$$

which is greater than some constant by Lemma 5 of Taylor (1953) for $N = 1$.

We will assume that $P(D_k) > \beta$ for all $k \geqq 1$ in the rest of this paper. This can be done by choosing m_1 large enough.

LEMMA 3.2. *Let $\{B_n\}$ be a subsequence of the sequence of cubicles $\{C_n\}$ with $B_{n+1} \subset B_n \in G(m_n)$ for all $n \geqq 1$. Let t^i denote the least vertex of cube B_i and μ_i be a positive number. Define*

$$\begin{aligned} \Lambda_i &= [\sup_{t \in B_i} |W(t) - W(t^i)| > \mu_i] \\ \Lambda &= \textstyle\bigcap_{i=1}^M \Lambda_i \qquad \text{where} \quad M \quad \text{is a positive integer.} \end{aligned}$$

Corresponding to each B_k, define D_k as in (3.1). *Let $K > 0$. Then there exists an integer $k > K$ such that*

$$P(D_k \Lambda) > 2^{-1} P(D_k) P(\Lambda)\,.$$

PROOF. Pick B_k with $k > M$ so that B_k is contained in each B_i, $1 \leqq i \leqq M$. Let $t \in B_i$ where $1 \leqq i \leqq M$ and consider $W(t) - W(t^i)$. We now use the property that W has independent increments to write $W(t) - W(t^i)$ as the sum of two independent random variables, say $Z_{ik}(t)$ and $Y_{ik}(t)$ chosen in such a way that $Y_{ik}(t)$ and D_k are independent; also $Z_{ik}(t)$ converging to zero almost surely as k goes to infinity. To do this, write the symmetric difference $S(t^i, t)$ as the union of two parts:

$$\begin{aligned} &S(t^i, t) \cap S(t^k, s^k) \\ &S(t^i, t) \cap [S(t^k, s^k)]' \end{aligned}$$

where $[S(t^k, s^k)]'$ is the complement of $S(t^k, s^k)$ and t^k, s^k are respectively the least and largest vertex of B_k. Both of these two parts can be written as the union of a finite number of nonoverlapping intervals. Now, let $Z_{ik}(t)$ and $Y_{ik}(t)$ be respectively the sum of increments of W over intervals that make up $S(t^i, t) \cap S(t^k, s^k)$ and $S(t^i, t) \cap [S(t^k, s^k)]'$. Let $\varepsilon > 0$. Define

$$J_i = [\sup_{t \in B_i} |Z_{ik}(t)| < \varepsilon]\,.$$

Clearly,

$$\begin{aligned} P(D_k \Lambda) &\geqq P(D_k \textstyle\bigcap_{i=1}^M [\sup_{t \in B_i} |Y_{ik}(t) + Z_{ik}(t)| > \mu_i] J_i) \\ &\geqq P(D_k \textstyle\bigcap_{i=1}^M [\sup_{t \in B_i} |Y_{ik}(t)| > \mu_i + \varepsilon] \textstyle\bigcap_{i=1}^M J_i) . \\ &\geqq P(D_k \textstyle\bigcap_{i=1}^M J_i) P(\textstyle\bigcap_{i=1}^M [\sup_{t \in B_i} |Y_{ik}(t)| > \mu_i + \varepsilon]) \end{aligned}$$

since $D_k \bigcap_{i=1}^M J_i$ and $\bigcap_{i=1}^M [\sup_{t \in B_i} |Y_{ik}(t)| > \mu_i + \varepsilon]$ are independent.

Next, $P(D_k \Lambda)$ is greater than or equal to

$$P(D_k \textstyle\bigcap_{i=1}^M J_i) P(\textstyle\bigcap_{i=1}^M [\sup_{t \in B_i} |W(t) - W(t^i)| > \mu_i + 2\varepsilon] \textstyle\bigcap_{i=1}^M J_i) .$$

Now, $P(\bigcap_{i=1}^M \sup_{t \in B_i} |W(t) - W(t^i)| > \mu_i + 2\varepsilon])$ converges to $P(\Lambda)$ as ε goes to zero since almost all sample functions of W are continuous. Observe that $P(D_k) > \beta$ by Lemma 3.1 and for each ε fixed, $P(\bigcap_{i=1}^M J_i)$ goes to 1 as k goes to infinity. The lemma now follows by first choosing ε small enough for $P(\bigcap_{i=1}^M [\sup_{t \in B_i} |W(t) - W(t^i)| > \mu_i + 2\varepsilon])$ to be greater than $\frac{5}{6}P(\Lambda)$ and then k large enough so that $P(\bigcap_{i=1}^M J_i)$ is greater than both $1 - \frac{1}{6}P(\Lambda)$ and $1 - (\beta/4)$.

Let $k_1 = 1$. Assume that $k_1, \cdots, k_m$ have been chosen with $k_1 < k_2 < \cdots < k_m$. Let $\Lambda_m = D'_{k_1} D'_{k_2} \cdots D'_{k_m}$. Clearly, Λ_m is a set of the type considered in Lemma 3.2. Therefore we can apply that lemma to get an integer k_{m+1} such that

$$P(D_{k_{m+1}} \Lambda_m) > 2^{-1} P(D_{k_{m+1}}) P(\Lambda_m) .$$

LEMMA 3.3. *Let $\{D_k\}$ be the sequence of events defined in* (3.1) *corresponding to the subsequence $\{B_n\}$ of cubicles defined in Lemma* 3.2. *Then*

$$P(D_k \text{ infinitely often}) = 1 .$$

PROOF. The proof of this lemma is essentially the same as that of Lemma 7 of Taylor (1953). Consider the subsequence $\{D_{k_i}\}$. Clearly, it is enough to show that $P(D_{k_i}$ infinitely often$) = 1$. Let

$$\Lambda = \textstyle\bigcap_{i=1}^\infty D'_{k_i} .$$

Since

$$\Lambda_m = \textstyle\bigcap_{i=1}^m D'_{k_i} ,$$

$$\begin{aligned} P(\Lambda_{m+1}) &= P(\Lambda_m D'_{k_{m+1}}) \\ &= P(\Lambda_m) - P(\Lambda_m D_{k_{m+1}}) \\ &< P(\Lambda_m) - 2^{-1} P(\Lambda_m) P(D_{k_{m+1}}) \\ &< P(\Lambda_m)(1 - 2^{-1}\beta) \\ &< P(\Lambda_1)(1 - 2^{-1}\beta)^m . \end{aligned}$$

Since $\beta > 0$ by Lemma 3.1, $P(\Lambda) = 0$ and Lemma 3.3 follows.

THEOREM 3.1. *$\{W_t : t \in R_+{}^N\}$ has zero 2N-measure almost surely when $d \geqq 2N$.*

PROOF. By the scaling property of W, it is enough to show that $\{W_t : t \in \Delta(\langle \frac{1}{2} \rangle, \langle 1 \rangle)\}$ has zero $2N$-measure almost surely. Consider the collection of cubicles $\{C_n\}$ of $\bigcup_{i=1}^\infty G(m_k)$ ordered as described in the paragraph preceding Lemma 3.1. For each cubicle C_k with sides equal to a_k and least vertex t^k, let

S_k be the cube in R^d with center at $W(t^k)$, sides parallel to coordinate axes and equal to $\mu a_k^{\frac{1}{2}}$ where μ is a small positive number to be chosen later. Denote the empty set by $\emptyset$ and $\{W_t : t \in C_k\}$ by $R(C_k)$.

Define

$$T_1 = S_1 \quad \text{if} \quad S_1 \supset R(C_1)$$
$$= \emptyset \quad \text{otherwise.}$$

For $k > 1$, define

$$(3.2) \qquad T_k = S_k \quad \text{if} \quad S_k \supset R(C_k) \quad \text{and} \quad R(C_k) \not\subset \bigcup_{j=1}^{k-1} T_j$$
$$= \emptyset \quad \text{otherwise.}$$

Let K be a positive integer. For brevity, let $\gamma_K = \sum_{k=1}^{K} (m_k)^N$ and $\psi_K = \bigcup_{k=1}^{\gamma_K} T_k$. Let r_K be the proportional of cubicles of $G(m_K)$ with images not yet covered by ψ_K. In other words,

$$r_K = (m_K{}^N)^{-1}(\# \text{ of } C_k \in G(m_K) \text{ with } R(C_k) \not\subset \psi_K)\,.$$

Claim that $P(R(C_k) \subset \psi_K)$ goes to 1 as $K \to \infty$ uniformly for all $C_k \in G(m_K)$. Let n be a positive integer and A_{n+1}, A_n be two cubicles with $A_{n+1} \in G(m_{n+1})$, $A_n \in G(m_n)$. Observe that if $A_{n+1} \subset A_n$, then $P(R(A_n) \not\subset \psi_n) \geqq P(R(A_{n+1}) \not\subset \psi_{n+1})$. Now, every cubicle of $G(m_{n+1})$ is contained in a cubicle of $G(m_n)$. If the claim is not true, then for some $\varepsilon > 0$ there exists a subsequence $\{A_n\}$ of $\{C_n\}$ with $A_n \in G(m_n)$ and with $P(R(A_n) \not\subset \psi_n) \geqq \varepsilon$ for all $n \geqq 1$. Now, $G(m_1)$ contains a finite number of cubicles and each cubicle of $\{A_n\}$ is a subset of some cubicle of $G(m_1)$ and so one of the cubicles of $G(m_1)$ must contain an infinite number of cubicles of $\{A_n\}$. Let B_1 be such a cubicle of $G(m_1)$. Again B_1 contains a finite number of cubicles of $G(m_2)$ so there exists a $B_2 \in G(m_2)$ with $B_2 \subset B_1$ such that B_2 contains an infinite number of cubicles of $\{A_n\}$. This process can be continued indefinitely. Therefore, there exists a subsequence $\{B_n\}$ of $\{C_n\}$ such that $B_{n+1} \subset B_n \in G(m_n)$ for all $n \geqq 1$ and each cubicle B_n contains an infinite number of cubicles of $\{A_n\}$. From the way ψ_n was constructed, it is clear that $P(R(B_n) \not\subset \psi_n) \geqq \varepsilon$ for all $n \geqq 1$. Consider the corresponding sequence of events $\{D_n\}$ defined as in (3.1) for this subsequence. Clearly $P(D_n \text{ i.o.}) \leqq 1 - \varepsilon$. This contradicts Lemma 3.3. It now follows from the claim that $Er_K \to 0$ as $K \to \infty$.

Consider the class $\xi(m_K{}^{-\frac{1}{2}})$ of cubes in R^d with edges parallel to coordinate axes, sides of length $m_K{}^{-\frac{1}{2}}$ and vertices of the form $(k_1 m_K{}^{-\frac{1}{2}}, \cdots, k_d m_K{}^{-\frac{1}{2}})$ with $k_1, \cdots, k_d$ integers. Let C_k be any cubicle of $G(m_K)$ with $R(C_k) \not\subset \psi_K$. We now follow the method used in the proof of Theorem 3.4 of Orey and Pruitt (1973) and count in a central block of $(2M + 1)^d$ cubes centered at E_k where E_k is the element of $\xi(m_K{}^{-\frac{1}{2}})$ which contains $W(t^k)$, and M is a large integer to be chosen later. Now, for all cubicles C_k of $G(m_K)$, without taking into account whether $R(C_k)$ has been converved by ψ_K or not, we add any cubes of $\xi(m_K{}^{-\frac{1}{2}})$ outside this central block which are intersected by $R(C_k)$. Denote the number of cubes added by N_k. We have now covered $\{W_t : t \in \Delta(\langle\frac{1}{2}\rangle, \langle 1\rangle\}$ with three collections of cubes.

Let the diameter of a cube be the distance from the least to the largest vertex. If cube is in R^d, then diam (cube) $< d$(side of cube). Define a function $f(k)$ by

$$f(k) = 1 \quad \text{if} \quad T_k \neq \emptyset$$
$$= 0 \quad \text{otherwise.}$$

Then,

$$\sum_{k=1}^{\gamma_K} (\text{diam } T_k)^{2N} < \sum_{k=1}^{\gamma_K} (d\mu\, a_k^{\frac{1}{2}})^{2N} f(k) .$$

Observe that a_k^N is the volume of C_k. Also, if $T_i \neq \emptyset$ and $T_j \neq \emptyset$ with $i \neq j$, then from the definition of T_k in (3.2), the interior of C_i and C_j are disjoint. Therefore, the total volume of all C_k's with $T_k \neq \emptyset$ must be smaller than the volume of $\Delta(\langle\frac{1}{2}\rangle, \langle 1\rangle)$ which is simply 2^{-N}. Therefore

$$\sum_{k=1}^{\gamma_K} (\text{diam } T_k)^{2N} < 2^{-N} d^{2N} \mu^{2N} . \tag{3.3}$$

Now, consider the cubicles of $G(m_K)$ only. The number of cubes added in the central blocks is $r_K m_K^N (2M + 1)^d$. Each of these cubes has sides equal to $m_K^{-\frac{1}{2}}$. Therefore, with respect to this collection of cubes, we obtain

$$\begin{aligned} E[\textstyle\sum (\text{diam})^{2N}] &< [Er_K] m_K^N (2M + 1)^d (d m_K^{-\frac{1}{2}})^{2N} \\ &< [Er_K](2M + 1)^d d^{2N} , \end{aligned} \tag{3.4}$$

which goes to zero as $K \to \infty$ since $Er_K \to 0$ as $K \to \infty$. As for the cubes added outside the central blocks, we have

$$\sum (\text{diam})^{2N} < \sum_{\gamma_{K-1}+1}^{\gamma_K} N_k (d m_K^{-\frac{1}{2}})^{2N} .$$

The summation is taken between $\gamma_{K-1} + 1$ and γ_K since we only consider cubicles of $G(m_K)$. We now use a computation of Orey and Pruitt (1973, Theorem 3.4) to obtain an upper bound for EN_k. We get

$$EN_k < c' \sum_{n \geq M} (2n + 3)^d n^{d-2} e^{-cn^2}$$

where c' and c are constants. Therefore, given $\varepsilon > 0$, we can choose M so large that $EN_k < \varepsilon$, independent of k. Then, with respect to the collection of cubes added outside the central blocks, we have

$$\begin{aligned} E[\textstyle\sum (\text{diam})^{2N}] &< \textstyle\sum_{\gamma_{K-1}+1}^{\gamma_K} \varepsilon (d m_K^{-\frac{1}{2}})^{2N} \\ &< \varepsilon d^{2N} m_K^{-N} m_K^N \\ &< \varepsilon d^{2N} \end{aligned}$$

since $\gamma_K - \gamma_{K-1} = m_K^N$ by definition of γ_K. Now, ε and μ can be chosen arbitrarily small. The theorem follows from (3.3), (3.4) and (3.5).

COROLLARY 3.1. *Almost all sample functions of $W^{(N,d)}$ have zero d-dimensional volume when $d \geq 2N$.*

REMARK. A different proof of this corollary can be found in Orey and Pruitt (1973, page 160).

REFERENCES

[1] CHENTSOV, N. N. (1956). Wiener random fields depending on several parameters. *Dokl. Akad. Nauk SSSR* **106** 607–609.

[2] DELPORTE, J. (1966). Fonctions aléatoires de deux variables presque surement à échantillons continus sur un domaine rectangulaire borne. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **6** 181–205.

[3] KAMETANI, S. (1946). On Hausdorff's measures and generalised capacities with some of their applications to the theory of functions. *Japan. J. Math.* **19** 217–257.

[4] KITAGAWA, T. (1951). Analysis of variance applied to function spaces. *Mem. Fac. Sci. Kyushu Univ. Ser. A* **6** 41–53.

[5] OREY, S. and PRUITT, W. E. (1973). Sample functions of the N-parameter Wiener process. *Ann. Probability* **1** 138–163.

[6] PARK, C. (1969). A generalized Paley–Wiener–Zygmund integral and its applications. *Proc. Amer. Math. Soc.* **23** 388–400.

[7] PARK, W. J. (1970). A multi-parameter Gaussian process. *Ann. Math. Statist.* **41** 1582–1595.

[8] TAYLOR, S. J. (1953). The Hausdorff α-dimensional measure of Brownian paths in n-space. *Proc. Cambridge Philos. Soc.* **49** 31–39.

[9] TAYLOR, S. J. (1955). The α-dimensional measure of the graph and set of zeros of a Brownian path. *Proc. Cambridge Philos. Soc.* **51** 265–274.

[10] YEH, J. (1960). Wiener measure in a space of functions of two variables. *Trans. Amer. Math. Soc.* **95** 443–450.

[11] YEH, J. (1963a). Cameron–Martin translation theorems in the Wiener space of functions of two variables. *Trans. Amer. Math. Soc.* **107** 409–420.

[12] YEH, J. (1963b). Orthogonal developments of functionals and related theorems in the Wiener space of functions of two variables. *Pacific J. Math.* **13** 1427–1436.

[13] ZIMMERMAN, G. (1972). Some sample function properties of the two-parameter Gaussian process. *Ann. Math. Statist.* **43** 1235–1246.

DEPARTMENT OF MATHEMATICS
SWAIN HALL-EAST
INDIANA UNIVERSITY
BLOOMINGTON, INDIANA 47401