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JOINT ORDERS IN COMPARATIVE PROBABILITY

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Comparative probability (CP) is a theory of probability in which uncertainty is measured by a CP ordering of events, rather than by a probability measure. A CP order is *additive* iff it has an agreeing probability measure. This paper deals with the formation of joint CP orders from given marginals, both with and without a certain independence condition, and with emphasis on the nonadditive case. Among the results are these: a CP model for many independent and identically distributed trials of a single experiment must be additive, with an agreeing probability measure of product type; there are CP marginals that have no joint CP order at all; there is a class of CP models, strictly containing all the additive ones, which are well behaved with respect to the formation of joint orders. We present as well several sufficient conditions, and one necessary condition, under which given marginals have a joint CP order.

1. Introduction.

A. Comparative probability (abbreviated CP) is a theory of probability in which probability assessments are expressed through a collection of statements of the form $A \lesssim B$ (read "event B is at least as probable as event A"). Its axioms date from de Finetti [2] (though there was even earlier consideration of CP by S. M. Bernstein and J. M. Keynes). It will be assumed here that the family of events (denoted $\mathscr{F}$) is an algebra of subsets of a sample space X. The axioms are then the following, where A, B, C run over all sets in $\mathscr{F}$, and ϕ is the null set:

(Nontriviality)	Not $(X \lesssim \phi)$
(Nonnegativity)	$\phi \lesssim A$
(Transitivity)	$A \lesssim B$, $B \lesssim C$ imply $A \lesssim C$
(Disjoint unions)	$A \cap C = B \cap C = \phi$ implies $(A \lesssim B$ iff $A \cup C \lesssim B \cup C)$
(Completeness)	$A \lesssim B$ or $B \lesssim A$.

Any ordering $\lesssim$ of sets in $\mathscr{F}$ which satisfies these axioms will be called a *CP order*, and the corresponding triple $(X, \mathscr{F}, \lesssim)$ a *CP space*.

Every probability measure P on $\mathscr{F}$ induces a CP order $\lesssim$ on $\mathscr{F}$ via

$$A \lesssim B \quad \text{iff} \quad P(A) \leqq P(B) \qquad A, B \in \mathscr{F}.$$

In the case that a given CP order $\lesssim$ arises in this way from some probability

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measure P, it will be said that $\lesssim$ is *additive*, and that P *agrees* with it. If at least the weaker relation

$$A \lesssim B \quad \text{implies} \quad P(A) \leqq P(B) \qquad\qquad A, B \in \mathscr{F}$$

prevails, for some probability measure P, then $\lesssim$ is *almost additive*, and P *almost agrees* with it. That these distinctions are nontrivial was shown by Kraft, Pratt and Seidenberg [10]; the facts are these:

(i) All CP orders on four or fewer atoms are additive.

(ii) There is a CP order on five atoms that is almost additive, but not additive.

(iii) There is a CP order on six atoms that is not almost additive.

B. The literature of comparative probability is small, and devoted mainly to the problem of finding conditions under which CP orders are additive; we refer particularly to references [10] through [15], and to the survey in [5]. For motivation, see [5] and [13].

The concern of the present paper is not with additivity, but with the properties of the CP axioms in the absence of extra restrictions. The following can be said in favour of considering CP in its own right, rather than as a step on the way to quantitative probability:

(1) The foundations of subjective probability ([13]) encourage the notion that a qualitative view of probability is a natural one.

(2) If you can produce probability numbers, then certainly you can produce probability orders. Thus comparative models for probability ought to be easier to estimate or learn than quantitative ones, especially in applications where there is little prior information.

(3) The necessary conditions required to reduce CP to quantitative probability do not, in our view, have the universality desirable in axioms capable of characterizing the manifold forms of random phenomena and uncertainty (see Chapters 2 and 3 of [5]).

Our purpose is specifically to present some results on the subject of joint orders in CP, with and without an independence condition. The following remarks are intended for motivation.

The underlying problem is to describe the structure of CP. One approach is to organize the body of CP orders into subclasses, according to their extension properties. (The results on additivity may be interpreted so as to address a special case: namely, the characterization in terms of extensions of the subclass consisting of the additive orders.) The formation of joint CP orders from given marginals is a special kind of extension (some others are discussed in [9]).

The problem of extending CP orders arises in a natural way, in contexts that have nothing to do with additivity. Suppose, for example, that $\mathscr{F}$ is the algebra of events from the simultaneous performance of n experiments. It is common practice in quantitative probability to form at least a gross description of a joint measure on $\mathscr{F}$ by specifying the n marginals, together with a law of interaction (independence, say, or exchangeability) between them. It is tempting to try the same in CP; or at least, to begin the description of a joint order by prescribing the marginals.

In quantitative probability, the marginals can be chosen independently of each other; in CP they cannot. Generally speaking, it is not easy in CP to decide whether or not a given family of inequalities is consistent in the sense that it extends to a CP order (actually trying to complete a partial order by scanning all possible extensions is generally impossibly arduous, even for small sample spaces). In this paper, we give some general conditions for consistency in the event that the given inequalities constitute the marginals for a product space; possible interpretations for the case that the marginals do not have a joint order are discussed at the conclusion.

Our chief motivation for studying CP is the hope that it someday might be useful: for example, in applications where, because of lack of data or the limited nature of the experiment, it is difficult to specify, or make sense of, a probability measure; or in quantum mechanics, where the existence of complementary observables suggests the need for a theory of probability that restricts the formation of joint orders. The problem of how to calculate (make decisions) with CP is not treated here (there are some preliminary results in [5]). We point out merely that one would expect that applications of CP would be guided by an understanding of its properties.

Our contribution to CP, and its relation to previous work, is summarized in the following:

(1) We present (Section 2) two qualitative axioms for i.i.d. (independent and identically distributed) trials in CP, both of which seem intuitively necessary; their main property is that the low-dimensional subspaces of large i.i.d. product spaces are additive, with agreeing probability measures of *product* type. In particular, the only CP spaces which are amenable to arbitrary i.i.d. repetition are additive; this establishes the general impossibility of forming i.i.d. product spaces with given CP marginals.

The result is presented in two parts. The first (Section 2B) applies to the case that the marginals are finite; the proof relies on the characterization of the additive orders in [10], and on an algebraic condition, due to Domotor [3], under which a (non-CP) ordering of pairs from a finite algebra $\mathscr{F}$ can be represented by a product of two identical probability measures on $\mathscr{F}$. The second part (Section 2C) applies when the marginals are continuous; the definition and properties of continuous CP orders are due to Villegas [15].

(2) CP spaces are said to be *compatible* if and only if they have at least one joint order. Section 3B presents a necessary condition for compatibility of two CP orders; it is based on the algebraic characterization of the almost additive CP orders given in [10]. A sufficient condition in Savage [13] for almost additivity is a special case. A corollary is that the almost additive CP orders are precisely those that are compatible with every finite, additive, antisymmetric CP space (a CP space is antisymmetric if it has no equivalences).

(3) The fact that it is generally impossible to complete a joint order follows from Section 3B, provided that one of the marginals is not almost additive. In Section 3C we extend the conclusion, by means of an example, to the case that both marginals are almost additive.

(4) CP orders $\lesssim_0$, $\lesssim_1$ are *commensurate* if and only if they accommodate a certain *partial* joint order. We show (Section 3D) that if $\lesssim_0$, $\lesssim_1$ have positive almost agreeing probability measures, then commensurability and compatibility are equivalent. Some applications are given to the problem of deciding whether or not given marginals have a joint order.

(5) Let Σ stand for the class of CP orders compatible with all the finite, additive ones. We show (Section 3E) that Σ contains infinitely many nonadditive orders, and that all the orders in Σ are mutually compatible (hence that the additive orders are not "closed" with respect to the formation of joint orders). The paper concludes with a test for membership in Σ for CP orders with a positive almost agreeing probability measure: compatibility with *all* finite, additive orders is shown to follow from compatibility with a certain one of them.

C. Our notational conventions are these: capital letters $A, B, \cdots$ stand for sets; $\bar{A}$, $|A|$, and I_A denote, respectively, the complement, cardinality, and indicator function of A. As usual, $A < B$, $A \sim B$ mean not $(B \lesssim A)$ and $(A \lesssim B \wedge B \lesssim A)$, respectively. The notation $(X, \lesssim)$ will denote a CP space in which the algebra of events is the power set of X.

The following property of CP orders will be used without comment (see [13] for the proof):

$$\text{If } A \cap B = C \cap D = \phi, \text{ then } A \lesssim C, \; B \lesssim D \text{ imply } A \cup B \lesssim C \cup D; \text{ if also } A < C \text{ or } B < D, \text{ then } A \cup B < C \cup D.$$

2. Independent joint orders. The content of this section is roughly that a joint order corresponding to many i.i.d. trials of a single experiment has a representing probability measure of *product* type. A qualitative definition for the "independent and identically distributed" property is presented first.

A. Let $I = \{1, 2, \cdots, |I|\}$, where $|I|$ may be infinity. Suppose that $(X_i, \mathscr{F}_i, \lesssim_i)$ is a CP space for each $i \in I$. Let X^* be the cartesian product of all the X_i. For each $A \in \mathscr{F}_i$ let A^i be the set of points in X^* with ith coordinate in A. For each $\alpha \subseteq I$ let $\mathscr{F}(\alpha)$ be the algebra generated by the sets A^i, where A runs through $\mathscr{F}_i$ and i through α. If $m < |I|$, X^m is the cartesian product of $X_1, \cdots, X_m$ and $\mathscr{F}^m = \mathscr{F}(\{1, \cdots, m\})$. Write $\mathscr{F}^*$ for $\mathscr{F}(I)$.

DEFINITION. The $(X_i, \mathscr{F}_i, \lesssim_i)$, $i \in I$, will be said to be *independently compatible* if and only if there is a CP order $\lesssim^*$ on $\mathscr{F}^*$ with the following properties:

(1) $\lesssim^*$ has marginals $\lesssim_i$; that is $A^i \lesssim^* B^i$ iff $A \lesssim_i B$, where $A, B \in \mathscr{F}^i$ and $i \in I$.

(2) If $A \in \mathscr{F}(\alpha)$, $B \in \mathscr{F}(\beta)$, $C \in \mathscr{F}(\gamma)$, $D \in \mathscr{F}(\delta)$, where $\alpha, \beta, \gamma, \delta \subseteqq I$ and $\alpha \cap \beta = \gamma \cap \delta = \phi$, then $A \lesssim^* C$, $B \lesssim^* D$ implies $A \cap B \lesssim^* C \cap D$. If also $A <^* C$ or $B <^* D$, and $\phi <^* C$, $\phi <^* D$, then $A \cap B <^* C \cap D$.

DEFINITION. If for each $i \in I (X_i, \mathscr{F}_i, \lesssim_i)$ is identical to $(X, \mathscr{F}, \lesssim)$, and if $\lesssim^*$ is a CP order on $\mathscr{F}^*$ which satisfies (1), (2), and

(3) $A^i \sim_* A^j$ ($A \in \mathscr{F}$ and $i, j \in I$), then $(X^*, \mathscr{F}^*, \lesssim^*)$ will be said to be an $|I|$-fold i.i.d. product space for $(X, \mathscr{F}, \lesssim)$.

A CP order $\lesssim^*$ on $\mathscr{F}^*$ will be said to be of independent (or of i.i.d.) type according to whether only the first (or both) of (2) and (3) holds. Conditions (2) and (3) have a couple of easy consequences that are needed shortly. The first amounts to the qualitative analogue of exchangeability [4]. For the proof of the second, see [8].

PROPOSITION 1. *Let* $(X^*, \mathscr{F}^*, \lesssim^*)$ *be an* i.i.d. *product space for* $(X, \mathscr{F}, \lesssim)$. *Then for each sequence* $\alpha = \{i_1, \cdots, i_n\}$ *of distinct elements in* I, *and for each permutation* $\{j_1, \cdots, j_n\}$ *of* α,

$$A_1^{i_1} \cap \cdots \cap A_n^{i_n} \sim^* A_1^{j_1} \cap \cdots \cap A_n^{j_n} \qquad A_1, \cdots, A_n \in \mathscr{F}.$$

PROPOSITION 2. *Let* $(X^{mn}, \mathscr{F}^{mn}, \lesssim^{mn})$ *be an mn-fold* i.i.d. *product space for* $(X, \mathscr{F}, \lesssim)$. *Let* $\lesssim^m$ *be the restriction to* $\mathscr{F}^m$ *of* $\lesssim^{mn}$. *Then* $(X^{mn}, \mathscr{F}^{mn}, \lesssim^{mn})$ *is an n-fold* i.i.d. *product space for* $(X^m, \mathscr{F}^m, \lesssim^m)$.

REMARKS. 1. Conditions (2) and (3) are valid when $\lesssim^*$ is induced by an i.i.d. product probability measure on $\mathscr{F}^*$; in this regard, the condition for independence in CP is contained in the condition for independence in quantitative probability.

2. It is not intended that independence (in any intuitive sense) be inferred when the joint order for the experiments $(X_i, \mathscr{F}_i, \lesssim_i)$, $i \in I$, satisfies (2); (2) is meant only as a *necessary* property of a joint order for independent experiments. For a more extensive discussion of axiomatic questions related to independence, see [3], [5], [7] and [8].

B. Assume that X is finite. Denote by X^m the m-fold cartesian product of X with itself, where m is a positive integer.

THEOREM 1. *The following conditions are equivalent*:

(i) $(X, \lesssim)$ *has an n-fold* i.i.d. *product space for every finite n*.
(ii) $(X, \lesssim)$ *is independently compatible with every finite*, *additive* CP *space*.
(iii) $\lesssim$ *is additive*.

THEOREM 2. *Let* $\lesssim^m$ *be a* CP *order on* X^m. *The following conditions are equivalent*:

(i) *For all* $n > m$, $(X, \lesssim)$ *has an n-fold* i.i.d. *product space* $(X^n, \lesssim^n)$ *in which* $\lesssim^n$ *extends* $\lesssim^m$; *that is*, $A \times X^{n-m} \lesssim^n B \times X^{n-m}$ iff $A \lesssim^m B$, *whenever* $A, B \subseteqq X^m$.

(ii) *There is a probability measure P on the power set of X, with corresponding* i.i.d. *product measure* P_m *on the power set of* X^m, *such that P agrees with* $\lesssim$ *and* P_m *with* $\lesssim^m$.

REMARKS. (a) One of the assumptions on which our conclusions are based is that $(X, \lesssim)$ is amenable to many i.i.d. repetitions; this is not an axiom, it being easy to imagine experiments or decision-problems in which repeated trials are uninteresting or nonexistent. Another assumption is that CP orders are complete; the appropriateness of the completeness axiom is discussed in the conclusion. A third assumption is that corresponding events in different trials are exactly equiprobable. We do not know in what form Theorem 1 would survive in the absence of (3); that is, when the necessary properties of a CP model for i.i.d. trials are simply that (2) holds and that the marginals are identical.

(b) Theorems 1 and 2 are false if X is infinite. However, the conclusions may be qualified in an obvious way so as to hold for arbitrary X. For example, if for every n $(X, \lesssim)$ has an n-fold i.i.d. product space, then $\lesssim$ is additive on finite sub-algebras, hence is almost additive on X.

The proof of Theorem 1 is based on the following lemma. For proof of the lemma, see [8]; the line of reasoning is similar to that employed in [10] in connection with the polarizability condition.

LEMMA. *If for each k* $(X, \lesssim)$ *is independently compatible with a* CP *space containing k equiprobable, nonnull, and disjoint events, then* $\lesssim$ *is additive.*

PROOF OF THEOREM 1. The equivalence of (ii) and (iii) is plain from the lemma. Obviously (iii) implies (i). It remains to show that (i) implies (iii). Suppose (the conclusion is trivial otherwise) that there exists $A \subseteqq X$ such that $\phi < A < X$. Note that every k-fold i.i.d. extension $(X^k, \lesssim^k)$ of $(X, \lesssim)$ contains k disjoint, nonnull, equiprobable events: namely, the subsets of X^k with ith coordinate in A^c and all other coordinates in A $(i \leqq k)$. Since by assumption there is at least one such extension, for each k, which is independently compatible with $(X, \lesssim)$, the conclusion follows from the lemma.

PROOF OF THEOREM 2. We show that (i) implies (ii). Assume that (i) is true, and observe that, X being finite, there are only finitely many CP orders on X^{2m}. There is, therefore, a CP order $\lesssim^{2m}$ on X^{2m} which extends $\lesssim^m$, and which for each $n > 2m$ extends to an n-fold i.i.d. product space for $(X, \lesssim)$. By Proposition 2 and Theorem 1, $\lesssim^{2m}$ is additive. By Theorem 7 of [3] (see the remark below), there is a probability measure P_m on X^m such that for all $A, B, C, D \subseteqq X^m$

$$A \times B \lesssim^{2m} C \times D \qquad \text{iff} \quad P_m(A)P_m(B) \leqq P_m(C)P_m(D) .$$

The proof is completed by observing that P_m agrees with $\lesssim^m$, and that P_m is of i.i.d. type. To prove the latter, take $A_1, \cdots, A_m \subseteqq X$. By Proposition 1,

$$(A_1 \times \cdots \times A_m) \times X^m \sim^{2m} (A_1 \times X^{m-1}) \times (A_2 \times \cdots \times A_m \times X) ,$$

whence

$$P_m(A_1 \times \cdots \times A_m) = P_m(A_1 \times X^{m-1}) P_m(A_2 \times \cdots \times A_m \times X) .$$

The conclusion follows by repeating the argument $(m - 1)$ times.

REMARK. In light of the fact that $\lesssim^{2m}$ is an additive CP order, all that must be checked, in order to apply the theorem in [3], is the following: if $A_1, \cdots, A_n$ and $B_1, \cdots, B_n$ are sets in X^m, and if α, β are permutations of $\{1, \cdots, n\}$, then the conditions $A_i \times B_i >^{2m} \phi$ and $A_i \times B_i \lesssim^{2m} A_{\alpha_i} \times B_{\beta_i}$ (all $i < n$) imply $A_{\alpha_n} \times B_{\beta_n} \lesssim^{2m} A_n \times B_n$. But suppose to the contrary. With $C = \times_1^n A_i \times B_i$, $D = \times_1^n A_{\alpha_i} \times B_{\beta_i}$, and $\lesssim^{2mn}$ a CP order of i.i.d. type on X^{2mn} extending $\lesssim^{2m}$, you get the contradictory conclusions $C <^{2mn} D$ (from the independence property) and $C \sim^{2mn} D$ (by Proposition 1).

C. Without the finiteness condition on X, Theorems 1 and 2 are false (at least for measures taking values in the standard reals). There are a number of technical hypotheses on the strength of which one could proceed to related results for the infinite case. The one we mention here is based on the CP analogue, due to Villegas [15], of countable additivity. It is defined as follows:

A CP order $\lesssim$ on a *σ-algebra* $\mathscr{F}$ *is monotonely continuous* iff for each increasing sequence $A_1 \subseteqq A_2 \subseteqq \cdots$ of events in $\mathscr{F}$, and each B in $\mathscr{F}$, $B \geqq A_i$ for all i implies $B \geqq \bigcup_i A_i$.

The general inapplicability of Theorems 1 and 2 in the infinite case persists (though not obviously) even when the CP orders are monotonely continuous. Theorem 3 below is obtained by passing to the *infinite*-dimensional product space. The notation is the same as before, except that all algebras assumed to be *σ-algebras*; specifically, $\mathscr{F}$ is a σ-algebra of subsets of an arbitrary set X, X^* is the countably infinite cartesian product of X with itself, and $\mathscr{F}^*$ is the minimal σ-algebra in X^* generated by the A^i, $A \in \mathscr{F}$ and i a positive integer.

THEOREM 3. *Suppose that $\lesssim^*$ on $\mathscr{F}^*$ is a monotonely continuous joint* CP *order for infinitely many* i.i.d. *copies of $(X, \mathscr{F}, \lesssim)$. Then there is a unique probability measure P^* on $\mathscr{F}^*$ agreeing with $\lesssim^*$, and P^* is an* i.i.d. *product measure.*

It follows that there is a one-to-one correspondence between probability measures P agreeing with $\lesssim$ and CP orders $\lesssim^*$ on $\mathscr{F}^*$ extending $\lesssim$ according to the axioms for i.i.d. trials. Choosing a particular representing measure P is equivalent to choosing a particular infinite-dimensional i.i.d. extension for $\lesssim$.

The following is an outline of the proof for Theorem 3 (see [8] for details). Assume that there is a set $A \in \mathscr{F}$ such that $\phi < A < X$ (the theorem is trivial otherwise). The first step is to show that then $(X^*, \mathscr{F}^*, \lesssim^*)$ is *atomless*; that is, that every nonnull event in $\mathscr{F}^*$ contains a strictly less probable event that is also nonnull. By a theorem of Villegas [15], every atomless, monotonely continuous CP space has a unique agreeing probability measure which is countably additive. It follows that $\lesssim^*$ has a unique representing measure P^*. To show that P^* is a product measure, one observes from the axioms for independence that there is a single-valued function g, defined on some domain D in the plane, such that

$$P^*(A \cap B) = g(P^*(A), P^*(B))$$

whenever $A \in \mathscr{F}(\alpha)$, $\beta \in \mathscr{F}(\beta)$, and α, β are disjoint sets of positive integers. It turns out that in fact: (1) D is the whole unit square; (2) $g(0, v) = 0$ and $g(1, v) = v$ for all v; (3) $g(u + v, w) = g(u, w) + g(v, w)$ whenever $u + v \leqq 1$; and (4) g is symmetric and continuous in each variable. It ensues that $g(u, v) = uv$, as claimed.

3. Joint orders without independence.

A. It follows from the previous section that if $(X_0, \lesssim_0)$ is a CP space in which X_0 is finite and $\lesssim_0$ nonadditive, and if $(X_1, \lesssim_1)$ contains sufficiently many disjoint, equivalent events, then any attempt to form a joint CP order for the two leads to a violation either of the axioms for CP or of the independence property. In fact, more is true. Nonexistence of joint orders is a pervasive phenomenon in CP. It does not rely on independence or on the presence of equivalent events; it can occur in the absence of both. The purpose of this section is to prove this fact, and to present one necessary and a number of sufficient conditions in order that specified marginals have a joint CP order. Though some of the results may be generalized, they are essentially for finite CP spaces. All sample spaces in this section are finite, and all algebras are power sets.

Terminology. CP spaces $(X_i, \lesssim_i)$, $i \in I$, will be said to be *compatible* if they have a joint order; that is, if there is a CP order $\lesssim$ on the cartesian product of the X_i such that $A^i \lesssim B^i$ if and only if $A \lesssim_i B$ $(A, B \subseteqq X_i$ and $i \in I)$.

CP spaces $(X_0, \lesssim_0)$, $(X_1, \lesssim_1)$ will be said to be *commensurate* if a certain partial joint order exists. Let C be the class of sets in $X_0 \times X_1$ of the form $A \times X_1$ or $X_0 \times B$, where $A \subseteqq X_0$ and $B \subseteqq X_1$. A *scale* for $(X_0, \lesssim_0)$, $(X_1, \lesssim_1)$ is a CP like ordering $\lesssim$ of C with marginals $\lesssim_0$, $\lesssim_1$; that is,

(1) $A \times X_1 \lesssim B \times X_1$ iff $A \lesssim_0 B$ $(A, B \subseteqq X_0)$, $X_0 \times A \lesssim X_0 \times B$ iff $A \lesssim_1 B$ $(A, B \subseteqq X_1)$;

(2) $A \lesssim B$, $B \lesssim C$ implies $A \lesssim C$ $(A, B \in C)$;

(3) If $A, B, C, D \in C$ are such that $A \cap B = C \cap D = \phi$ and $A \lesssim C$, $B \lesssim D$, then $A \cup B \lesssim C \cup D$; if also $A < C$ or $B < D$, then $A \cup B < C \cup D$.

(4) $\lesssim$ is complete: $A \lesssim B$ or $B \lesssim A$ $(A, B \in C)$.

$(X_0, \lesssim_0)$, $(X_1, \lesssim_1)$ will be said to be commensurate if they have a scale. It is easy to see that compatible CP spaces are commensurate.

B. We proceed to derive a necessary condition for compatibility in the case that one of the CP orders is not almost additive. Recall that we assume that all sets are finite.

Suppose that $(X_0, \lesssim_0)$ is a CP space, where $\lesssim_0$ is not almost additive. Suppose that $(X_1, \lesssim_1)$ is also a CP space, and that $\lesssim$ is a scale for the two. We will define a number d_0 that depends on $\lesssim_0$, a number λ that depends on $\lesssim_1$, and a number Δ that depends on $\lesssim_0$, $\lesssim_1$ and $\lesssim$. It will turn out that under the right conditions, d_0, λ and Δ are simply related.

Because $\lesssim_0$ is not almost additive, there are, by the algebraic condition for almost additivity given in [10], a positive number d, and sets $A_1, \cdots, A_l, B_1, \cdots, B_l$ in X_0, such that

$$A_i \lesssim_0 B_i\,, \qquad i = 1, \cdots, l$$
$$(1) \qquad \sum_{i=1}^{l} I_{A_i} = \sum_{i=1}^{l} I_{B_i} + I_{X_0}$$
$$\sum_{i=1}^{l} |A_i| = \frac{1}{d}\,;$$

d_0 is the largest such d.

DEFINITION. Let $M_0, \cdots, M_l$ be a sequence of sets in X_1 satisfying

$$\phi \sim_1 M_0 <_1 \cdots <_1 M_l \sim_1 X_1\,.$$

(a) If $A \gtrsim_1 M_i$, $B \gtrsim_1 M_j$ imply $A \cup B \gtrsim_1 M_{i+j}$ whenever $i + j \leqq l$ and $A, B \subseteqq X_1$ are *disjoint*, $\{M_i\}_i$ is said to be a *lower series*.

(b) If $A \lesssim_1 M_i$, $B \lesssim_1 M_j$ imply $A \cup B \lesssim_1 M_{i+j}$ whenever $i + j \leqq l$ and $A, B \subseteqq X_1$, $\{M_i\}_i$ is said to be an *upper series*.

REMARK. Think of the M_i as intervals, with M_{i+j} the "sum" of M_i and M_j. The content of (a), colloquially, is that the "width" of a sum is no greater than the sum of the widths. The content of (b) is that the width of a sum is no less than the sum of the widths.

Suppose that λ is chosen so that there is a lower series $M_0, \cdots, M_\lambda$, and an upper series $N_0, \cdots, N_\lambda$, in $(X_1, \lesssim_1)$. These, together with $\lesssim$, give rise to a pair of integer-valued functions m, n on the power set of X_0:

$$m(A) = \max\{i : X_0 \times M_i \lesssim A \times X_1\}$$
$$n(A) = \min\{i : X_0 \times N_i \gtrsim A \times X_1\}\,, \qquad A \subseteqq X_0\,.$$

Δ is now defined by

$$\Delta = \max_{x \in X_0} (n(x) - m(x))\,.$$

Each of the functions m, n is a one-sided version of a probability measure (see (2) and (3) below). If the M_i, N_i are well interleaved, in a certain sense, the two functions together form a probability-like approximation to $\lesssim_0$. Specifically, to each $A \subseteqq X_0$ there attaches the interval $(m(A), n(A))$, of "typical" width Δ. The idea is that because $\lesssim_0$ is not almost additive, the ratio of Δ to λ cannot be too small.

THEOREM 4. *Suppose that $N_i \lesssim_1 M_i$, $i = 1, \cdots, \lambda$. If $\lesssim$ is a scale for $(X_0, \lesssim_0)$, $(X_1, \lesssim_1)$, where $\lesssim_0$ is not almost additive, and if m, n, Δ, and d_0 are defined as above, then $\Delta/\lambda \geqq d_0$.*

PROOF. The functions m, n have the following properties, where $A, B \subseteq X_0$:

$$\text{If} \quad A \cap B = \phi\,, \quad \text{then} \quad m(A \cup B) \geqq m(A) + m(B)\,. \tag{2}$$

$$n(A \cup B) \leqq n(A) + n(B)\,. \tag{3}$$

$$\text{If} \quad A \lesssim_0 B\,, \quad \text{then} \quad m(A) \leqq m(B) \quad \text{and} \quad n(A) \leqq n(B)\,. \tag{4}$$

$$m(A) \leqq n(A)\,. \tag{5}$$

Properties (2) and (3) are immediate from the definition of lower and upper series, respectively. Property (4) follows from the fact that the M_i and N_i are increasing with respect to $\lesssim_1$. To prove (5), let $i = m(A)$ and $j = n(A)$; then $X_0 \times M_i \lesssim A \times X_1 \lesssim X_0 \times N_j$, so $N_i \lesssim_1 M_i$ implies $N_i \lesssim_1 N_j$. Since the N_i are strictly increasing, it follows that $i \leqq j$.

Let $A_1, \cdots, A_l, B_1, \cdots, B_l$ be sets in X_0 satisfying (1) for $d = d_0$. From (2) through (5) and the first condition in (1) it follows that

$$(\forall\, i \leqq l)\ \textstyle\sum_{x \in A_i} m(x) \leqq m(A_i) \leqq m(B_i) \leqq n(B_i) \leqq \sum_{x \in B_i} n(x)\,,$$

hence

$$\textstyle\sum_{i=1}^{l} \sum_{x \in A_i} m(x) \leqq \sum_{i=1}^{l} \sum_{x \in B_i} n(x)\,.$$

From the second condition in (1),

$$\textstyle\sum_{i=1}^{l} \sum_{x \in A_i} n(x) = \sum_{i=1}^{l} \sum_{x \in B_i} n(x) + \sum_{x \in X_0} n(x)\,.$$

Substituting above gives

$$\textstyle\sum_{i=1}^{l} \sum_{x \in A_i} m(x) \leqq \sum_{i=1}^{l} \sum_{x \in A_i} n(x) - \sum_{x \in X_0} n(x)\,,$$

or

$$\textstyle\sum_{i=1}^{l} \sum_{x \in A_i} (n(x) - m(x)) \geqq \sum_{x \in X_0} n(x) \geqq n(X_0) = \lambda\,;$$

that is, $\Delta \sum_{i=1}^{l} |A_i| \geqq \lambda$. □

Here is an example of sequences $\{M_i\}_i$, $\{N_i\}_i$ satisfying the hypotheses of the theorem. Suppose (just for the sake of the example) that $\lesssim_1$ is positive; that is, $x >_1 \phi$ for each $x \in X_1$. Choose M_i among the *most* probable sets in X_1 of cardinality i, and N_i among the *least* probable. That is,

$$\begin{aligned} &|M_i| = i\,, \qquad |N_i| = i & & 0 \leqq i \leqq |X_1| \\ &N_i \lesssim_1 A \lesssim_1 M_i & & A \subseteq X_1,\ |A| = i\,. \end{aligned} \tag{6}$$

The M_i and N_i are strictly increasing, because $\lesssim_1$ is positive. It is easy to complete the argument that $\{M_i\}_i$ is a lower series and $\{N_i\}_i$ an upper series. The condition $N_i \lesssim_1 M_i$ is evident from the definition.

The choice of M_i, N_i in accordance with (6) is attractive in that Δ then has an upper bound which depends only on $\lesssim_1$, and not at all on $\lesssim$. The necessary condition in Theorem 4 can then be made a function of the properties of the marginals alone: specifically, of d_0 and a number r_1 associated with $(X_1, \lesssim_1)$ by the following prescription: r_1 is the smallest integer r satisfying

$$|A| < |B| - r \quad \text{implies} \quad A <_1 B \qquad A, B \subseteq X_1 .$$

(r_1 measures the "uniformity" of $\lesssim_1$; it is zero when $\lesssim_1$ is the uniform distribution on X_1.)

COROLLARY 1. *If* $(X_0, \lesssim_0)$, $(X_1, \lesssim_1)$ *are commensurate, and if* λ *is the number of* $x \in X_1$ *such that* $x >_1 \phi$, *then* $2(r_1 + 1)/\lambda \geqq d_0$.

PROOF. It is enough to prove the inequality for the case that $\lesssim_1$ is positive, the extension to the general case being obvious.

Suppose, then, that $\lesssim_1$ is positive, and that $\lesssim$ is a scale for $(X_0, \lesssim_0)$, $(X_1, \lesssim_1)$. Choose $M_i, N_i \subseteq X_1$ satisfying (6), $i = 0, \cdots, |X_1|$, and define functions m, n on the power set of X_0 as before. From the theorem, $\Delta/|X_1| \geqq d_0$. It remains only to prove that $\Delta \leqq 2(r_1 + 1)$.

To this end, fix $A \subseteq X_0$, and choose $B \subseteq X_1$ satisfying

$$\begin{aligned} &X_0 \times B \lesssim A \times X_1 \\ &C >_1 B \quad \text{implies} \quad X_0 \times C > A \times X_1 \qquad C \subseteq X_1 . \end{aligned}$$

Then for all $C \subseteq X_1$,

$$\begin{aligned} |C| \leqq |B| - r_1 - 1 \quad &\text{implies} \quad X_0 \times C < A \times X_1 \\ |C| \geqq |B| + r_1 + 1 \quad &\text{implies} \quad X_0 \times C > A \times X_1 . \end{aligned}$$

In particular, $X_0 \times M_{|B|-r_1-1} \lesssim A \times X_0 \lesssim X_0 \times N_{|B|+r_1+1}$, that is, $m(A) \geqq |B| - r_1 - 1$ and $n(A) \leqq |B| + r_1 + 1$. Subtracting the first of these inequalities from the second gives

$$n(A) - m(A) \leqq 2(r_1 + 1) . \qquad \square$$

The following may be read directly from the corollary.

(1) There are incommensurate (and hence incompatible) CP orders; for example, if $\lesssim_0$ is not almost additive, it is incommensurate with the uniform distribution on n atoms whenever $n > 2d_0^{-1}$. In fact:

(2) If for every $\varepsilon > 0$ there is a finite CP space $(X_1, \lesssim_1)$, commensurate with $(X_0, \lesssim_0)$ and satisfying

$$\frac{r_1 + 1}{\lambda} < \varepsilon , \quad \lambda = |\{x \in X_1 : x >_1 \phi\}| ,$$

then $\lesssim_0$ is almost additive.

The last conclusion, together with its converse, constitutes

THEOREM 5. $\lesssim_0$ *is almost additive if and only if* $(X_0, \lesssim_0)$ *is compatible with every finite, additive, antisymmetric* CP *space.*

The proof requires the following construction, used repeatedly in the sequel:

DEFINITION. Let $\lesssim_1, \cdots, \lesssim_n$ be CP orders on a set X. Their *lexicographic* *composition* (denoted $L(\lesssim_1, \cdots, \lesssim_n)$) is the ordering $\lesssim$ on X defined by

$$A \sim_i B\,, \quad i = 1, \cdots, j-1\,, \quad \text{and} \quad A <_j B \quad \text{implies} \quad A < B$$
$$A \sim_i B\,, \quad i = 1, \cdots, n\,, \quad \text{implies} \quad A \sim B\,.$$

It is easy to verify that $L(\lesssim_1, \cdots, \lesssim_n)$ is a CP order.

PROOF. We wish to prove that the condition is necessary. Suppose that $\lesssim_0$ is almost additive, with almost agreeing probability measure P. Suppose that $(X_1, \lesssim_1)$ is finite, additive and anti-symmetric. There is a CP order $\lesssim$ on $X_0 \times X_1$ with marginals $\lesssim_P$, $\lesssim_1$, where $\lesssim_P$ is the CP order on X_0 induced by P. There is another CP order $\lesssim'$ on $X_0 \times X_1$ with marginal $\lesssim_0$ on X_0 (the marginal on X_1 is arbitrary; it may be the trivial CP order corresponding to unit mass on one atom). $L(\lesssim, \lesssim')$ is a CP order on $X_0 \times X_1$ with marginals $\lesssim_0$, $\lesssim_1$, as required. (Antisymmetry of $\lesssim_1$ ensures that the X_1-marginal of $\lesssim$ is unaffected by the lexicographic composition.) □

REMARK. A CP space is almost additive if and only if all its finite subexperiments are almost additive. Though we will not prove this here, we mention it because it entails that Theorem 4 holds for arbitrary X_0, finite or not. It entails also that Savage's condition for almost additivity ([13], page 34) can be inferred from Theorem 4. The condition to which we refer is that the CP space in question (call it $(X_0, \lesssim_0)$) has an n-fold "almost uniform" partition for infinitely many values of n. Pick any such n and denote by $(X_1, \lesssim_1)$ the CP space generated by the corresponding n-fold partition, so that (trivially) $(X_0, \lesssim_0)$, $(X_1, \lesssim_1)$ are commensurate; the requirement that the partition be "almost uniform" means $r_1 = 0$. To see now that $\lesssim_0$ is necessarily almost additive, let n increase without bound and apply the second comment following Corollary 1.

We remark also that the sufficiency part of Theorem 5 can, with a little extra effort, be obtained from Savage's result.

C. We shall prove, by means of a single example, that incommensurability occurs in the almost additive case, and that the antisymmetry condition in Theorem 5 is essential. It is enough to exhibit CP orders $\lesssim_0$, $\lesssim_1$ such that $\lesssim_1$ is additive, $\lesssim_0$ is almost additive, and the two are incommensurate. Let $X = \{a, b, c, d, e\}$. Consider the following family of equivalences:

(7) $$c \sim ab$$

(8) $$ac \sim d$$

(9) $$e \sim abc$$

(10) $$cd \sim ae$$

(11) $$be \sim acd\,.$$

We need only prove

(a) there are two CP orders on X which extend (7) through (11); one is additive, and the other, almost additive;

(b) if $\lesssim$ is a scale on $X \times X$ with marginals satisfying (7) through (11), then for all $A \subseteq X$ $A \times X \sim X \times A$.

It will follow from (b) that the orders promised in (a) are incommensurate.

Proof of (a). The problem is to exhibit $\lesssim_0$, $\lesssim_1$ satisfying (7) through (11). There is a unique probability measure P agreeing with (7) through (11), given, up to a normalizing constant, by $P(a) = 1$, $P(b) = 2$, $P(c) = 3$, $P(d) = 4$, $P(e) = 6$. Let $\lesssim_1$ be the CP order induced by P, and let $\lesssim_0$ be the CP order defined by the following chain:

$$\phi <_0 a <_0 b <_0 c \sim_0 ab <_0 ac \sim_0 d <_0 ad <_0 bc <_0 e \sim_0 abc \sim_0 bd$$
$$<_0 cd \sim_0 ae \sim_0 abd <_0 be \sim_0 acd \,. \tag{12}$$

$\lesssim_0$ differs from $\lesssim_1$ only in the relations $ad \sim_1 bc$, $ad <_0 bc$, and in the corresponding relations for the complementary sets. In particular, $\lesssim_0$ is almost additive (the almost agreeing probability measure is unique and identical to P) and satisfies (7) through (11).

Proof of (b). We temporarily abuse our notation by writing A for $A \times X$ and A' for $X \times A$. Suppose that $\lesssim$ is a scale with marginals satisfying (7) through (11). Then

$$be \sim b'e' \tag{13}$$

$$acd \sim a'c'd' \tag{14}$$

$$ac \sim a'c' \tag{15}$$

$$d \sim d' \,. \tag{16}$$

To verify (13), notice that if $be < b'e'$, then also (by (11)) $acd < a'c'd'$; hence $be \cup acd < b'e' \cup a'c'd'$, or $X \times X < X \times X$, which is nonsense. Thus $be < b'e'$ is false; by symmetry, also $be > b'e'$ is false, and the conclusion follows. The proofs of the other three are similar.

Now suppose that $b < b'$. From (13) and (9), $b < b'$ implies $e > e'$ implies $abc > a'b'c'$ implies $ac > a'c'$. The last inequality violates (15), hence $b \gtrsim b'$. Similarly, $b \lesssim b'$, hence $b \sim b'$. From (13) again, $e \sim e'$, hence from (9), $abc \sim a'b'c'$; from (7), $c \sim c'$ and $ab \sim a'b'$. The last relation, acting with $b \sim b'$, gives $a \sim a'$. In summary, $x \sim x'$ for all $x \in X$, as claimed.

If $\lesssim$ were a scale for $\lesssim_0$, $\lesssim_1$, then it would follow that $bc \sim b'c' \sim a'd' \sim ad$; that is, that $ad \sim_0 bc$, in violation of (12). Thus, as claimed, $(X, \lesssim_0)$ and $(X, \lesssim_1)$ are incommensurate.

Remark. Much use has been made of the equivalences in $\lesssim_0$, $\lesssim_1$. In fact, one can find incommensurate $\lesssim_0$, $\lesssim_1$ which are *antisymmetric* as well as almost additive. The example is less transparent than the one given here, and the calculations are lengthier, so we omit it.

D. The CP marginals shown to be incompatible in Section 3C are also incommensurate. This is not exceptional. In this subsection we show that the equivalence between incompatibility and incommensurability prevails for a certain broad class of marginals. The result is of some value for the problem of showing that given orders are incompatible, it being at least occasionally the case that scales are easy to guess and easy to describe; an application is given in subsection E.

A CP order will be said to be *positively* almost additive if it has a *positive* almost agreeing probability measure. The main result is

THEOREM 6. *If* $\lesssim_0$, $\lesssim_1$ *are positively almost additive, then* $(X_0, \lesssim_0)$, $(X_1, \lesssim_1)$ *are compatible if and only if they are commensurate.*

Actually a little more will be proved. In the following sequence of steps, we describe the substance of the proof. For notation we use X for the cartesian product $X_0 \times X_1$, $\mathscr{C}$ for the class of sets in X of the form $A \times X_1$ or $X_0 \times B$ ($A \subseteqq X_0$, $B \subseteqq X_1$), and $\mathscr{C}^*$ for the set of vectors $I_B - I_A$ ($A, B \in \mathscr{C}$). When $\lesssim'$, $\lesssim''$ are CP orders on the same set, such that for all subsets A, B $A <' B$ implies $A <'' B$, we say that $\lesssim'$ *almost agrees* with $\lesssim''$; this extends the terminology of almost agreement in a natural way.

Clearly only sufficiency needs proof. Assume that $\lesssim$ is a scale for $\lesssim_0$, $\lesssim_1$.

STEP 1. Let π stand for the set of probability measures on X with marginals almost agreeing with $\lesssim_0$, $\lesssim_1$. Let S be the set of pairs (A, B), $A, B \subseteqq X$, such that $P(A) = P(B)$ for all $P \in \pi$.

CLAIM. $(A, B) \in S$ implies $I_B - I_A \in \mathscr{C}^*$.

That is, the events that are equiprobable for all choices of $P \in \pi$ essentially belong to $\mathscr{C}$. It is crucial to the validity of the claim that there are positive probability measures in π; that is, that $\lesssim_0$, $\lesssim_1$ are *positively* almost additive.

STEP 2. By the finiteness of X_1 there is a sequence $P_1, \cdots, P_n \in \pi$ such that for all $A, B \subseteqq X$, either $(A, B) \in S$, or else $P_i(A) \neq P_i(B)$ for some $i \leqq n$. Let $\lesssim^i$ be the CP order on X induced by P_i. Define $\lesssim_S = L(\lesssim^1, \cdots, \lesssim^n)$, where L stands for the lexicographic composition (see subsection B). $\lesssim_S$ has the following two properties: first, its marginals almost agree with $\lesssim_0$, $\lesssim_1$; second, for all $A, B \subseteqq X$, $A \sim_S B$ iff $(A, B) \in S$.

STEP 3. $\lesssim_S$ is not generally a joint order for $(X_0, \lesssim_0)$, $(X_1, \lesssim_1)$; its marginals may have equivalence where $\lesssim_0$, $\lesssim_1$ do not. Our intention is to modify $\lesssim_S$ so that these discrepancies disappear. In general, any perturbation in the marginals of a joint order must propagate through the order, so that the axioms for CP remain satisfied. What makes the theorem work is that for our problem, the propagation is confined to S:

FACT. If $\lesssim_0$, $\lesssim_1$ have at least one joint CP order $\lesssim'$, they have also a joint CP order $\lesssim''$ which agrees with $\lesssim_S$ outside of S.

PROOF. $\lesssim'' = L(\lesssim_S, \lesssim')$ has the asserted property.

STEP 4. By Step 1, if $(A, B) \in S$ then essentially $A, B \in \mathscr{C}$. By hypothesis, $\lesssim$ is an ordering of $\mathscr{C}$ with the required marginals. Together these suggest that we modify $\lesssim_S$ on S according to $\lesssim$. Consider the order $\lesssim_*$ prescribed on X in the following way:

(1) If $(A, B) \in S$, choose $A', B' \in \mathscr{C}$ so that $I_B - I_A = I_{B'} - I_{A'}$; define $A \lesssim_* B$ iff $A' \lesssim B'$.

(2) If $(A, B) \notin S$, define $A \lesssim_* B$ iff $A \lesssim_S B$.

CLAIM. (1) $\lesssim_*$ is well defined; that is, if $I_{B'} - I_{A'} = I_{B''} - I_{A''}$, where $A', B'\ A'', B'' \in \mathscr{C}$, then $A' \lesssim B'$ iff $A'' \lesssim B''$.

(2) $\lesssim_*$ is a CP order.

The proof of the theorem is concluded as soon as our claims are proved. The details may be found in [9].

A slightly stronger version of Theorem 6 is accessible with only a little extra effort. It follows upon the observation that the construction of $\lesssim_*$ does not make use of the completeness of $\lesssim$ on $\mathscr{C}$; in fact, it is sufficient that $\lesssim$ be defined only for those pairs $A, B \in \mathscr{C}$ for which $(A, B) \in S$. For each such pair, one of three things can happen:

(a) One of A, B belongs to $\mathscr{F}_0$; the other to $\mathscr{F}_1$.

(b) There exists $i \in \{0, 1\}$, $C \in \mathscr{F}_{1-i}$, such that both of A, B belong to $\mathscr{F}_i$, and (A, C), (B, C) belong to S.

(c) Both of A, B belong to $\mathscr{F}_i$, where $i \in \{0, 1\}$, but (b) fails.

Here $\mathscr{F}_i$ is the class of sets A^i, $A \subseteqq X_i$, where, as before, A^i is the set of $x \in X$ with ith coordinate in A.

Let T stand for set of pairs (A, B) satisfying (a) or (b).

THEOREM 7. *Suppose that there is an order $\lesssim$, defined for all pairs in T, agreeing with $\lesssim_0$, $\lesssim_1$ where defined, and such that*

(1) *$A \lesssim B$, $B \lesssim C$ implies $A \lesssim C$*

(2) *$A \cap B = C \cap D = \phi$ and $A \lesssim C$, $B \lesssim D$ implies $A \cup B \lesssim C \cup D$; if also $A < C$ or $B < D$, then $A \cup B < C \cup D$.*

Then $(X_0, \lesssim_0)$, $(X_1, \lesssim_1)$ are compatible.

PROOF. It is enough, by Theorem 6, to show that $\lesssim$ extends from T to a scale on $\mathscr{C}$. Suppose that $\lesssim_S$ is the CP order on X defined before. Define $\lesssim'$ on $\mathscr{C}$ as follows:

$$\lesssim' \text{ has marginals } \lesssim_0, \lesssim_1$$
$$A <_S B \text{ implies } A <' B$$
$$(A, B) \in T \text{ implies } A \lesssim' B \text{ iff } A \lesssim B .$$

(The effect of the last condition is to modify $\lesssim_S$ for those pairs $A, B \in \mathscr{C}$ such that $A \sim_S B$.) The three conditions are consistent. The proof that $\lesssim'$ has all the properties of a scale is straightforward. □

Denote by $\lesssim_i | T$ the restriction of $\lesssim_i$ to pairs $A, B \subseteq X_i$ satisfying $(A^i, B^i) \in T$. It follows from the theorem that if $\lesssim_0 | T$, $\lesssim_1 | T$ extend to compatible CP orders on X_0, X_1 respectively, then $(X_0, \lesssim_0)$, $(X_1, \lesssim_1)$ are compatible. By way of application, let K_i be the set of $A \subseteq X_i$ such that $P(A) =$ constant as P runs through all probability measures on X_i almost agreeing with $\lesssim_i$. Suppose that $i, j \in \{0, 1\}$ and $A \subseteq X_i$, $B \subseteq X_j$. A necessary condition in order that $(A^i, B^j) \in T$ is that $A \in K_i$, $B \in K_j$. Hence

COROLLARY 2. *If $\lesssim_0$ restricted to pairs from K_0, and $\lesssim_1$ restricted to pairs from K_1, extend to compatible* CP *orders on X_0, X_1 respectively, then $(X_0, \lesssim_0)$, $(X_1, \lesssim_1)$ are compatible.*

APPLICATION. If $\lesssim_0$ is the almost additive ordering on five atoms constructed in [10], then $\lesssim_0$ is positively almost additive, and K_0 is an algebra with three atoms. Since all CP orders on four or fewer atoms are additive, it follows that $\lesssim_0$ on K_0 extends to an additive CP order on X_0, hence, by Corollary 2, that $(X_0, \lesssim_0)$ is compatible with every finite, additive CP space.

REMARK. It is not known how far, if at all, Theorem 6 extends beyond the case that both marginals are positively almost additive.

E. The purpose of this subsection is to balance the negative results of Sections 3B and 3C by exhibiting a large class of mutually compatible CP orders. Specifically, let Σ stand for the class of finite CP spaces compatible with all the additive ones; we shall prove some simple properties of Σ.

The following are immediate:

(a) All CP orders in Σ are almost additive.
(b) There are almost additive CP orders which do not belong to Σ.
(c) There are nonadditive CP orders which *do* belong to Σ.

In connection with the last, recall from Section 3D that the almost additive ordering in [10] belongs to Σ; more generally, by Corollary 2, $(X_0, \lesssim_0)$ belongs to Σ whenever $\lesssim_0$ is positively almost additive and the restriction of $\lesssim_0$ to pairs from K_0 extends to an additive order on X_0.

The main property of Σ in connection with the formation of joint orders is

(d) Every n-tuple of CP orders from Σ has a joint order, and in fact, a joint order which also belongs to Σ.

The proof is based on the following proposition; recall from Section D the definition for almost agreement of CP orders.

PROPOSITION. *Let $\lesssim_i$, $\lesssim_i'$ be* CP *orders on X_i such that $\lesssim_i'$ almost agrees with $\lesssim_i$, $i = 1, \cdots, n$. If for each i there is a joint* CP *order $\lesssim^{(i)}$ on $X_1 \times \cdots \times X_n$ with marginals $\lesssim_j'$ on X_j $(j \neq i)$ and $\lesssim_i$ on X_i, then $(X_1, \lesssim_1), \cdots, (X_n, \lesssim_n)$ are compatible.*

The proposition is proved by noticing that $L(\lesssim^{(1)}, \cdots, \lesssim^{(n)})$ is a CP order with marginals $\lesssim_1, \cdots, \lesssim_n$.

PROOF OF (d). Suppose that $(X_i, \lesssim_i) \in \Sigma$, $i = 1, \cdots, n$. Write $E(k)$ for the CP space corresponding to the uniform distribution on $k!$ atoms. The first step is to show that $(X_1, \lesssim_1), \cdots, (X_n, \lesssim_n)$, $E(k)$ are compatible; this is done by taking for $\lesssim_i'$ any additive CP order on X_i almost agreeing with $\lesssim_i$ (that there is at least one follows from (a)), observing that $(X_i, \lesssim_i)$, $(X_j, \lesssim_j')$, $j \neq i$, $E(k)$ are compatible (since $\lesssim_i$ belongs to Σ), and then applying the proposition. It ensues that for each k there is a joint CP order for $(X_i, \lesssim_i)$, $i = 1, \cdots, n$, which is compatible with $E(k)$. Consider the sequence of such orders, indexed by k. At least one element of the sequence (call it $\lesssim^*$) occurs infinitely often, there being only finitely many distinct CP orders on a finite set. Since each finite, additive CP space is a subspace of $E(k)$ for all k sufficiently large, it follows that $\lesssim^*$ belongs to Σ. □

Note that Σ is the largest class of mutually compatible CP orders containing all the additive ones. The *antisymmetric* orders in Σ (of which the order in [10] is an example) are especially amenable to the formation of joint orders:

(e) The antisymmetric orders in Σ are compatible with *all* almost additive ones, in Σ or not.

PROOF. Suppose that $\lesssim_0$ belongs to Σ and that $\lesssim_1$ is almost additive. Then there is a joint order $\lesssim'$ with marginals $\lesssim_0$, $\lesssim_1'$ where $\lesssim_1'$ is additive and almost agrees with $\lesssim_1$. Let $\lesssim''$ be any CP order at all on $X_0 \times X_1$ with marginal $\lesssim_1$ on X_1. If $\lesssim_0$ is antisymmetric, then $L(\lesssim', \lesssim'')$ has marginals $\lesssim_0$, $\lesssim_1$, as required. □

The next result is an application of Theorem 7. Its impact is that a positively almost additive CP order is compatible with *all* the finite, additive CP spaces, provided that it is compatible with a certain one of them.

(f) If $\lesssim_0$ on X_0 is positively almost additive, then a necessary and sufficient condition in order that it belong to Σ is that there exist an *additive* CP order $\lesssim_0'$ on X_0, almost agreeing with $\lesssim_0$ and such that $(X_0, \lesssim_0)$, $(X_0, \lesssim_0')$ are compatible.

PROOF. Only sufficiency needs proof. Assume that there is available a CP order $\lesssim_0''$ on $X_0 \times X_0$ with marginal $\lesssim_0$ on the first coordinate and $\lesssim_0'$ on the second, where $\lesssim_0'$ is additive and almost agrees with $\lesssim_0$. We wish to show that $(X_0, \lesssim_0)$, $(X_1, \lesssim_1)$ are compatible whenever $\lesssim_1$ is additive. To this end let T be the class of pairs (A, B), $A, B \subseteq X_0 \times X_1$, defined in connection with Theorem 7, and construct $\lesssim$ for pairs in T in the following way: first, $\lesssim$ is to agree with $\lesssim_0$, $\lesssim_1$; second, for pairs in T of the form $(A \times X_1, X_0 \times B)$, where $A \subseteq X_0$ and $B \subseteq X_1$, $A \times X_1 \lesssim X_0 \times B$ iff $A \times X_0 \lesssim_0'' X_0 \times A$.

The proof is completed by verifying that $\lesssim$ satisfies the hypotheses of Theorem 7; the details can be found in [9]. □

REMARK. It seems that the almost additive CP orders *not* in Σ are those with well-specified almost agreeing probability measure. We have no example of a nonadditive CP order in Σ with a *unique* almost agreeing probability measure.

4. Conclusion. As shown, one of the properties of the CP axioms is that there are incompatible marginals. Since this is not a property of the additive theory, it might seem (at least to one brought up in the quantitative tradition) problematical in *any* theory of probability. We discuss below two ways in which the nonexistence of joint orders in CP might be reconciled with intuition.

One way is to surrender the completeness axiom for CP. There is some precedent for incomplete probability descriptions: complimentarity in quantum mechanics in an instance in physical theory where marginals exist, but joint distributions do not exist in an experimentally measurable sense; see [1] for an analysis of complimentarity in which algebras of events are replaced with propositional lattices, and [6] for a sketch of a possible relationship between CP and quantum mechanics. In any case, abandoning the completeness requirement would mean formal acceptance (which seems realistic) of a world in which not all acts or decisions are comparable; the role of our results would then be to describe classes of comparisons that do, or do not, lead to contradictions.

A second way is to observe that to some extent the issue is terminological. Our speaking of "compatibility of experiments" encourages the view that marginals are logically prior to joint orders; that is, that a priori there are two separate experiments, each with its own probability description, and that the problem is to describe them together. Were we to consider instead that the marginals are simply different aspects of the same experiment, there seems little reason to suppose that they can be chosen independently; the impossibility of extending "given" marginals would seem no more unusual than the fact that errors can occur in the course of construction of a CP order.

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