The Annals of Probability
1977, Vol. 5, No. 3, 430–450

BIRTH, DEATH AND CONDITIONING OF MARKOV CHAINS[1]

BY M. JACOBSEN AND J. W. PITMAN

University of Copenhagen and Cambridge University

Given a Markov chain with stationary transition probabilities, we study random times τ determined by the evolution of the Markov chain for which either the pre-τ or post-τ process is Markovian with stationary transition probabilities. A complete description is given of all such random times which admit a conditional independence property analogous to the strong Markov property at a stopping time.

1. Introduction. Given a Markov chain $X_0, X_1, \cdots$ with stationary transition probabilities, we investigate random times τ with the property that the joint distribution of the pre-τ fragment $(X_0, \cdots, X_{\tau-1})$ and the post-τ fragment $(X_\tau, X_{\tau+1}, \cdots)$ can be described by saying that one or other of these fragments is Markovian with stationary transition probabilities, and that the two fragments are conditionally independent given the position of the inner endpoint of the Markovian fragment at $\tau - 1$ or τ. Such a description of the joint law of the pre-τ and post-τ processes for a random time τ will be called a *path decomposition*. For some examples of more sophisticated path decompositions which provided motivation for the present study see Williams [9], [10], Jacobsen [3], Pitman [5], [6], Pittenger and Shih [7]. Following Meyer, Smythe and Walsh [4] we refer to those random times τ for which the post-τ fragment is Markov as *birth times*, and to those for which the pre-τ fragment is Markov as *death times*. We show that for discrete time Markov chains with countable state space the analogues of the types of birth times and death times considered by Meyer, Smythe and Walsh for continuous time processes, namely optional, cooptional, terminal and coterminal times, all admit the additional conditional independence property described above; and that from these special types of random times it is possible to construct the most general random times determined by the evolution of the Markov chain which allow this kind of path decomposition.

Let $(X_n, n \in N)$ be the coordinate process defined on the space Ω of all sequences in a countable set J indexed by the nonnegative integers N, and equip Ω with the usual product σ-field $\mathscr{F}$. A probability P on $(\Omega, \mathscr{F})$ is *Markov*, or *Markov* (p), if P makes (X_n) a Markov chain with stationary transition probabilities p. For background see Freedman [1].

A random time $\tau = \tau(\omega)$ is now an $\mathscr{F}$-measurable function of sequences

Received October 24, 1975.

[1] This research was prepared with the support of National Science Foundation Grant MPS 74-18967 and the Danish Natural Science Research Council.

AMS 1970 *subject classifications*. Primary 60J10; Secondary 60G40.

Key words and phrases. Markov chain, path decomposition, birth time, death time, conditional independence of path fragments.

$\omega \in \Omega$ with values in the extended time set $N \cup \{\infty\}$. Given a Markov (p) probability P, a random time τ is a *birth time for P* if the P distribution of the post-τ process is Markov (q) for some transition matrix q, and τ is a regular birth time for P if in addition the pre-τ and post-τ processes are conditionally independent given X_τ on $(\tau < \infty)$. According to the strong Markov property each optional (stopping) time τ is a regular birth time for every Markov probability P, and in this case $q = p$. If all states are recurrent it will be seen that every regular birth time is a.s. equal to an optional time, but if there are transient states there will usually be many regular birth times τ for which q differs from p; e.g., the last time τ that a certain set of states H is visited, when the post-τ process is like the original process conditioned never to hit H.

It turns out quite generally that the Markov chain which emerges at a regular birth time τ can be described by conditioning a Markov chain with the same transition probabilities as the original. With this in mind we determine in Section 2 the collection of all events $C \in \mathscr{F}$ with the property that when a Markov probability P on $(\Omega, \mathscr{F})$ is conditioned on C, another Markov probability results. Then in Section 3 it is shown that there is a class of random times $\mathscr{B}$ with the property that for each Markov probability P

(i) every $\tau \in \mathscr{B}$ is a regular birth time for P,
(ii) every regular birth time for P is P-a.s. equal to a random time in $\mathscr{B}$.

Roughly speaking, $\mathscr{B}$ comprises "optional times after coterminal times." It is interesting that the conditional independence hypothesis involved in regularity is essential for this type of result. We show that there exists no such canonical collection of plain birth times by exhibiting two Markov probabilities P and Q with the same null sets together with a random time τ which is a birth time for P but not for Q.

In Sections 4 and 5 we consider death times. A notion of regularity for death times is introduced in Section 5, and it is shown that there is a canonical class $\mathscr{D}$ of regular death times which roughly speaking comprises "cooptional times prior to terminal times." This result is like a dual to the existence of the class $\mathscr{B}$ of regular birth times, but owing to the impossibility of reversing on $(\tau = \infty)$ we are unable to bridge between the two results by any direct use of time reversal. For the death time theorem we instead make use of a new method developed in Section 4, exploiting a functional equation satisfied by certain conditional probabilities associated with any death time, regular or not. Once again we show that there is no canonical collection of plain death times.

Section 6 is devoted to random times which are both regular birth times and regular death times. We show that for nice transition matrices p these times are essentially either terminal times or coterminal times, and give a detailed description of the associated path decompositions. Finally, in Section 7 we discuss possible extensions to Markov processes with more general time set or state space.

We set out now the basic notation. Except where otherwise specified, P is a fixed Markov probability on the sequence space $(\Omega, \mathscr{F})$ with arbitrary transition matrix $p = (p(x, y),\ x, y \in J)$. For $x \in J$, P^x is the probability on $(\Omega, \mathscr{F})$ which is Markov (p) with starting state x. For use after killing operations we introduce a coffin state Δ: let $J_\Delta = J \cup \{\Delta\}$, and let Ω_Δ be the space of all sequences $\omega = (\omega_0, \omega_1, \cdots)$ in J_Δ which satisfy the coffin condition: $\omega_n = \Delta$ implies $\omega_m = \Delta$ for all $m \geqq n$. For $n \in \bar{N} = N \cup \{\infty\}$ define coordinate maps $X_n : \Omega_\Delta \to J_\Delta$, killing operators $K_n : \Omega_\Delta \to \Omega_\Delta$ and shift operators $\theta_n : \Omega_\Delta \to \Omega_\Delta$ as follows:

$$X_n(\omega) = \omega_n\,, \qquad K_n(\omega) = (\omega_0, \cdots, \omega_{n-1}, \Delta, \Delta, \cdots)\,,$$
$$\theta_n(\omega) = (\omega_n, \omega_{n+1}, \cdots)\,, \qquad n \in N\,,$$
$$X_\infty(\omega) = \Delta\,, \qquad K_\infty(\omega) = \omega\,, \qquad \theta_\infty(\omega) = \omega_\Delta\,,$$

where $\omega_\Delta = (\Delta, \Delta, \cdots)$ is the dead sequence.

Equip Ω and Ω_Δ with the usual σ-fields $\mathscr{F}$ and $\mathscr{F}_\Delta$ generated by the coordinates, and for random times $\tau : \Omega \to \bar{N}$ define measurable mappings $X_\tau : \Omega \to J_\Delta$, $K_\tau : \Omega \to \Omega_\Delta$ and $\theta_\tau : \Omega \to \Omega_\Delta$ in the obvious way, e.g., $X_\tau(\omega) = X_{\tau(\omega)}(\omega)$, $\omega \in \Omega$. For $n \in N$ let $\mathscr{F}_n$ be the sub-σ-field of $\mathscr{F}$ generated by $X_0, \cdots, X_n$, and let $\mathscr{A}_n$ be the countable collection of the atoms of $\mathscr{F}_n$, i.e., all events A of the form $A = (X_k = x_k,\ 0 \leqq k \leqq n)$ for some $x_0, \cdots, x_n \in J$. For a random time τ define $\mathscr{F}_\tau$, the σ-field of events up to and including time τ, to be the σ-field generated by $K_{\tau+1}$. This agrees with the usual definition for an optional time τ, and especially $\mathscr{F}_\tau = \mathscr{F}_n$ for the constant time $\tau = n$. For each $n \in \bar{N}$ the trace of $\mathscr{F}_\tau$ on the event $(\tau = n)$ is identical to the trace of $\mathscr{F}_n$ on $(\tau = n)$ and the event $(\tau < \infty)$ is the union of the countable collection $\{A(\tau = n),\ A \in \mathscr{A}_n,\ n \in N\}$ of atoms of $\mathscr{F}_\tau$. Here and throughout, AB denotes the intersection of events A and B in $\mathscr{F}$.

2. Conditioned Markov chains. Given a Markov chain with stationary transition probabilities, on what events determined by the evolution of the Markov chain can one condition to obtain a new Markov chain with stationary transition probabilities? For a probability P on $(\Omega, \mathscr{F})$ and $C \in \mathscr{F}$ with $P(C) > 0$ let P_C or $P(\cdot \mid C)$ denote the probability on $(\Omega, \mathscr{F})$ obtained by conditioning on C:

$$P_C(F) = P(F \mid C) = P(FC)/P(C)\,, \qquad F \in \mathscr{F}\,.$$

Thus the problem becomes: given that P is Markov, for which $C \in \mathscr{F}$ is P_C again Markov? We start by defining various collections of events contained in $\mathscr{F}$:

(2.1) DEFINITION. Let

$$\mathscr{C}_0 = \{C : C \in \mathscr{F},\ C = (X_0 \in H) \text{ for some } H \subset J\}\,,$$
$$\mathscr{C}_* = \{C : C \in \mathscr{F},\ C = [(X_n, X_{n+1}) \in V,\ n \in N] \text{ for some } V \subset J \times J\}\,,$$
$$\mathscr{C}_\infty = \{C : C \in \mathscr{F},\ C = (\theta_1 \in C)\}\,.$$

Thus $\mathscr{C}_0 = \mathscr{F}_0$ is the σ-field of *initial events* generated by X_0, $\mathscr{C}_\infty$ is the σ-field of *invariant events*, but the collection $\mathscr{C}_*$ of events which constrain all the transitions to be of a certain type is not a σ-field at all.

(2.2) DEFINITION. Let

$$\mathscr{C}_+ = \{C : C \in \mathscr{F}, C = C_* C_\infty \text{ for some } C_* \in \mathscr{C}_*, C_\infty \in \mathscr{C}_\infty\},$$
$$\mathscr{C} = \{C : C \in \mathscr{F}, C = C_0 C_* C_\infty \text{ for some } C_0 \in \mathscr{C}_0, C_* \in \mathscr{C}_*, C_\infty \in \mathscr{C}_\infty\}.$$

Events in $\mathscr{C}_+$ will be called *coterminal events*, anticipating the connection between these events and coterminal times which is described in the next section. Events in $\mathscr{C}$ are intersections of initial events and coterminal events.

Now each of the collections $\mathscr{C}_\beta$, $\beta = 0, *, \infty$, is readily seen to have the property that if P is Markov then so is P_C whenever $C \in \mathscr{C}_\beta$ and $P(C) > 0$, and it follows by repeated conditioning that the class $\mathscr{C}$ of all intersections of events from these collections must again have this property.

(2.3) THEOREM. *Suppose P is Markov and C is an event with $P(C) > 0$. Then P_C is Markov if and only if C is P-equivalent to an event in $\mathscr{C}$.*

The theorem is an immediate consequence of Lemma (2.5) and Proposition (2.10) below. Proofs of these results take up the remainder of the section, but we mention first a simple corollary:

(2.4) COROLLARY. *Suppose the Markov probability P makes all states recurrent. Then P_C is Markov if and only if C is P-equivalent to an initial event.*

PROOF. If P makes all states recurrent then every coterminal event is P-equivalent to an initial event (see Freedman [1], 1.120).

(2.5) LEMMA. *If C is a coterminal event, then there are events $F_n \in \mathscr{F}_n$ such that*

(i) $$C = F_n(\theta_n \in C), \qquad n \in N.$$

Conversely, if C is an event such that

(ii) $$C = F_1(\theta_1 \in C)$$

for some $F_1 \in \mathscr{F}_1$, then C is a coterminal event. Furthermore, if (ii) *holds only P^x-*a.s. *for all $x \in J$, where P^x is Markov (p) starting at x, then there is a coterminal event which is P^x-equivalent to C for all x.*

PROOF. The first assertion is obvious. For the converse suppose that $C \in \mathscr{F}$ satisfies (ii). Since $F_1 = ((X_0, X_1) \in V)$ for some $V \subset J \times J$,

$$C = ((X_0, X_1) \in V)(\theta_1 \in C),$$

whence

$$C = ((X_{k-1}, X_k) \in V, 1 \leqq k \leqq n)(\theta_n \in C), \qquad n \geqq 1,$$

by iteration. But intersecting this identity over all $n \geqq m$ gives

$$C = C_V(\theta_n \in C, n \geqq m)$$

where $C_V = ((X_{k-1}, X_k) \in V,\ 1 \leqq k < \infty) \in \mathscr{C}_*$, and taking the union of this identity over all m gives $C = C_V C_\infty$ where $C_\infty = \liminf_{n\to\infty} (\theta_n \in C)$ is invariant, and thus C is a coterminal event. For the final assertion the same sequence of identities is justified P^x-a.s. for all x by using the fact that if two events F_1 and F_2 agree P^x-a.s. for all x then so too do the events $(\theta_n \in F_1)$ and $(\theta_n \in F_2)$ for each $n \in N$.

(2.6) NOTATION. Recall that $\mathscr{A}_n$ denotes the countable collection of all atoms of $\mathscr{F}_n$. Now for $y \in J$ let $\mathscr{A}_{ny}$ denote the subcollection of $\mathscr{A}_n$ comprising those atoms contained in the event $(X_n = y)$.

We observe that a probability P on $(\Omega, \mathscr{F})$ is Markov if and only if for each $y \in J$ the P_A distribution of the post-n process θ_n remains constant as A varies over all events in $\mathscr{A}_{ny}$ with $P(A) > 0$ and n varies over N. When P is Markov (p) this constant distribution is of course P^y.

(2.7) DEFINITION. For each event A in $\mathscr{A}_n$, and each event $F \in \mathscr{F}$, define a set $F_A \subset \Omega$, the *section of F beyond A* as follows: for $A = (X_k = x_k, 0 \leqq k \leqq n) \in \mathscr{A}_n$, F_A comprises those sequences $\omega = (\omega_0, \omega_1, \cdots)$ such that $\omega_0 = x_n$ and $(x_0, \cdots, x_n, \omega_1, \omega_2, \cdots) \in F$.

Then F_A is an event in $\mathscr{F}$ and we shall make repeated use of the identity

$$AF = A(\theta_n \in F_A)\,, \qquad A \in \mathscr{A}_n,\ F \in \mathscr{F}\,. \tag{2.8}$$

Notice that if $A \in \mathscr{A}_{ny}$ then $F_A \subset (X_0 = y)$, $F \in \mathscr{F}$.

(2.9) LEMMA. *Suppose P is Markov (p), $C \in \mathscr{F}$ with $P(C) > 0$. Then for $A \in \mathscr{A}_{ny}$ with $P(AC) > 0$, the P_{AC} distribution of θ_n is $P^y_{C_A} = P^y(\cdot \mid C_A)$.*

PROOF. For $B \in \mathscr{F}$, $A \in \mathscr{A}_{ny}$ with $P(AC) > 0$, we have

$$\begin{aligned} P_{AC}(\theta_n \in B) &= \frac{P[AC(\theta_n \in B)]}{P(AC)} = \frac{P[A(\theta_n \in C_A B)]}{P[A(\theta_n \in C_A)]} \\ &= \frac{P_A(\theta_n \in C_A B)}{P_A(\theta_n \in C_A)} = \frac{P^y(C_A B)}{P^y(C_A)} = P^y(B \mid C_A)\,. \end{aligned}$$

(2.10) PROPOSITION. *Suppose P is Markov (p), $C \in \mathscr{F}$ with $P(C) > 0$. Then P_C is Markov if and only if there exists an event $D \in \mathscr{F}$ such that*

(i) $$C = C_0 D \quad P\text{-a.s.}$$

for some initial event $C_0 \in \mathscr{C}_0$, and

(ii) *for each $n \in N$ there is an event $F_n \in \mathscr{F}_n$ with*

$$D = F_n(\theta_n \in D) \quad P^x\text{-a.s.}$$

for all $x \in J$.

PROOF. Fix P and $C \in \mathscr{F}$ with $P(C) > 0$, and define $I \subset J$ to be the essential range of (X_n) under P_C:

$$I = \{y \in J\colon P_C(X_n = y) > 0 \text{ for some } n\}\,.$$

For $y \in J$ define $\mathscr{A}_{ny}^{+}$ to be the collection of all atoms A in $\mathscr{A}_{ny}$ with $P_C(A) > 0$, and set $\mathscr{A}_y^{+} = \bigcup_n \mathscr{A}_{ny}^{+}$. Thus $\mathscr{A}_y^{+}$ is nonempty if and only if $y \in I$. Now P_C is Markov if and only if for each $y \in I$ the P_{AC} distribution of θ_n is constant as A varies over $\mathscr{A}_{ny}^{+}$ and n varies over N. Thus by (2.9) P_C is Markov if and only if for each $y \in I$ the probabilities $P^y(\cdot \mid C_A)$ are identical, $A \in \mathscr{A}_y^{+}$, i.e., if and only if the events C_A are P^y-a.s. identical, $A \in \mathscr{A}_y^{+}$. But if there is a D satisfying (i) and (ii) then clearly for all $A \in \mathscr{A}_y^{+}$

$$C_A = (X_0 = y)D \quad P^y\text{-a.s.},$$

hence P_C is Markov. Conversely, if P_C is Markov, say Markov (q), then for $y \in I$ select a representative event C_A with $A \in \mathscr{A}_y^{+}$, and call it D_y. Let $D = \bigcup_{y \in I} D_y$. Then for $y \in I$

$$(2.11) \qquad P^y(\cdot \mid D) = P^y(\cdot \mid D_y) = P^y(\cdot \mid C_A) = Q^y(\cdot)\,, \qquad A \in \mathscr{A}_y^{+}\,,$$

where Q^y is Markov (q) starting at y. Obviously this D satisfies (i) with $C_0 = (X_0 \in H)$ for $H = \{y : P_C(X_0 = y) > 0\}$, and it will now be shown that this D also satisfies (ii). For $y \in I$, $P_D{}^y$ is Markov (q) so that $D = D_A$ P^z-a.s. for $A \in \mathscr{A}_{nz}$ with $P^y(AD) > 0$. Consequently $AD = A(\theta_n \in D)$ P^y-a.s. for $A \in \mathscr{A}_n$ with $P^y(AD) > 0$, and taking the union over all such atoms A gives a representation $D = F_{ny}(\theta_n \in D)$ P^y-a.s. with $F_{ny} \in \mathscr{F}_n$. Let $F_n = \bigcup_{y \in I} (X_0 = y)F_{ny}$. Then $D = F_n(\theta_n \in D)$ P^x-a.s., $x \in I$, and since $P^x(D) = P^x(F_n) = 0$ for $x \notin I$, D satisfies (ii). The proof is complete.

3. Regular birth times. The main result of this section is Theorem (3.9) which describes all regular birth times for a Markov probability P in terms of certain fundamental birth times associated with the coterminal events of the previous section. Using the notation defined in the introduction, a random time τ is a regular birth time for P if and only if a P conditional distribution of θ_τ given $\mathscr{F}_\tau$ is equal to Q^x on $(\tau < \infty, X_\tau = x)$, where Q^x is Markov (q) with starting state x for some transition matrix q. Put another way, τ is a regular birth time for P if and only if under P the post-τ sequence $(X_{\tau+n}, n \in N)$ is Markov (q) with respect to the increasing sequence of σ-fields $(\mathscr{F}_{\tau+n}, n \in N)$.

Suppose now that C is a coterminal event as defined in (2.2), i.e.,

$$(3.1) \qquad C = C_V C_\infty$$

where

$$(3.2) \qquad C_V = [(X_{n-1}, X_n) \in V\,, 1 \leqq n < \infty]$$

for some $V \subset J \times J$, and C_∞ is invariant.

(3.3) DEFINITION. The *coterminal time associated with* C is the random time τ_C defined by

$$\tau_C = \inf\{n \in N : \theta_n \in C\}\,.$$

NOTE. Here and elsewhere we use the convention $\inf \varnothing = \infty$, $\sup \varnothing = 0$.

The connection between these coterminal times and the coterminal times of Meyer, Smythe and Walsh [4] will be pointed out later in the section. Since for coterminal events C

(3.4) $$(\theta_k \in C) \subset (\theta_m \in C)\,, \qquad 0 \leqq k \leqq m < \infty\,,$$

we have the identity

(3.5) $$(\tau_C \leqq n) = (\theta_n \in C)\,, \qquad n \in N\,.$$

In particular, if $C = C_V$, then τ_C is the time that the last transition in V^c is completed:

(3.6) $$\tau_{C_V} = \sup\{n \geqq 1 : (X_{n-1}, X_n) \in V^c\}\,,$$

while if $C = C_\infty$ is invariant, then

(3.7) $$\tau_{C_\infty} = 0 \quad \text{on} \quad C_\infty\,, \qquad \infty \quad \text{on} \quad C_\infty{}^c\,,$$

and in general if $C = C_V C_\infty$ then τ_C is the maximum of τ_{C_V} and τ_{C_∞}.

Now each coterminal time τ_C is a birth time for the Markov probability P: indeed Lemma (3.12) below shows that for $\tau = \tau_C$ a P conditional distribution of θ_τ given $\mathscr{F}_\tau$ equals $P_C{}^x$ on $(X_\tau = x)$, where $P_C{}^x = Q^x$ is Markov (q) for some q by (2.3). This is the analogue in the present context of Theorem 5.1 of Meyer, Smythe and Walsh [4]. For a more detailed description of the path decomposition at τ_C giving the transition probabilities q of the post-τ_C process, see Section 6.

(3.8) DEFINITION. Let $\mathscr{B}$ be the class of all random times τ of the form

$$\tau = \tau_C + \rho$$

where τ_C is the coterminal time associated with a coterminal event C, and ρ is an optional time for the increasing sequence of σ-fields $(\mathscr{F}_{\tau_C+n}, n \in N)$, i.e., $(\rho = n) \in \mathscr{F}_{\tau_C+n}$, $n \in N$.

Once it is known that each τ_C is a regular birth time for P, it follows from the strong Markov property of the sequence $(X_{\tau_C+n}, n \in N)$ adapted to $(\mathscr{F}_{\tau_C+n}, n \in N)$ that each $\tau \in \mathscr{B}$ is again a regular birth time for P. This proves one of the implications of the following theorem. Proof of the converse implication takes up the rest of the section.

(3.9) THEOREM. *A random time τ is a regular birth time for a Markov probability P if and only if τ is P-equivalent to a random time in $\mathscr{B}$.*

(3.10) COROLLARY. *If P makes all states recurrent then every regular birth time for P is P-equivalent to a stopping time.*

PROOF. Just as for (2.4).

(3.11) DEFINITION. A random time τ is a *conditional independence time* for P if under P the pre-τ and post-τ processes are conditionally independent given X_τ, i.e., if there is a conditional distribution of θ_τ given $\mathscr{F}_\tau$ within $(\tau < \infty)$

which is a function of X_τ alone.

(3.12) LEMMA. *A random time τ is a conditional independence time for P if and only if there are events $F_n \in \mathscr{F}_n$, $n \in N$, and $G \in \mathscr{F}$ such that*

$$(\tau = n) = F_n(\theta_n \in G)\,, \quad P\text{-a.s.}, \qquad n \in N\,, \tag{3.13}$$

and there is then a conditional distribution of θ_τ given $\mathscr{F}_\tau$ which equals $P_G{}^x$ on $(\tau < \infty, X_\tau = x)$.

REMARK. The proof can be sharpened to show that τ is a conditional independence time for P if and only if τ is P a.s. equal to a τ^* with $(\tau^* = n) = F_n(\theta_n \in G)$ exactly for some $F_n \in \mathscr{F}_n$, $G \in \mathscr{F}$. Every such time is thus a.s. equal to a *splitting time*, defined in Jacobsen [3] as a random time τ for which $(\tau = n) = F_n(\theta_n \in G_n)$ for some $F_n \in \mathscr{F}_n$, $G_n \in \mathscr{F}$. The present argument will also show that splitting times are characterized by conditional independence of the pre-τ and post-τ processes given both X_τ and τ.

PROOF. Working on atoms as in the proof of (2.10), let $\mathscr{A}_{nx}^+$ be the collection of all atoms A of $\mathscr{F}_n$ contained in $(X_n = x)$ with $P(AG_n) > 0$, where $G_n = (\tau = n)$, so that $(X_n = x)G_n$ is P-a.s. equal to the union of the sets AG_n over all A in $\mathscr{A}_{nx}^+$. Defining G_{nA} as in (2.7) to be the section of G_n beyond A, we have from (2.9) that the P conditional distribution of θ_n given AG_n is $P^x(\cdot\,|\,G_{nA})$, $A \in \mathscr{A}_{nx}^+$. But τ is a conditional independence time if and only if this conditional distribution is a function of x alone for all $A \in \mathscr{A}_{nx}^+$, $n \in N$, i.e., if and only if for each $x \in J$ there exists $G_x \in \mathscr{F}$ with $G_x \subset (X_0 = x)$ and

$$G_{nA} = G_x \quad P^x\text{-a.s.}, \qquad A \in \mathscr{A}_{nx}^+\,. \tag{3.14}$$

But (3.13) implies (3.14) with $G_x = G(X_0 = x)$, while (3.14) implies (3.13) with $G = \bigcup_{x \in J} G_x$, and F_n the union of all $A \in \mathscr{A}_n$ with $P(AG_n) > 0$.

PROOF OF THEOREM (3.9). That each $\tau \in \mathscr{B}$ is a regular birth time has been argued already. For the converse, suppose τ is a regular birth time for P. Then by (3.12) there are events $F_n \in \mathscr{F}_n$, $G \in \mathscr{F}$ with

$$(\tau = n) = F_n(\theta_n \in G) \quad P\text{-a.s.}, \tag{3.15}$$

and the conditional distribution of θ_τ given $\mathscr{F}_\tau$ is $P_G{}^x$ on $(\tau < \infty, X_\tau = x)$. Moreover this probability is Markov(q) for some q not depending on x, and thus Theorem (2.3) shows that for every x with $P(\tau < \infty, X_\tau = x) > 0$, G is P^x-a.s. equal to a coterminal event C which because of (2.10)(ii) and (2.5)(ii) may be chosen so as not to depend on x. Thus the set $G_x = G(X_0 = x)$ which appears in (3.14) can be replaced by the set $C_x = C(X_0 = x)$ to give

$$(\tau = n) = F_n(\theta_n \in C) \quad P\text{-a.s.}, \qquad n \in N\,. \tag{3.16}$$

But now let

$$C_n = F_n(\theta_n \in C) \tag{3.17}$$

and define a random time τ' by setting

$$\begin{aligned} \tau' &= n \qquad \text{on} \quad C_n \backslash \textstyle\bigcup_{k=0}^{n-1} C_k\,, \qquad\qquad n \in N \\ &= \infty \qquad \text{on} \quad (\textstyle\bigcup_{k=0}^{\infty} C_k)^c\,. \end{aligned}$$

(The sets C_n might not be disjoint.) Clearly $\tau' = \tau$ P-a.s., and we conclude by showing that $\tau' \geqq \tau_C$ and that $\rho = \tau' - \tau_C$ is an optional time for $(\mathscr{F}_{\tau_C+m}, m \in N)$. But from (3.17), (3.4) and (3.5),

$$(\tau' \leqq n) = \textstyle\bigcup_{k=0}^{n} C_k \subset \bigcup_{k=0}^{n} (\theta_k \in C) = (\tau_C \leqq n) \qquad n \in N\,,$$

which implies $\tau' \geqq \tau_C$. Now put $\rho = \tau' - \tau_C$. Then for $m \in N$ we have the identities

$$(3.18) \qquad (\rho \leqq m) = \textstyle\bigcup_{n=m}^{\infty} (\tau_C = n - m, \tau' \leqq n)\,,$$

$$(3.19) \qquad (\tau_C = n - m, \tau' \leqq n) = (\tau_C = n - m)[\textstyle\bigcup_{k=0}^{n} C_k] = \bigcup_{k=0}^{n} B_k$$

where $B_k = (\tau_C = n - m)C_k$. If $k < n - m$, (3.17) and (3.5) show $C_k \subset (\theta_k \in C) = (\tau_C \leqq k)$ so that $B_k = \emptyset$, while if $n - m \leqq k \leqq n$ then (3.5) and (3.4) make $(\tau_C = n - m) \subset (\theta_{n-m} \in C) \subset (\theta_k \in C)$ so that (3.17) yields $B_k = (\tau_C + m = n)F_k$ where $F_k \in \mathscr{F}_k \subset \mathscr{F}_n$. In either case $B_k \in \mathscr{F}_{\tau_C+m}$, and thus working back through (3.19) to (3.18) shows $(\rho \leqq m) \in \mathscr{F}_{\tau_C+m}$, which is to say that ρ is an optional time of $(\mathscr{F}_{\tau_C+m}, m \in N)$. The proof is complete.

We now point out the connection between the random times τ_C of Definition (3.3) and the coterminal times of Meyer, Smythe and Walsh [4] (MSW). A MSW coterminal time is a random time τ defined on Ω_Δ satisfying

$$(3.20) \qquad \begin{aligned} &\text{(i)} \quad \tau \circ \theta_n = (\tau - n)^+\,, && n \in N\,, \\ &\text{(ii)} \quad \tau \circ K_n = \tau \quad \text{on} \quad (\tau < n)\,, && n \in N\,, \\ &\text{(iii)} \quad \tau \circ K_n \leqq n\,, && n \in N\,. \end{aligned}$$

The random time τ_C of (3.3) is defined only on Ω. If (3.3) is interpreted as a definition on Ω_Δ, the resulting random time will not be MSW coterminal because both (ii) and (iii) above may be violated. However, every τ_C admits an extension to Ω_Δ which is MSW coterminal and the restriction of every MSW coterminal time to Ω is of the form τ_C for a coterminal event C, as will now be shown.

For $V \subset J \times J$ define $\tilde{V} \subset J_\Delta \times J_\Delta$ by

$$(3.21) \qquad \tilde{V} = V \cup (\pi(V) \times \{\Delta\}) \cup \{(\Delta, \Delta)\}\,,$$

where $\pi(V) = \{x \in J: (x, y) \in V \text{ for some } y \in J\}$. Now for $C = C_V C_\infty$ as in (3.1) define $\tilde{C} \in \mathscr{F}_\Delta$ by

$$(3.22) \qquad \tilde{C} = [(X_{n-1}, X_n) \in \tilde{V}, n \geqq 1][C_\infty \cup (\Omega_\Delta \backslash \Omega)]\,,$$

and define $\tau_{\tilde{C}}$ on Ω_Δ by

$$(3.23) \qquad \tau_{\tilde{C}} = \inf\{n: \theta_n \in \tilde{C}\}\,.$$

Then it is easily checked that $\tau_{\tilde{C}} = \tau_C$ on Ω, and that $\tau_{\tilde{C}}$ is MSW coterminal.

Conversely, if τ is a MSW coterminal time, then $C = (\tau = 0)\Omega$ is a coterminal event as in (3.1), and $\tau = \tau_C$ on Ω. To see this observe first that (3.20) (i) implies

$$(3.24) \qquad (\tau \leqq n) = (\tau \circ \theta_n = 0) = (\theta_n \in (\tau = 0)), \qquad n \in N .$$

Thus by (3.5) it suffices to show that $(\tau = 0)\Omega = C$ for a coterminal event C. But by (3.20)(ii) for $n = 2$ and (3.24) for $n = 1$,

$$\begin{aligned}(\tau = 0) &= (\tau = 0, \tau < 2) = (\tau \circ K_2 = 0, \tau < 2) \\ &= (\tau \circ K_2 = 0, \theta_1 \in (\tau = 0)) .\end{aligned}$$

Since $(\tau \circ K_2 = 0)\Omega \in \mathscr{F}_1$, (2.5)(ii) now shows that $(\tau = 0)\Omega$ is a coterminal event.

We conclude this section with some remarks about plain birth times. For an interesting example fix $m \geqq 1$ and define

$$\tau = \inf\{n \geqq 1 : (X_n, \cdots, X_{n+m}) = (X_0, \cdots, X_m)\} .$$

If P makes all states recurrent then τ is P-a.s. finite and since $\tau + m$ is a stopping time and $(X_\tau, \cdots, X_{\tau+m}) = (X_0, \cdots, X_m)$ on $(\tau < \infty)$ it is easy to see that the P distribution of θ_τ is P. Thus τ is a birth time for P, but certainly not a regular birth time, since knowledge of the pre-τ process completely determines the first m-moves of the post-τ process. Examples show that τ can fail to be a birth time if there are transient states. But it is of greater interest to use the idea behind this example to construct two Markov probabilities P and R with the same null sets and a τ which is a birth time for P but not for R, since this shows that there exists no canonical class $\mathscr{B}^*$ of plain birth times with the property that τ is a birth time for a Markov probability P if and only if τ is P-equivalent to a time in $\mathscr{B}^*$.

(3.25) EXAMPLE. Let $J = \{1, 2, 3\}$. Define transition matrices p and r on J by

$$p = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \qquad r = \begin{bmatrix} 0 & \frac{1}{4} & \frac{3}{4} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

Let P and R be Markov (p) and Markov (r) respectively, both starting at 1. Then P and R have the same null sets but the random time τ defined by

$$\begin{aligned}\tau &= \inf\{n : X_n = 2, X_{n+1} = 2\} \qquad \text{if} \quad X_1 = 2 \\ &= \inf\{n : X_n = 2, X_{n+1} = 3\} \qquad \text{if} \quad X_1 = 3\end{aligned}$$

is a birth time for P only.

4. Death times. Given a Markov (p) probability P on $(\Omega, \mathscr{F})$ we now investigate death times for P, i.e., random times τ such that under P the distribution of the pre-τ fragment $(X_0, \cdots, X_{\tau-1})$ is Markov with stationary substochastic transition probabilities, or, what is the same, that the killed process $K_\tau = (X_0, \cdots, X_{\tau-1}, \Delta, \Delta, \cdots)$ is Markov with stationary transition probabilities.

Let τ be a death time for P. If we let J_+ denote the essential range of the pre-τ path,,

$$J_+ = \{x \in J\colon P(\tau > n, X_n = x) > 0 \text{ for some } n \in N\},$$

then the transition probabilities $q(x, y)$ of the killed chain are well defined for all $x, y \in J_+ \cup \{\Delta\}$ and induce a family of probabilities $\{Q^x, x \in J_+ \cup \{\Delta\}\}$ on Ω which concentrate on paths ω which remain forever within this restricted state space $J_+ \cup \{\Delta\}$. Clearly if $P(\tau > 0, X_0 = x) > 0$ then Q^x is identical to the P^x distribution of K_τ given $(\tau > 0)$. This conclusion may be either false or meaningless if $P(\tau > 0, X_0 = x) = 0$, but we shall see that matters can be rectified by redefining $\tau(\omega)$ properly for paths ω starting at such points x.

We consider now the whole family of probabilities $\{P^x\}_{x \in J}$ where P^x on $(\Omega, \mathscr{F})$ is Markov (p) with starting state x. For each random time τ define $J_\tau \subset J$ by

$$J_\tau = \{x \in J\colon P^x(\tau > 0) > 0\},$$

and say that τ is a *death time for the family* $\{P^x\}_{x \in J}$ if there is a transition matrix q on $J_\tau \cup \{\Delta\}$ such that for each $x \in J_\tau$ the P^x distribution of K_τ given $(\tau > 0)$ is Q^x, where Q^x on $(\Omega, \mathscr{F})$ is Markov (q) with starting state x. In particular Q^x concentrates on the set of sequences in $J_\tau \cup \{\Delta\}$ which satisfy the coffin condition.

Obviously every death time for the family $\{P^x\}_{x \in J}$ is a death time for each Markov (p) probability P. Conversely:

(4.1) PROPOSITION. *Let τ be a death time for a Markov (p) probability P. Then there is a death time τ^* for the family $\{P^x\}_{x \in J}$ such that $\tau^*(\omega) = \tau(\omega)$ for all paths ω starting at points $x \in J$ with $P(\tau > 0, X_0 = x) > 0$.*

REMARK. If $P = P^y$ for a fixed state y then τ^* and τ agree P-a.s.

PROOF. As before let J_+ be the essential range of the pre-τ path under P, and suppose K_τ is Markov (q) under P. For those paths ω starting at an $x \in J \backslash J_+$ set $\tau^*(\omega) = 0$, while for those starting at an $x \in J_+$ with $P(\tau > 0, X_0 = x) > 0$ put $\tau^*(\omega) = \tau(\omega)$. Finally, if $x \in J_+$ and $P(\tau > 0, X_0 = x) = 0$ find an $m \geqq 1$ and $x_0, \cdots, x_m \in J_+$ with $x_m = x$ such that $P[A(\tau > m)] > 0$ where $A = (X_0 = x_0, \cdots, X_m = x_m)$, and then for $\omega = (\omega_0, \omega_1, \cdots)$ with $\omega_0 = x$ define

$$\tau^*(\omega) = (\tau(x_0, \cdots, x_{m-1}, \omega_0, \omega_1, \cdots) - m)^+ .$$

Obviously $J_{\tau^*} = J_+$ and it remains only to check that the P^x distribution of K_{τ^*} given $(\tau^* > 0)$ is Q^x for $x \in J_+$ with $P(\tau > 0, X_0 = x) = 0$, where Q^x is Markov (q) starting at x. For this it suffices to show that for $n \in N$, $y_0 = x, y_1, \cdots, y_n \in J_+$, $B = (X_0 = y_0, \cdots, X_n = y_n)$,

$$P^x[B(\tau^* > n) \mid \tau^* > 0] = Q^x(B),$$

where $Q^x(B) = q(y_0, y_1) \cdots q(y_{n-1}, y_n)$ by definition.

But let m and n be as above. Then, first using the Markov property of P and then the fact that $\tau^* \circ \theta_m = (\tau - m)^+$ on A,

$$\begin{aligned} P^x[B(\tau^* > n)] &= P[A(\theta_m \in B, \tau^* \circ \theta_m > n)]/P(A) \\ &= P[A(\theta_m \in B, \tau > m + n)]/P(A) . \end{aligned}$$

Similarly $P^x(\tau^* > 0) = P[A(\tau > m)]/P(A)$, whence

$$P^x[B(\tau^* > n) \mid \tau^* > 0] = P[A(\theta_m \in B,\ \tau > m + n)]/P[A(\tau > m)]$$

which equals $Q^x(B)$ because the P distribution of K_τ is Markov (q).

This result reduces the problem of describing the death times for a given Markov probability P to that of characterizing the death times for a family $\{P^x\}_{x \in J}$. Such a characterization is provided by the following proposition. Write $P(\cdot \mid \mathscr{F}_n)$ for any of the identical conditional probabilities $P^x(\cdot \mid \mathscr{F}_n)$, and given a random time τ introduce

$$f(x) = P^x(\tau > 0)\,, \qquad Z_n = P(\tau > n \mid \mathscr{F}_n)\,, \qquad n \in N\,.$$

Note that $Z_0 = f(X_0)$ and recall that $J_\tau = \{x \in J : f(x) > 0\}$.

(4.2) PROPOSITION. *A random time τ is a death time for the family $\{P^x\}_{x \in J}$ if and only if for $m, n \in N$ the identity*

$$\begin{aligned} Z_{m+n} &= Z_m Z_n \circ \theta_m / f(X_m) && \text{on } (X_m \in J_\tau) \\ &= 0 && \text{on } (X_m \notin J_\tau) \end{aligned}$$

holds P^x-a.s., $x \in J$.

PROOF. With Q^x the P^x distribution of K_τ given $(\tau > 0)$, the condition that τ be a death time for $\{P^x\}$ is equivalent to requiring that for all $m, n, k \in N$, $x_0, \cdots, x_{m+n}, x \in J_\tau$,

$$\text{(4.3)} \qquad \begin{aligned} &Q^{x_0}(X_0 = x_0, \cdots, X_{m+n} = x_{m+n}) \\ &\quad = Q^{x_0}(X_0 = x_0, \cdots, X_m = x_m) Q^{x_m}(X_0 = x_m, \cdots, X_n = x_{m+n})\,, \end{aligned}$$

$$\text{(4.4)} \qquad Q^x(X_k \notin J_\tau \cup \{\Delta\}) = 0\,.$$

Introducing the atoms $A = (X_0 = x_0, \cdots, X_m = x_m)$, $B = (X_0 = x_m, \cdots, X_n = x_{m+n})$, it is seen that (4.3) and (4.4) are equivalent to

$$\text{(4.5)} \qquad P^{x_0}[A(\theta_m \in B,\ \tau > m + n)] = P^{x_0}[A(\tau > m)] P^{x_m}[B(\tau > n)]/f(x_m)$$

$$\text{(4.6)} \qquad P^x(X_k \notin J_\tau,\ \tau > k) = 0\,.$$

But the left side of (4.5) equals

$$P^{x_0}[Z_{m+n}\,;\ A(\theta_m \in B)]$$

while the right side becomes

$$P^{x_0}(Z_m\,;\ A) P^{x_m}(Z_n\,;\ B)/f(x_m) = P^{x_0}[Z_m Z_n \circ \theta_m / f(X_m)\,;\ A(\theta_m \in B)]\,.$$

Since A and B are arbitrary atoms for paths within J_τ and since (4.6) is equivalent to demanding that $Z_k, Z_{k+1}, \cdots$ vanish P^x-a.s. on $(X_k \notin J_\tau)$ the result follows.

We finish this section with an example of two Markov probabilities P and R, with the same null sets, and a random time τ which is a death time for P but not for R, thus proving that there exists no collection of random times $\mathscr{D}^*$ such that τ is a death time for a Markov probability P if and only if τ is P-a.s. equal to a random time in $\mathscr{D}^*$ (cf. Example (3.25)).

(4.7) EXAMPLE. Let $J = \{1, 2, 3, 4\}$. Define transition matrices p and r on J by

$$p = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad r = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{4} & \frac{3}{4} \\ 0 & \frac{3}{4} & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Let P and R be Markov (p) and Markov (r) respectively, both starting at 1. Clearly P and R have the same null sets. Define ρ as the time of entry into $\{2, 3\}$, σ as the time to absorption in state 4, put $\delta = \sigma - \rho$ and finally let τ be the minimum of ρ and δ.

Under both P and R the random times ρ and δ are independent, under P they both have a geometric distribution, whence so too does τ, but under R the random time ρ is geometric while δ is not, and thus τ is not geometric either. Since the pre-τ process never leaves state 1 it is therefore Markov under P but not under R.

5. Regular death times. A random time τ is a *regular death time* for the Markov probability P if the P distribution of the pre-τ process $(X_0, \cdots, X_{\tau-1})$ is Markov (r) for some substochastic transition matrix r, and in addition the pre-τ and post-τ processes are conditionally independent given $X_{\tau-1}$ on $(0 < \tau < \infty)$. Since the time reversal of a Markov fragment with finite lifetime and stationary transition probabilities is again Markov with stationary transition probabilities, the regularity condition for a death time is equivalent to demanding that there is some substochastic transition matrix $\hat{r}$ such that conditional on the post-τ process, $0 < \tau < \infty$ and $X_{\tau-1} = x$ (for each state x), the reversed pre-τ fragment $(X_{\tau-1}, X_{\tau-2}, \cdots, X_0)$ is Markov $(\hat{r})$ with starting state x. Thus the notion of a regular death time may be viewed as the dual under time reversal to the notion of a regular birth time, but we shall not make any use of this fact.

(5.1) DEFINITION. Let $\mathscr{D}$ be the class of all random times τ of the form

$$\tau = \tau_{V,F} = \sup\{n : 1 \leqq n \leqq \tau_V, \theta_{n-1} \in F\}$$

for some $V \subset J \times J$, $F \in \mathscr{F}$, where τ_V is the terminal time associated with V:

$$\tau_V = \inf\{n : n \geqq 1, (X_{n-1}, X_n) \in V\}.$$

According to the following theorem, $\mathscr{D}$ is a complete canonical collection of regular death times for each Markov probability P^x on $(\Omega, \mathscr{F})$ with a fixed starting state x. For random starts $\mathscr{D}$ must be enlarged to include all times of the form $\tau 1(X_0 \in H)$ for $\tau \in \mathscr{D}$, $H \subset J$, but we shall ignore this trivial complication.

(5.2) THEOREM. *A random time τ is a regular death time for a Markov probability P^x if and only if τ is P^x-equivalent to a random time in $\mathscr{D}$.*

The proof of this theorem will be taken up later in the section under (5.7) and (5.9). We consider first various characterizations of $\mathscr{D}$.

(5.3) PROPOSITION. *Each of the following three conditions for a random time τ is equivalent to $\tau \in \mathscr{D}$:*

(a) *For every $n \in N$*

$$\text{(i)} \qquad (\tau > n) = F_n(\tau \circ \theta_n > 0) \qquad \text{for some} \quad F_n \in \mathscr{F}_n ,$$

$$\text{(ii)} \qquad \tau \circ \theta_n = \tau - n \qquad \text{on} \quad (\tau > n) .$$

(b) *For each $n \in N$ there is an $F_n \in \mathscr{F}_n$ such that*

$$(\tau > m + n) = F_n(\tau \circ \theta_m > n) , \qquad m \in N .$$

(c) *There is an $F_1 \in \mathscr{F}_1$ such that for*

$$\bar{F}_n = (\theta_m \in F_1, 0 \leqq m < n) ,$$

$$\text{(i)} \qquad (\tau = n + 1) = \bar{F}_n(\tau \circ \theta_n = 1) , \qquad n \geqq 1 ,$$

$$\text{(ii)} \qquad (\tau = \infty) = \bar{F}_n(\tau \circ \theta_n = \infty) , \qquad n \geqq 1 .$$

REMARK. It will follow from the proof that if τ satisfies either (a), (b) or (c), then $\tau = \tau_{V,F}$ where

$$\text{(5.4)} \qquad F = (\tau > 0) , \qquad F_1 = ((X_0, X_1) \in V^c) .$$

The representation of $\tau \in \mathscr{D}$ as $\tau_{V,F}$ is far from unique however, and in particular it is not the case in general that $(\tau_{V,F} > 0) = F$.

PROOF. It is plain that (a) and (b) are equivalent and that (c) implies (b). Since it is also easily checked that any $\tau = \tau_{V,F} \in \mathscr{D}$ satisfies (c), it is enough to show that any τ satisfying (b) is in $\mathscr{D}$. So suppose τ satisfies (b). Define F and V according to (5.4), and consider that for $n \in N$

$$\begin{aligned}(\tau > n) &= F_1(\tau \circ \theta_1 > n - 1) = F_1(\theta_1 \in F_1)(\tau \circ \theta_2 > n - 2) = \cdots \\ &= (\theta_m \in F_1, 0 \leqq m < n) = (\tau_V > n, \theta_n \in F) ,\end{aligned}$$

and hence for $m \in N$

$$(\tau > m) = \textstyle\bigcup_{n \geqq m} (\tau > n) = \bigcup_{n \geqq m} (\tau_V > n, \theta_n \in F) .$$

But the definition of $\tau_{V,F}$ identifies this last event as $(\tau_{V,F} > m)$, and the conclusion that $\tau = \tau_{V,F}$ is now apparent.

For an optional time τ we have $(\tau > n) \in \mathscr{F}_n$ so that condition (a) of (5.3) collapses to (a)(ii) alone. Since an optional time satisfying (a)(ii) is by definition a terminal time, an optional time is in $\mathscr{D}$ if and only if it is a terminal time. Furthermore, it is plain from the original definition of $\mathscr{D}$, that τ is a terminal time if and only if τ is the first time that the path either enters H or completes a jump in V for some $H \subset J$, $V \subset J \times J$ (cf. Walsh and Weil [8]).

For a cooptional time, $\tau \circ \theta_n = (\tau - n)^+$ by definition, and since this obviously implies (5.3)(a), $\mathscr{D}$ includes all cooptional times. Moreover, every cooptional time τ can be represented as $\tau = \sup \{n \geqq 1 : \theta_{n-1} \in F\}$ where $F = (\tau > 0)$, and since also $(\tau > 0) = (\theta_1 \in G)$ for some $G \in \mathscr{F}$, it follows that τ is a cooptional time if and only if $\tau = \sup\{n \in N: \theta_n \in G\}$ for some $G \in \mathscr{F}$.

Finally we mention two important closure properties of $\mathscr{D}$ which are readily checked using (5.3)(b): (i) if σ and τ are in $\mathscr{D}$, then so is their minimum $\sigma \wedge \tau$; (ii) if τ is in $\mathscr{D}$, and σ is in $\mathscr{D}_\Delta$, the class of random times on Ω_Δ obtained by applying Definition (5.1) to Ω_Δ rather than Ω, then the random time $\sigma \circ K_\tau$ obtained by using σ after killing at τ is in $\mathscr{D}$ provided $\sigma \leqq \inf\{n \in N: X_n = \Delta\}$.

(5.5) LEMMA. *A random time τ is a regular death time for P if and only if τ is a death time for P and there are events $F_n \in \mathscr{F}_n$, $n \in N$ and $G \in \mathscr{F}$ such that*

$$(5.6) \qquad (\tau = n+1) = F_n(\theta_n \in G), \quad P\text{-a.s.}, \qquad n \in N.$$

PROOF. Define $\hat\tau$ to equal τ on $(0 < \tau < \infty)$, ∞ on $(\tau = 0 \text{ or } \infty)$. Then τ is a regular death time if and only if τ is a death time and $\hat\tau$ is a conditional independence time. Now apply (3.12).

(5.7) PROOF OF THEOREM (5.2), FIRST HALF. Let $\tau = \tau_{V,F}$. Then (5.3)(b) implies that for $Z_n = P(\tau > n \mid \mathscr{F}_n)$, $Z_{m+n} = 1_{F_m} Z_n \circ \theta_m$ P^x-a.s., $x \in J$. But Proposition (4.2) now shows that τ is a death time for the family $\{P^x\}_{x \in J}$, and regularity follows from (5.3)(c) and Lemma (5.5). Hence if τ is P^x-equivalent to a member of $\mathscr{D}$, then τ is a regular death time for P^x.

We now prepare for the proof of the converse.

NOTATION. Henceforth "a.s." means "P^x-a.s. for all $x \in J$."

The arguments below lean heavily on the fact that for $F, F' \in \mathscr{F}$, $F = F'$ a.s. implies $(\theta_n \in F) = (\theta_n \in F')$ a.s. Note that this would fail if a.s. just meant P^x-a.s. for a single x.

(5.8) LEMMA. *Let τ be a random time such that for some $F_1 \in \mathscr{F}_1$ the identities of (5.3)(c) hold* a.s. *Then τ is* a.s. *equal to a member of $\mathscr{D}$.*

PROOF. Let τ satisfy the identities of (5.3)(c) for all n except on a P^x null set L_x. Then $L = \bigcup_{x \in J} (X_0 = x)L_x$ is a null set for all P^x, $x \in J$ simultaneously, and thus for $\Omega^* = \bigcap_{n \in N} (\theta_n \in L^c)$ the identities

$$(\tau = n+1)\Omega^* = \bar F_n(\tau \circ \theta_n = 1)\Omega^*, \qquad (\tau = \infty)\Omega^* = \bar F_n(\tau \circ \theta_n = \infty)\Omega^*$$

hold exactly, $n \in N$. But because $\Omega^* \subset (\theta \in \Omega^*)$, the proofs used for Proposition (5.3) apply here to show that for the V and F of (5.4), $\tau = \tau_{V,F}$ on Ω^*.

(5.9) PROOF OF THEOREM (5.2), SECOND HALF. Fix $x \in J$, and let τ be a regular death time for P^x. Then Lemma (5.5) provides $F_n \in \mathscr{F}_n$, $G \in \mathscr{F}$ such that (5.6) holds with $P = P^x$.

Replacing τ by the τ^* constructed in Proposition (4.1) we get a random time which is P^x-a.s. equal to τ and which is a death time for the family $\{P^y\}_{y \in J}$. Furthermore, it is not difficult to verify that $(\tau^* = n+1) = F_n{}^*(\theta_n \in G)$ a.s., $n \in N$, where $F_n{}^*$ is the union of the events $F_{n,y}$ over y in J_{τ^*}, with $F_{n,y}$ the section of F_{m+n} beyond the atom $A_y = (X_0 = x_0, \cdots, X_m = y)$ used to define τ^* on paths starting at $y \in J_{\tau^*}$ (see (2.7) and the proof of (4.1)). Thus we may as well drop the stars and take it from the start that τ is a death time for the family $\{P^y\}_{y \in J}$ such that there are events $F_n \in \mathscr{F}_n$, $G \in \mathscr{F}$ with

(5.10) $$(\tau = n + 1) = F_n(\theta_n \in G) \quad \text{a.s.}$$

We now aim to show that such a τ satisfies the hypothesis of Lemma (5.8). From (5.10) it follows that

$$P(\tau = n + 1 \,|\, \mathscr{F}_n) = 1_{F_n} P^{X(n)} G \quad \text{a.s.},$$

while by Proposition (4.2) this conditional probability can be expressed (using the terminology of that proposition) as

$$Z_n - P(Z_{n+1} \,|\, \mathscr{F}_n) = 1_{(X_n \in J_\tau)} Z_n P^{X(n)}(\tau = 1 \,|\, \tau > 0) \quad \text{a.s.}$$

Therefore

(5.11) $$F_n(P^{X(n)}G > 0) = (Z_n > 0, P^{(n)}(\tau = 1) > 0) \quad \text{a.s.},$$

since (4.2) implies $(Z_n > 0) \subset (X_n \in J_\tau)$ a.s. Now by (5.10) for $n = 0$, $F_0 \supset (\tau = 1)$ a.s., hence $F_0 \supset (P^{X(0)}(\tau = 1) > 0)$ a.s., and also $(\theta_n \in F_0) \supset (P^{X(n)}(\tau = 1) > 0)$ a.s. But with (5.11) this implies

$$\begin{aligned}(\tau = n + 1) &= F_n(\theta_n \in G, P^{X(n)}G > 0) = (Z_n > 0, P^{X(n)}(\tau = 1) > 0, \theta_n \in G) \\ &= (Z_n > 0, P^{X(n)}(\tau = 1) > 0, \theta_n \in F_0 G) \\ &= (Z_n > 0, \tau \circ \theta_n = 1) \quad \text{a.s.}\end{aligned}$$

and (4.2) now shows that τ satisfies the first identity of (5.3)(c) a.s. with $F_1 = (Z_1 > 0)$. To establish the second identity of (5.3)(c) observe that by the martingale convergence theorem $1(\tau = \infty) = \lim_{m \uparrow \infty} Z_m$ a.s. But according to Proposition (4.2) that limit equals

$$Z_n[f(X_n)]^{-1} 1(X_n \in J_\tau) \lim_m Z_m \circ \theta_m = Z_n[f(X_n)]^{-1} 1(X_n \in J_\tau, \tau \circ \theta_n = \infty) \quad \text{a.s.}$$

for every $n \in N$, whence $(\tau = \infty) = (Z_n > 0, \tau \circ \theta_n = \infty)$ a.s. That τ is a.s. equal to a member of $\mathscr{D}$ now follows from (5.8).

As a final comment on death times, it may be observed that there exist random times τ which are death times for all Markov probabilities simultaneously without being regular. A simple example is obtained by taking an integer $a \geqq 2$ and defining

$$(\tau > 0) = \Omega\,, \qquad (\tau > n) = (X_0 = \cdots = X_{an})\,, \qquad n \geqq 1\,.$$

6. Birth and death times. We now consider random times which are both regular death times and regular birth times for each Markov (p) probability P. For such a random time τ it is seen that a path decomposition specifying the joint distribution of the pre-τ and post-τ processes can be given in terms of just four quantities determined by τ and p, namely the function $f\colon J \to [0, 1]$, the substochastic matrix q and two stochastic matrices r and s on J, such that under the probability P^x on $(\Omega, \mathscr{F})$ which makes (X_n) Markov (p) starting at x, we have

$$
\text{(6.1)}\quad
\begin{array}{ll}
\text{(i)} & P^x(\tau > 0) = f(x)\,, \quad x \in J\,, \\
\text{(ii)} & \text{conditional on } \tau > 0 \text{ the } P^x \text{ distribution of the pre-}\tau \text{ process is Markov } (q) \text{ starting at } x\,, \\
\text{(iii)} & \text{conditional on } 0 < \tau < \infty \text{ and a pre-}\tau \text{ path with } X_{\tau-1} = y \text{ the } P^x \text{ distribution of } X_\tau \text{ is } r(y, \cdot)\,, \\
\text{(iv)} & \text{conditional on } \tau < \infty\,, \text{ the pre-}\tau \text{ path and } X_\tau = z\,, \text{ the } P^x \text{ distribution of the post-}\tau \text{ process is Markov } (s) \text{ starting at } z\,.
\end{array}
$$

With f, q, r and s specified by (6.1) the path decomposition involved in (6.1) can be expressed more intuitively by saying that the following probabilistic motion describes a Markov chain with stationary transition probabilities p: start at x, and then with probability $f(x)$ move off according to a Markov chain with transition probabilities q; when (if ever) this chain dies look back at the position y where the chain was at the instant before it died and instead of dying make a single transition according to $r(y, \cdot)$; if this gets you to state z (where $z = x$ if there was no motion according to q) complete the motion by moving forevermore according to a Markov chain with transition probabilities s starting at z.

There are two basic kinds of random times which induce a path decomposition of this kind: terminal times and coterminal times. We first indicate how the parameters f, q, r and s are obtained for these times, and then show that for nice transition matrices p these are essentially the only random times inducing such a path decomposition.

Suppose first that τ is a terminal time. Then as mentioned below (5.3) there is a subset H of J and a subset V of $J \times J$ with $V^c \subset H^c \times H^c$ such that

$$
\begin{aligned}
(\tau > 0) &= (X_0 \in H^c)\,, \\
(\tau > n) &= ((X_k, X_{k+1}) \in V^c, 0 \leqq k \leqq n-1)\,, \qquad n \geqq 1\,.
\end{aligned}
$$

With H and V so defined it easily is checked that the parameters f, q, r and s are given by

$$
\begin{aligned}
f(x) &= 1_{H^c}(x)\,, \\
q(x, y) &= p(x, y) 1_{V^c}(x, y)\,, \\
r(x, y) &= p(x, y) 1_V(x, y) / \textstyle\sum_z p(x, z) 1_V(x, z)\,, \qquad x \in H^c \\
s(x, y) &= p(x, y)\,,
\end{aligned}
$$

where $r(x, y)$ may be defined arbitrarily for $x \in H$, and 1_B is the indicator of a subset B of either J or $J \times J$.

For τ a coterminal time there is a subset V of $J \times J$ and an invariant event C_∞ such that

$$
(\tau \leqq n) = (\theta_n \in C)\,, \qquad n \in N\,,
$$

where C is the coterminal event $((X_k, X_{k+1}) \in V, k \in N) C_\infty$. Define functions f,

$\bar{f}$, g and h from J to $[0, 1]$ by setting for $x \in J$

$$
\begin{aligned}
f(x) &= P^x(\tau > 0) = P^x(C^c)\,, \\
\bar{f}(x) &= P^x(\tau = 0) = P^x(C)\,, \\
g(x) &= P^x(\tau = 1) = P^x[(X_0, X_1) \in V^c,\ \theta_1 \in C]\,, \\
h(x) &= P^x(\tau = \infty) = P^x[((X_n, X_{n+1}) \in V^c \text{ infinitely often}) \cup C_\infty{}^c]\,.
\end{aligned}
$$

Then f, $\bar{f}$ and g are related by the identities $f + \bar{f} = 1$ and

$$g(x) = \textstyle\sum_y p(x, y) 1_{V^c}(x, y) \bar{f}(y)\,, \qquad x \in J\,.$$

and it may also be observed that f is p-excessive, that h is p-harmonic, and that the Riesz decomposition of f is $f = Ug + h$ where $U = \sum_{n=0}^{\infty} p^n$ is the potential operator associated with p. The parameters f, q, r and s for the path decomposition are now readily seen to be specified as follows: f has already been defined,

$$
\begin{aligned}
q(x, y) &= p(x, y) f(y)/f(x) \\
r(x, y) &= p(x, y) 1_{V^c}(x, y) \bar{f}(y)/g(x) \\
s(x, y) &= p(x, y) 1_V(x, y) \bar{f}(y)/\bar{f}(x)\,,
\end{aligned}
$$

where the definition of any of these quantities is arbitrary if the denominator on the right-hand side is zero. In this case parts (i), (ii) and (iv) of the path decomposition statement (6.1) are the discrete analogues of Theorems (2.1) and (5.1) of Meyer, Smythe and Walsh [4]. Part (iii) provides the inner link between the pre-τ and post-τ processes which is required for the full statement of the path decomposition.

We now establish a characterization of terminal and coterminal times by the path decomposition (6.1).

(6.2) THEOREM. *Suppose p is irreducible and let $P = P^x$ for some fixed $x \in J$. A random time τ is both a regular birth time and a regular death time for P if and only if τ is P-equivalent to either a terminal time or a coterminal time.*

REMARK. The characterization fails without some hypothesis on p. If p induces two closed communicating classes A and B and a transient state x from which absorption into either A or B is possible, examples can be given where τ equals a terminal time on paths entering A and a coterminal time on paths entering B.

PROOF. The "if" part is contained in Theorems (3.9) and (5.2). For the "only if" part, observe that if τ is a regular birth time and a regular death time for P, then by the same results

(6.3) $$\tau = \tau_C + \rho \quad P\text{-a.s.},$$

(6.4) $$\tau = \tau_{V,F} \quad P\text{-a.s.}$$

where τ_C is a coterminal time associated with some coterminal event C, ρ is an optional time for $\{\mathscr{F}_{\tau_C+n}, n \geqq 0\}$, and $\tau_{V,F}$ is given as in (5.1) for some $V \subset J \times J$, $F \in \mathscr{F}$. In particular, if $\tau_V \geqq 1$ is the terminal time associated with V, we have

$$(6.5) \qquad \tau_C \leqq \tau \leqq \tau_V \quad P\text{-a.s.}$$

Now C is the intersection of an invariant event C_∞ with an event requiring that all transitions belong to some subset, W^c say, of $J \times J$. By (6.5) therefore $\tau_C = \tau = \tau_V = \infty$, P-a.s. on $C_\infty{}^c$ so that $C_\infty{}^c \subset ((X_n, X_{n+1}) \in V^c, n \in N)$, P-a.s., which because P is irreducible is possible only if either $PC_\infty{}^c = 0$ or transitions in V are impossible. Thus either (i) $\tau_C = \sigma_W$ P-a.s., where σ_W is the last time a transition in W is completed, or (ii) $\tau_V = \infty$ P-a.s. In case (i) the inequality $\sigma_W \leqq \tau_V$ shows it is impossible to perform a W transition after a V transition, so that either W or V must consist of transitions which are impossible under p (using irreducibility again). Thus we may assume $\tau_C = 0$ P-a.s., i.e., that τ is P equivalent to an optional time ρ, since otherwise we are back to case (ii). It was observed below Proposition (5.3) that an optional time satisfying (6.4) exactly must be terminal, but here we only have (6.4) with $\tau = \rho$ P-a.s. It is still true that τ is P equivalent to a terminal time, but the definition of this time now depends on the transition matrix p. From $\tau = \rho$ P-a.s. for optional ρ it follows that $P(\tau > n \mid \mathscr{F}_n) = 1_{(\tau > n)}$ P-a.s., while by (6.4) and (5.3) the same conditional probability becomes $1_{(\tau_V > n)} a(X_n)$ where $a(y) = P^y(\tau_{V,F} > 0)$. Thus for $n \in N$

$$(\tau > n) = (\tau_V > n,\ a(X_n) > 0) \quad P\text{-a.s.},$$

and hence also

$$(\tau > n) = \textstyle\bigcap_{m \leqq n} (\tau > m) = (\tau_V > n,\ a(X_m) > 0,\ 0 \leqq m \leqq n) \quad P\text{-a.s.}$$

which shows that τ is P-equivalent to a terminal time.

Similarly in case (ii), $\tau = \tau_{\varnothing,F}$ P-a.s. implies $P(\tau \leqq n \mid \mathscr{F}_n) = b(X_n)$, where $b(y) = P^y(\tau_{\varnothing,F} = 0)$, while (6.3) gives $(\tau \leqq n) = F_n(\theta_n \in C)$ for some $F_n \in \mathscr{F}_n$ so that the conditional probability equals $1_{F_n} P^{X(n)}C$. Thus $F_n(P^{X(n)}C > 0) = (b(X_n) > 0)$ P-a.s., whence

$$(\tau \leqq n) = F_n(\theta_n \in C,\ P^{X(n)}C > 0) = (\theta_n \in C,\ b(X_m) > 0,\ m \geqq n) \quad P\text{-a.s.},$$

showing that τ is P-equivalent to a coterminal time $\tau_{\hat{C}}$ for a new coterminal event $\hat{C}$. The theorem is proved.

7. Possible extensions. Though we have restricted ourselves in this paper to Markov processes in discrete time with countable state space, the concepts of birth times, death times, and conditional independence times can all be formulated for Markov processes with more general time set or state space. We conclude in this section with some comments on the difficulties involved in extending our results to apply to these situations.

A few of the results do carry over to apply to Markov processes with abstract measurable state space and time set T either N or $[0, \infty)$. Copying the definition from Section 4 of a death time for a family of Markov probabilities, a generalization of Proposition (4.2) remains valid, and by adopting an analogous definition of a conditioning event for a family rather than a single Markov probability, it can be shown that F is such an event if and only if there are events $C \in \mathscr{F}$, $F_t \in \mathscr{F}_t$, $t \in T$ such that $F = F_t(\theta_t \in C)$, P^x-a.s. for every x and every $t \geqq 0$ (cf. (2.10) and Jacobsen [3], Lemma 1).

As for our other results, the assumption of a countable state space is used chiefly to avoid measure theoretical problems in the proofs of the harder "only if" assertions of Theorems (2.3), (3.9) and (5.2), while the restriction to discrete time is essential for our treatment of conditional independence across a random time. The basic criterion for deciding whether a random time possesses the conditional independence property is Lemma (3.12), the proof of which relies on the fact that within the set $A(\tau = n)$, where A is an arbitrary atom in $\mathscr{F}_n$, the conditional probability law of θ_τ given the pre-τ field may be determined as a conditional P^y-probability given the event G_{nA} which is the section of $(\tau = n)$ beyond A. A generalization of this to processes in continuous time fails, partly because the conditioning event G_{nA} may now have measure 0 for more than a negligible collection of atoms A, and partly because, even when this is not the case, it is no longer obvious that the desired conditional probability given $\mathscr{F}_\tau$ results. For some criteria for conditional independence and some of the subtleties involved see Getoor and Sharpe [2], Jacobsen [3], Pittenger and Shih [7].

Finally it may be observed that results for inhomogeneous Markov chains with countable state space can be read off from the present results for homogeneous chains by using the device of considering the space-time chain.

Acknowledgment. We would like to thank the referee for a number of helpful criticisms.

REFERENCES

[1] FREEDMAN, D. (1971). *Markov Chains*. Holden-Day, San Francisco.

[2] GETOOR, R. K. and SHARPE, M. J. (1973). Last exit decompositions and distributions. *Indiana Univ. Math. J.* **23** 377–404.

[3] JACOBSEN, M. (1974). Splitting times for Markov processes and a generalized Markov property for diffusions. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **30** 27–43.

[4] MEYER, P. A., SMYTHE, R. T. and WALSH, J. B. (1972). Birth and death of Markov processes. *Proc. Sixth Berkeley Symp. Math. Statist. Prob.* **3** 295–305, Univ. of California Press.

[5] PITMAN, J. W. (1974). Path decomposition for conditional Brownian motion. Inst. of Math. Statist., Univ. of Copenhagen, Preprint No. 11.

[6] PITMAN, J. W. (1975). 1-dimensional Brownian motion and the 3-dimensional Bessel process. *Advances in Appl. Probability* **7** 511–526.

[7] PITTENGER, A. O. and SHIH, C. T. (1973). Coterminal families and the strong Markov property. *Trans. Amer. Math. Soc.* **182** 1–42.

[8] WALSH, J. and WEIL, M. (1972). Représentation de temps terminaux et applications aux fonctionelles additives et aux systèmes de Lévy. *Ann. Sci. École Norm. Sup.* **5** 121–155.

[9] WILLIAMS, D. (1970). Decomposing the Brownian path. *Bull. Amer. Math. Soc.* **76** 871–873.

[10] WILLIAMS, D. (1974). Path decomposition and continuity of local time for one-dimensional diffusions. *Proc. London Math. Soc.* **28** 738–768.

INSTITUTE OF MATHEMATICAL STATISTICS
UNIVERSITY OF COPENHAGEN
5, UNIVERSITETPARKEN
2100 COPENHAGEN Ø
DENMARK

STATISTICAL LABORATORY
DEPARTMENT OF PURE MATHEMATICS AND MATHEMATICAL STATISTICS
16 MILL LANE
CAMBRIDGE CB2 1SB
ENGLAND