The Annals of Probability
1977, Vol. 5, No. 3, 387–394

SPACING DISTRIBUTION ASSOCIATED WITH A STATIONARY RANDOM MEASURE ON THE REAL LINE[1]

By Sidney C. Port and Charles J. Stone

University of California, Los Angeles

Let $\mathscr{N}$ denote the collection of all Radon measures n on $\mathbb{R}$ such that $0 < \lim_{x\to\infty} n((0, x])/x = \lim_{x\to-\infty} n((x, 0])/|x| < \infty$. For $n \in \mathscr{N}$, let $n^{-1} \in \mathscr{N}$ be the measure whose distribution function is the inverse of the distribution function of n. Given a random element N of $\mathscr{N}$ having distribution P, let P^I denote the distribution of N^{-1}. Let N be a random element of $\mathscr{N}$ having stationary distribution P and let P^T be the appropriately defined tagged distribution corresponding to P. It is shown that P^I has an asymptotically stationary distribution P^S on $\mathscr{N}$. Moreover $P = (P^S)^S$, $P^I = (P^S)^T$, and $P^T = (P^S)^I$. P^S is given explicitly in terms of P^T. In particular, if N is purely nonatomic with probability one, then $P^S = (P^T)^I$. If P is a stationary compound renewal process, then so is P^S.

1. Description of results. Let n be a Radon measure on the real line $\mathbb{R}$, that is, a measure on the Borel sets in $\mathbb{R}$ such that $n(C) < \infty$ whenever C is compact. For $x \in \mathbb{R}$, let n_x be the Radon measure determined by $n_x(A) = n(x + A)$. Here $x + A = \{x + y : y \in A\}$. Set $n(x) = n((0, x])$ for $x \geqq 0$ and $n(x) = -n((x, 0])$ for $x < 0$. Then $n(x)$, $x \in \mathbb{R}$, is determined uniquely by the requirements that $n(0) = 0$ and $n((a, b]) = n(b) - n(a)$ for $-\infty < a < b < \infty$ (using the same symbol to denote both the measure and the corresponding distribution function is convenient and should cause no confusion). Let $\bar{n} = \lim_{|x|\to\infty} n(x)/x$ if the indicated limit, finite or infinite, exists, in which case $\bar{n}_x = \bar{n}$ for all $x \in \mathbb{R}$.

Let n be such that $n([0, \infty)) = n((-\infty, 0]) = +\infty$. For $x \in \mathbb{R}$ set $l(x) = \sup[y : n(y) \leqq x]$ and let n^{-1} be the Radon measure on $\mathbb{R}$ determined by $n^{-1}((a, b]) = l(b) - l(a)$ for $-\infty < a \leqq b < \infty$. If $\bar{n}$ exists, then $\overline{(n^{-1})} = 1/\bar{n}$.

Let $\mathscr{N}$ denote the collection of all Radon measures n on $\mathbb{R}$ such that $\bar{n}$ exists and $0 < \bar{n} < \infty$. Then $\mathscr{N}$ can be made into a measure space by choosing the smallest σ-algebra on $\mathscr{N}$ such that for each Borel set $A \subset R$ and each $t \in [0, \infty]$, $\{n \in \mathscr{N} : n(A) \leqq t\}$ is measurable. Let $\mathscr{F}$ denote the collection of all bounded measurable real valued functions on $\mathscr{N}$.

Let N be a random element of $\mathscr{N}$ and let P denote the distribution of N. N and P are said to be *stationary* if N_x has distribution P for all $x \in \mathbb{R}$. Let $\mathscr{P}$ denote the collection of all such stationary probability distributions P on $\mathscr{N}$.

Given $P \in \mathscr{P}$ there is a unique probability distribution P^T on $\mathscr{N}$, called the

Received March 28, 1976.
[1] This research was supported in part by NSF Grant No. MPS 72-04591.
AMS 1970 *subject classifications.* Primary 60K99; Secondary 60K05.
Key words and phrases. Random measure, Palm measure, tagged distribution, spacing distribution, Poisson process, compound renewal process.

tagged distribution associated with P such that for all $f \in \mathscr{F}$ and all Borel subsets $A \subset \mathbb{R}$

$$(1) \qquad |A| \int P^T(dN)f(N) = \int P(dN)(1/\bar{N}) \int f(N_x)1_A(x)N(dx) .$$

The relation between this definition of P^T and the usual definition of the tagged distribution (Palm measure) used, for example, in Mecke (1967) and Port and Stone (1973) will be discussed at the end of this section.

Let N be distributed according to $P \in \mathscr{P}$. The distribution P^I of N^{-1} is called the *inverse* distribution associated with P. It is a probability distribution on $\mathscr{N}$ but it need not be stationary. According to Theorem 1, however, it is asymptotically stationary. That is, there is a probability distribution P^S on $\mathscr{N}$ such that

$$(2) \qquad \int P^S(dN)f(N) = \lim_{b\to\infty} \int P^I(dN)(1/b) \int_a^{a+b} f(N_x)\, dx$$

holds for all $f \in \mathscr{F}$ uniformly for $a \in \mathbb{R}$. If P^S satisfies (2), it is called the *spacing* distribution associated with P. The main purpose of this paper is to prove the following result, which asserts that a spacing distribution always exists. The proof of this result will be given in Section 2.

THEOREM 1. *Let $P \in \mathscr{P}$. Then there is a $P^S \in \mathscr{P}$ such that for $f \in \mathscr{F}$ and $a > 0$*

$$(3) \qquad \int P^S(dN)f(N) = \int P(dN)(1/a\bar{N}) \int_0^{N(a)} f((N^{-1})_x)\, dx .$$

P^S is the spacing distribution associated with P. Moreover $P = (P^S)^S$, $P^I = (P^S)^T$, and $P^T = (P^S)^I$. P^S is given explicitly in terms of P^T according to the formula

$$(4) \qquad \int P^S(dN)f(N) = \int P^T(dN)\Big[1_{\{N(\{0\})=0\}} \frac{1}{N(\{0\})} \int_{-N(\{0\})}^0 f((N^{-1})_x)\, dx + 1_{\{N(\{0\})=0\}} f(N^{-1})\Big] .$$

If P is concentrated on the purely nonatomic measures in $\mathscr{N}$, then $P^S = (P^T)^I$.

Suppose $P \in \mathscr{P}$. If P is concentrated on the purely atomic measures in $\mathscr{N}$, then P^T is concentrated on $\{n \in \mathscr{N}: n(\{0\}) > 0\}$. In this case (4) implies that P^S is determined from P^T according to

$$(5) \qquad \int P^S(dN)f(N) = \int P^T(dN) \frac{1}{N(\{0\})} \int_{-N(\{0\})}^0 f((N^{-1})_x)\, dx .$$

Let $\mathscr{N}_1$ denote the measures in $\mathscr{N}$ which are concentrated on the integers and let $\mathscr{N}_2$ denote the integer valued measures in $\mathscr{N}$. Let $P \in \mathscr{P}$ be concentrated on $\mathscr{N}_2$. It follows from (5) that

$$(6) \qquad \int P^S(dN)f(N) = \int P_1^S(dN) \int_{-1}^0 f(N_x)\, dx ,$$

where P_1^S is the distribution on $\mathscr{N}_1$ such that for $f \in \mathscr{F}$

$$(7) \qquad \int P_1^S(dN)f(N) = \int P^T(dN) \frac{1}{N(\{0\})} \sum_{i=0}^{N(\{0\})-1} f((N^{-1})_i) .$$

(In particular, if P is concentrated on the measures in $\mathscr{N}_2$ such that every atom has measure 1, then $P_1^S = (P^T)^I$.) Let N_1 be distributed according to P_1^S. Then N_1 can be written as $N_1(A) = \sum_i \xi_i 1_{\{i \in A\}}$, where $\{\xi_i\}_{-\infty}^{\infty}$ is a stationary sequence of nonnegative random variables. Let U denote a random variable which is independent of $\{\xi_i\}_{-\infty}^{\infty}$ and uniformly distributed on [0, 1] and let N be the random measure on $\mathbb{R}$ determined by $N(A) = \sum_i \xi_i 1_{\{U+i \in A\}}$. Then N has distribution P^S. In particular, if P is the distribution of the Poisson process on $\mathbb{R}$ with parameter λ, then P^S and P_1^S are of the above form, where the ξ_i's are independent exponential random variables with mean λ^{-1}. That a *stationary spacing sequence* arises naturally in connection with stationary integer valued random measures has been shown by Ryll-Nardzewski (1961), Slivnjak (1962), and Port and Stone (1973).

For an example of a $P \in \mathscr{P}$ for which P^S can be determined explicitly, consider a stationary compound renewal process in which the arrival times $\{x_i\}_{-\infty}^{\infty}$ correspond to a spacing sequence $\{\xi_i\}_{-\infty}^{\infty}$ consisting of i.i.d. nonnegative random variables having finite positive mean and at each arrival time x_i a mass η_i is put down, where $\{\eta_i\}_{-\infty}^{\infty}$ is another sequence of i.i.d. nonnegative random variables having finite positive mean and $\{\eta_i\}_{-\infty}^{\infty}$ is independent of $\{x_i\}_{-\infty}^{\infty}$. Then $N(A) = \sum_i \eta_i 1_{\{x_i \in A\}}$. Let P denote the distribution of N. Then $P \in \mathscr{P}$. It is not hard to show that P^S is a stationary compound renewal process with the roles of $\{\xi_i\}$ and $\{\eta_i\}$ reversed. A sketch of the proof of this result will be given in Section 3. If P is the distribution of a Poisson process on $\mathbb{R}$ with parameter λ, it is of the above form with the ξ_i's exponentially distributed with mean λ^{-1} and the η_i's identically equal to one.

The definitions of P^T and P^S given above are desirable from several viewpoints. Firstly, they are well defined even if $\alpha(P) = \int P(dN)\bar{N} = \infty$. Secondly, (2) holds. Thirdly, $(\lambda P_1 + (1-\lambda)P_2)^T = \lambda P_1^T + (1-\lambda)P_2^T$ for $0 < \lambda < 1$ and the same equation holds with T replaced by S.

The usual definition of the tagged particle distribution (Palm measure) requires that $\alpha(P) < \infty$ and replaces $\bar{N}$ by $\alpha(P)$ in (1). Let P^{T*} be the resulting distribution. Then $P^{T*}(dN) = (\bar{N}/\alpha(P))P^T(dN)$. In this context it is natural to define P^{S*} by $P^{S*}(dN) = (1/\bar{N}\alpha(P))P^S(dN)$. Then $P^{S*} \in \mathscr{P}$, $\alpha(P^{S*}) = 1/\alpha(P)$, $(P^{S*})^{S*} = P$, $(P^{S*})^{T*} = P^I$, and $(P^{S*})^I = P^{T*}$. Equation (4) holds with P^S and P^T replaced by P^{S*} and P^{T*} respectively. The same remark holds for equations (6) and (7) when $P \in \mathscr{P}$ is concentrated on $\mathscr{N}_2$.

2. Proof of Theorem 1. The following result is a straightforward consequence of the ergodic theorem.

PROPOSITION 1. *Let $\{X_i\}_{-\infty}^{\infty}$ be a stationary sequence of nonnegative random variables. Then $\bar{X} = \lim_n (X_1 + \cdots + X_n)/n$ exists almost surely. If $P(0 < \bar{X} < \infty) = 1$, then $\{X_i/\bar{X}\}_{-\infty}^{\infty}$ determines a stationary sequence of nonnegative random variables each having mean one.*

PROOF. Set $X = X_0$. Without loss of generality it can be assumed that $X_i(\omega) = X(T^i\omega)$, $\omega \in \Omega$, where T is a measure preserving shift transformation on Ω. Let $\mathscr{I}$ denote the σ-algebra of events in Ω which are invariant under transformation by T. Set $X^{(M)} = X1_{\{X \leq M\}}$ and $E[X|\mathscr{I}] = \lim_{M\to+\infty} E[X^{(M)}|\mathscr{I}]$. It follows from the ergodic theorem and an easy truncation argument that $\bar{X} = \lim_n (X_1 + \cdots + X_n)/n = \infty$ almost surely on the event $\{E[X|\mathscr{I}] = \infty\}$. For $0 < M < \infty$ set $Y^{(M)} = X1_{\{E[X|\mathscr{I}]\leq M\}}$. Then $EY^{(M)} \leq M < \infty$ and $E[Y^{(M)}|\mathscr{I}] = E[X|\mathscr{I}]1_{\{E[X|\mathscr{I}]\leq M\}}$. Application of the ergodic theorem to $Y^{(M)}(T^i\omega)$ yields the conclusion that $\bar{X} = \lim_n (X_1 + \cdots + X_n)/n = E[X|\mathscr{I}]$ almost surely on $\{E[X|\mathscr{I}] \leq M\}$. Consequently $\bar{X} = \lim_n (X_1 + \cdots + X_n)/n = E[X|\mathscr{I}]$ almost surely. Suppose $P(0 < \bar{X} < \infty) = 1$. Set $Y = X_0/\bar{X}$. It is not hard to show that $E[Y|\mathscr{I}] = 1$ almost surely and hence that $EY = 1$. This completes the proof of the proposition.

Observe that if $n \in \mathscr{N}$, then $(n_x)_y = n_{x+y}$, $n_x(y) = n(x+y) - n(x)$, and $(n_x)^{-1} = (n^{-1})_{n(x)}$ for $x, y \in \mathbb{R}$. Also if $g: \mathbb{R} \to \mathbb{R}$ is a bounded Borel function, then $\int g(y)n_x(dy) = \int g(y - x)n(dy)$.

PROPOSITION 2. *Let $P \in \mathscr{P}$, let $f \in \mathscr{F}$, and let $g: \mathbb{R} \to \mathbb{R}$ be a bounded Borel function. Then*

$$\int P^T(dN) \int f(N_x)g(x)\,dx = \int P(dN)(f(N)/\bar{N}) \int g(-x)N(dx) \tag{8}$$

and

$$\lim_{b\to\infty} \int P^T(dN)(1/b) \int_a^{a+b} f(N_x)\,dx = \int P(dN)f(N) \tag{9}$$

uniformly for $a \in \mathbb{R}$.

PROOF. By (1)

$$\begin{aligned}\int P^T(dN) \int f(N_x)g(x)\,dx &= \int P(dN)(1/\bar{N}) \iint f(N_{x+y})1_{(0,1)}(x)g(y)\,dy\,N(dx)\\ &= \int P(dN)(1/\bar{N}) \iint f(N_t)1_{(0,1)}(x)g(t-x)\,dt\,N(dx)\\ &= \int P(dN)(f(N)/\bar{N}) \iint 1_{(0,1)}(x)g(t-x)\,dt\,N_{-t}(dx)\\ &= \int P(dN)(f(N)/\bar{N}) \iint 1_{(0,1)}(x+t)\,dt\,g(-x)N(dx)\\ &= \int P(dN)(f(N)/\bar{N}) \int g(-x)N(dx)\end{aligned}$$

and hence (8) holds. It follows from (8) that

$$\begin{aligned}&\int P^T(dN)(1/b) \int_a^{a+b} f(N_x)\,dx\\ &\quad = \int P(dN)f(N)(1/b)\left(\frac{N(-a) - N(-a-b)}{\bar{N}}\right).\end{aligned} \tag{10}$$

Let M be an upper bound to $|f|$. Then by (10)

$$\begin{aligned}&\left|\int P^T(dN)(1/b) \int_a^{a+b} f(N_x)\,dx - \int P(dN)f(N)\right|\\ &\quad \leq M \int P(dN)\left|\frac{N(-a) - N(-a-b)}{b\bar{N}} - 1\right|\\ &\quad = M \int P(dN)\left|\frac{N(b) - N(0)}{b\bar{N}} - 1\right|.\end{aligned}$$

Equation (9) now follows from Proposition 1 and the ergodic theorem.

PROPOSITION 3. *If $P \in \mathscr{P}$, there is a $P^S \in \mathscr{P}$ satisfying* (3).

PROOF. Let $f \in \mathscr{F}$, $a \geqq 0$, and $b \geqq 0$. Then by the stationarity of P

$$\begin{aligned}\int P(dN)(1/\bar{N}) \int_{N(b)}^{N(a+b)} f((N^{-1})_x)\, dx &= \int P(dN)(1/\bar{N}) \int_{N_{-b}(b)}^{N_{-b}(a+b)} f(((N_{-b})^{-1})_x)\, dx \\ &= \int P(dN)(1/\bar{N}) \int_{-N(-b)}^{N(a)-N(b)} f((N^{-1})_{x+N(-b)})\, dx \\ &= \int P(dN)(1/\bar{N}) \int_0^{N(a)} f((N^{-1})_x)\, dx\,.\end{aligned}$$

Denote the last expression by $g(a)$. Then g is a continuous function on $[0, \infty)$ and $g(a+b) \equiv g(a) + g(b)$. Thus $g(a) \equiv ca$ for some constant c. In other words, the right side of (3) is independent of a for $a > 0$. Consequently, by Proposition 1, there is a probability distribution P^S on $\mathscr{N}$ satisfying (3). To see that P^S is stationary observe that for $t \in \mathbb{R}$

$$\begin{aligned}\int P^S(dN) f(N_t) &= \int P(dN)(1/a\bar{N}) \int_0^{N(a)} f(N^{-1})_{x+t})\, dx \\ &= \int P(dN)(1/a\bar{N}) \int_t^{t+N(a)} f((N^{-1})_x)\, dx\,.\end{aligned}$$

Thus

$$\begin{aligned}&\int P^S(dN) f(N_t) - \int P^S(dN) f(N) \\ &\quad = \int P(dN)(1/a\bar{N}) \int_{N(a)}^{t+N(a)} f((N^{-1})_x)\, dx - \int P(dN)(1/a\bar{N}) \int_0^t f((N^{-1})_x)\, dx \\ &\quad = \int P(dN)(1/a\bar{N}) \int_0^t f(((N_a)^{-1})_x)\, dx - \int P(dN)(1/a\bar{N}) \int_0^t f((N^{-1})_x)\, dx \\ &\quad = 0\,.\end{aligned}$$

Therefore P^S is stationary and hence $P^S \in \mathscr{P}$, as desired.

Given $N \in \mathscr{N}$, set $L(x) = \sup[y\colon N(y) \leqq x]$. Then $N^{-1}(x) = L(x) - L(0)$. Also $((N^{-1})_x)^{-1} = N_{L(x)}$. For $b > 0$

$$\begin{aligned}\{y\colon L(x) < y < L(x+b)\} &\subset \{y\colon x < L(y) \leqq x + b\} \\ &\subset \{y\colon L(x) \leqq y \leqq L(x+b)\}\,.\end{aligned}$$

Thus these three sets differ from each other only on a set having Lebesgue measure zero.

PROPOSITION 4. *If $P \in \mathscr{P}$, then $(P^S)^S = P$.*

PROOF. Choose $f \in \mathscr{F}$, $a > 0$, and $b > 0$. Then

$$\begin{aligned}&\int (P^S)^S(dN) f(N) \\ &\quad = \int P^S(dN)(1/b\bar{N}) \int_0^{N(b)} f((N^{-1}))_y)\, dy \\ &\quad = \int P(dN)(1/ab) \int_0^{N(a)} dx (\int_0^{(N^{-1})_x(b)} f(((N^{-1}))_x)^{-1})_y)\, dy) \\ &\quad = \int P(dN)(1/ab) \int_0^{N(a)} dx \int_{L(x)}^{L(x+b)} f(N_y)\, dy \\ &\quad = \int P(dN)(1/ab) \int\int f(N_y) 1_{(0,N(a))}(x) 1_{(x,x+b]}(N(y))\, dx\, dy \\ &\quad = \int P(dN)(1/ab)(f(N) \int\int 1_{(0,N_{-y}(a))}(x) 1_{(x,x+b]}(N_{-y}(y))\, dx\, dy \\ &\quad = \int\int P(dN) f(N)(1/ab) \int\int 1_{(0,N(a-y)-N(-y))}(x) 1_{(x,x+b]}(-N(-y))\, dx\, dy\,.\end{aligned}$$

To complete the proof of the proposition it suffices to show that

$$\int\int 1_{(0,N(a+y)-N(y))}(x) 1_{(x,x+b]}(-N(y))\, dx\, dy = ab\,. \tag{11}$$

The left side of (11) can be written as

$$\int [N(a+y) \wedge 0 - N(y) + (N(y)+b) \wedge 0]^+ \, dy = \int_{L(-b)-a}^{0} [N(a+y) \wedge 0 - N(y) + (N(y)+b) \wedge 0] \, dy \,,$$

where $c \wedge d = \min(c, d)$ and $c^+ = \max(c, 0)$. There are two cases to consider: $L(-b) \geqq -a$ and $L(-b) < -a$. In the first case $L(-b) - a \leqq -a \leqq L(-b) \leqq 0$ and the last integral can be written as

$$\int_{L(-b)-a}^{-a} (N(a+y) + b) \, dy + b \int_{-a}^{L(-b)} dy - \int_{L(-b)}^{0} N(y) \, dy = ab \,.$$

Thus (11) holds if $L(-b) \geqq -a$. The proof that (11) holds if $L(-b) < -a$ is similar. This completes the proof of (11) and hence also that of Proposition 4.

PROPOSITION 5. *If* $P \in \mathscr{P}$, *then* $P^I = (P^S)^T$ *and* $P^T = (P^S)^I$.

PROOF. Choose $f \in \mathscr{F}$ and $a > 0$. Then

$$\begin{aligned} \int P^I(dN) f(N) &= \int (P^S)^S(dN) f(N^{-1}) \\ &= \int P^S(dN)(1/a\bar{N}) \int_0^{N(a)} f(((N^{-1})_x)^{-1}) \, dx \\ &= \int P^S(dN)(1/a\bar{N}) \int_0^{N(a)} f(N_{L(x)}) \, dx \\ &= \int P^S(dN)(1/a\bar{N}) \int_0^a f(N_x) N(dx) \\ &= \int (P^S)^T(dN) f(N) \,, \end{aligned}$$

so that $P^I = (P^S)^T$. Thus $(P^S)^I = ((P^S)^S)^T = P^T$, which completes the proof of the proposition.

PROPOSITION 6. *If* $P \in \mathscr{P}$, *then* P^S *is the spacing process associated with* P.

PROOF. This result follows immediately from Propositions 2 and 5.

PROPOSITION 7. *If* $P \in \mathscr{P}$, *then* (4) *holds.*

PROOF. Set $A = (0, 1]$ in (1). Then the right side of (4) can be written as

$$\begin{aligned} &\int P(dN)(1/\bar{N}) \int_{(0,1]} N(dx)\Big[1_{\{N_x(\{0\})>0\}} \frac{1}{N_x(\{0\})} \int_{-N_x(\{0\})}^{0} f(((N_x)^{-1})_t) \, dt \\ &\quad + 1_{\{N_x(\{0\})=0\}} f((N_x)^{-1})\Big] \\ &\qquad = \int P(dN)(1/\bar{N}) \int_{(0,1]} N(dx)\Big[1_{\{N(\{x\})>0\}} \frac{1}{N(\{x\})} \int_{N(x-)}^{N(x)} f((N^{-1})_t) \, dt \\ &\qquad\quad + 1_{\{N(\{x\})=0\}} f((N^{-1})_{N(x)})\Big] \,. \end{aligned}$$

To complete the proof of the proposition, it suffices to prove that the last quantity equals the right side of (3) with $a = 1$. To do this it is enough to show that if g is a bounded Borel function on $[0, \infty)$, then

$$\begin{aligned} \text{(12)} \qquad &\int_{(0,1]} N(dx)\Big[1_{\{N(\{x\})>0\}} \frac{1}{N(\{x\})} \int_{N(x-)}^{N(x)} g(t) \, dt + 1_{\{N(\{x\})=0\}} g(N(x))\Big] \\ &\quad = \int_0^{N(1)} g(t) \, dt \,. \end{aligned}$$

In proving (12) it is enough to consider functions g of the form $g = 1_{(0,b]}$, where

$b > 0$. If $b > N(1)$, then (12) is obvious for such a function. Suppose $0 < b < N(1)$. Set $c = L(b)$. Then $N(c-) \leqq b \leqq N(c)$. The right side of (12) equals

$$\int_{(0,c)} N(dx) 1_{\{N(\{x\})>0\}} + (b - N(c-)) + \int_{(0,c)} N(dx) 1_{\{N(\{x\})=0\}}$$
$$= \int_{(0,c)} N(dx) + (b - N(c-)) = b \,,$$

which equals the right side of (12), as desired.

Theorem 1 follows from Propositions 3—7.

3. Compound renewal process. Let P correspond to the compound renewal process as described in the introduction. Let $R \in \mathscr{P}$ correspond to the compound renewal process obtained by reversing the roles of $\{\xi_i\}$ and $\{\eta_i\}$. The proof that $P^S = R$ will be sketched in the nonlattice case. A similar proof works in the lattice case.

Suppose then that the distribution of η_1 is nonlattice, i.e., that there is no proper closed subgroup (under addition) of $\mathbb{R}$ containing the support of the distribution of η_1. It is sufficient to show that for $f \in \mathscr{F}$

$$\lim_{a\to+\infty} \int P^I(dN) \int_a^{a+1} f(N_x)\, dx = \int R(dN) f(N) \,. \tag{13}$$

Set $\mu = E\eta_1$. In order to verify (13), it is enough to verify the following result: Let X_0 be independent of $\{\eta_i\}_{-\infty}^{\infty}$ and have an absolutely continuous distribution. Set $T_a = \min\,[k \geqq 1 : X_0 + \eta_1 + \cdots + \eta_k \geqq a]$ and $Z_a = X_0 + \eta_1 + \cdots + \eta_{T_a} - a$. Then as $a \to +\infty$ the distribution of Z_a converges in total variation to the distribution having density $\mu^{-1} P(\eta_1 \geqq x)$, $-\infty < x < \infty$.

This result in turn is implied by the following result: Let H denote the renewal measure associated with $\{\eta_i\}_{-\infty}^{\infty}$ and let g be an absolutely integrable function on $\mathbb{R}$ having compact support. Then

$$\lim_{x\to+\infty} \int_x^{x+1} dy \left| \int g(y-z) H(dz) - \mu^{-1} \int g(z)\, dz \right| = 0 \,. \tag{14}$$

To prove (14) choose $M > 0$ such that $g(y) = 0$ for $|y| > M$. Set

$$C = \sup_x \int_{|z-x| \leqq M+1} H(dz) < \infty \,.$$

Choose $\varepsilon > 0$. Let h be a continuous function on $\mathbb{R}$ supported on $[-M, M]$ and such that $\int h(y)\, dy = \int g(y)\, dy$ and $\int |h(y) - g(y)|\, dy \leqq \varepsilon$. It follows from the renewal theorem in the nonlattice case that

$$\lim_{x\to+\infty} \int h(x-z) H(dz) = \mu^{-1} \int h(y)\, dy \,. \tag{15}$$

Also

$$\int_x^{x+1} dy \left| \int g(y-z) H(dz) - \int h(y-z) H(dz) \right| \leqq \left(\int_{|z-x| \leqq M+1} H(dz) \right) \int |g(y) - h(y)|\, dy \leqq \varepsilon C \,. \tag{16}$$

By (15) and (16),

$$\limsup_{x\to+\infty} \int_x^{x+1} dy \left| \int_{-\infty}^{\infty} g(y-z) H(dz) - \mu^{-1} \int g(z)\, dz \right| \leqq \varepsilon C \,.$$

Since ε can be made arbitrarily small, (14) holds as desired.

REFERENCES

[1] MECKE, J. (1967). Stationäre Zufällige Masse auf lokalkompakten Abelschen Gruppen. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **9** 36–48.

[2] PORT, S. C. and STONE, C. J. (1973). Infinite particle systems. *Trans. Amer. Math. Soc.* **178** 307–339.

[3] RYLL-NARDZEWSKI, C. (1961). Remarks on processes of calls. *Proc. Fourth Berkeley Symp. Math. Statist. and Prob.* **2** 455–465. Univ. of California Press.

[4] SLIVNJAK, I. M. (1962). Some properties of stationary streams of homogeneous random events. *Theor. Probability Appl.* **7** 336–341.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA
LOS ANGELES, CALIFORNIA 90024