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SAMPLE-CONTINUITY OF SQUARE-INTEGRABLE PROCESSES

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Let $\{X(t) : t \in [0, 1]\}$ be a stochastic process and f a nonnegative function on $[0, 1]$ which is nondecreasing in a neighborhood of 0. Under the assumption that $E(X(t) - X(s))^2 \leqq f(|t - s|)$, we find best possible conditions for determining whether or not $X(t)$ is sample-continuous.

1. Introduction. Let $\{X(t), t \in [0, 1]\}$ be a stochastic process with the property that for some $\varepsilon > 0$

$$E(X(t) - X(s))^2 \leqq f(|t - s|) , \qquad |t - s| \leqq \varepsilon , \tag{1.1}$$

for some nonnegative function f on $[0, 1]$ which is nondecreasing on $[0, \varepsilon]$. We consider the problem of determining the sample-continuity properties of $X(t, \omega)$ solely from the information stored in (1.1). It was shown in Hahn (1975) that $X(t)$ is sample-continuous, i.e., there is a version of the process with continuous sample paths, if

$$\int_0 y^{-\frac{3}{2}} f^{\frac{1}{2}}(y)\, dy < \infty . \tag{1.2}$$

In Section 2, Theorem 1 improves the above result by weakening the hypothesis to require that (1.2) holds for a new function $\underline{f}$ which is derived from f and minorizes f. An example is given in Section 5 which shows that the new result is strictly stronger than previous results, specifically (1.2) and results of Garsia and Rodemich ((1974), page 104).

Furthermore, we show that if the only known information about a process $X(t, \omega)$ takes the form of condition (1.1), then Theorem 1 is the best possible result which can be obtained. This is shown in Section 4, where for each function f whose associated function $\underline{f}$ does not satisfy Theorem 1, we construct a discontinuous process $X(t)$ such that

$$E(X(t) - X(s))^2 \leqq \underline{f}(|t - s|) .$$

Not only does $X(t)$ fail to be sample-continuous but it also does not have a version with finite right and left limits at all points. This is unavoidable due to a theorem of Kallenberg (1973).

One reason for desiring sufficient conditions under which (1.1) implies sample-continuity is that such conditions are also sufficient for uniform tightness of the measures induced by a family of processes all satisfying (1.1) with the same f. Thus, for instance, we can now conclude that if $X_1, X_2, \cdots$ are

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independent, mean 0, stochastic processes on [0, 1] with the same distribution, say $\mathscr{L}(X)$, and if X satisfies (1.1) with an f satisfying Theorem 1, then

(i) $Z_n(t, \omega) = n^{-\frac{1}{2}} \sum_1^n X_i(t, \omega)$ is a sample-continuous process for each n;

(ii) there exists a continuous Gaussian process Z with the same covariance as X such that

$$\mathscr{L}(Z_n) \to \mathscr{L}(Z) \qquad \text{weakly in} \quad C[0, 1] ;$$

i.e., if $P = \mathscr{L}(Z)$ and $P_n = \mathscr{L}(Z_n)$ then for every bounded continuous real function F on $C[0, 1]$

$$\int_{C[0,1]} F(x)\, dP_n(x) \to \int_{C[0,1]} F(x)\, dP(x) .$$

For more specific details concerning this last application, see Hahn (1975).

2. Sufficient conditions for sample-continuity. We begin by associating to each nonnegative function f which is nondecreasing on $[0, \varepsilon]$ another function $\underline{f}$ defined by

$$\begin{aligned} \underline{f}(s) &= \inf_{y \geq 1} y^2 f(s/y) && \text{if} \quad s \in [0, \varepsilon] \\ &= f(s) && \text{if} \quad s > \varepsilon . \end{aligned} \tag{2.1}$$

Clearly, $\underline{f}(s)$ is nondecreasing on $[0, \varepsilon]$.

Using $\underline{f}$, a particularly simple condition for sample-continuity of a stochastic process can be formulated as follows:

THEOREM 1.[2] *Let f be a nonnegative function which is nondecreasing on* $[0, \varepsilon]$. *Suppose that*

$$\int_0 y^{-\frac{3}{2}} \underline{f}^{\frac{1}{2}}(y)\, dy < \infty . \tag{2.2}$$

If $\{X(t),\, t \in [0, 1]\}$ *is a stochastic process with*

$$E(X(t) - X(s))^2 \leqq f(|t - s|) , \qquad |t - s| \leqq \varepsilon , \tag{2.3}$$

then $X(t)$ is sample-continuous.

PROOF. In Hahn (1975) (Theorem 2.5) it was shown that condition (2.2) with $\underline{f}$ replaced by f is a sufficient condition for the sample-continuity of $X(t)$. We will show that

$$E(X(t) - X(s))^2 \leqq 4\underline{f}(|t - s|) , \qquad |t - s| \leqq \varepsilon .$$

Then the result just quoted from Hahn (1975) yields the desired conclusion.

If $|t - s| \leqq \varepsilon$ then for any integer $k \geqq 1$,

$$\begin{aligned} (X(t) - X(s))^2 &= \left(\textstyle\sum_{j=1}^k X(s + j(t-s)/k) - X(s + (j-1)(t-s)/k)\right)^2 \\ &\leqq k \textstyle\sum_{j=1}^k (X(s + j(t-s)/k) - X(s + (j-1)(t-s)/k))^2 . \end{aligned}$$

[2] By defining $\underline{f}_r(s) = \inf_{y \geq 1} y^r f(s/y)$, it can be shown in an analogous manner that $E|X(t) - X(s)|^r \leqq f(|t - s|)$ and $\int_0 y^{-(r+1)/r} \underline{f}_r^{1/r}(y)\, dy < \infty$ imply $X(t)$ is sample-continuous.

Taking expectations and using (2.3),

$$E(X(t) - X(s))^2 \leqq k \textstyle\sum_{j=1}^{k} f(|t-s|/k) = k^2 f(|t-s|/k) . \tag{2.4}$$

Since (2.4) holds for all $k \geqq 1$,

$$E(X(t) - X(s))^2 \leqq \inf_{k \in \mathbb{N}} k^2 f(|t-s|/k) . \tag{2.5}$$

If $u \in [0, \varepsilon]$ and y is such that $k \leqq y \leqq k + 1$, $k \in \mathbb{N}$, then

$$4y^2 f(u/y) \geqq 4y^2 f(u/(k+1)) \geqq 4k^2 f(u/(k+1)) \geqq (k+1)^2 f(u/(k+1)) . \tag{2.6}$$

Hence, using (2.5) and (2.6),

$$\begin{aligned} E(X(t) - X(s))^2 &\leqq \inf_{k \in \mathbb{N}} k^2 f(|t-s|/k) \\ &\leqq 4 \inf_{y \geqq 1} y^2 f(|t-s|/y) \\ &= 4\underline{f}(|t-s|) . \end{aligned} \tag{2.7}$$

□

The virtue of the above theorem over the one in Hahn (1975) is that (2.2) may be satisfied even though $\int_0 y^{-\frac{3}{2}} f^{\frac{1}{2}}(y)\, dy = \infty$ (see Section 5). The examples to be constructed in Section 4 will show that, under the given hypothesis, Theorem 1 provides an optimal sufficient condition for sample-continuity.

For applications the following corollary may be more useful because the conditions involve the given function f somewhat more directly.

COROLLARY 2. *Let f be a nonnegative function which is nondecreasing on* $[0, \varepsilon]$. *Suppose that either*

$$\int_0 y^{-\frac{3}{2}} f^{\frac{1}{2}}(y)\, dy < \infty \tag{2.8}$$

or

$$\liminf_{y \to 0} y^{-2} f(y) < \infty . \tag{2.9}$$

If $\{X(t), t \in [0, 1]\}$ *is a stochastic process with*

$$E(X(t) - X(s))^2 \leqq f(|t-s|) , \qquad |t-s| \leqq \varepsilon ,$$

then $X(t)$ *is sample-continuous.*

PROOF. Since $\underline{f}(s) \leqq f(s)$ for all s, condition (2.8) implies condition (2.2), hence sample-continuity of $X(t)$.

Assume (2.9). Since $\underline{f}(y) \leqq f(y)$, $\underline{f}$ satisfies condition (2.9) also. An important property of $\underline{f}$ to be used here and later is:

$$y^{-2}\underline{f}(y) \quad \text{is nonincreasing on} \quad (0, \varepsilon] . \tag{2.10}$$

Suppose not, then there exists $c > 1$ and $0 < y_1 < y_2 \leqq \varepsilon$ such that

$$y_2^{-2}\underline{f}(y_2) > c y_1^{-2}\underline{f}(y_1) .$$

By the construction of $\underline{f}$, there exists $1 \leqq y < \infty$ such that

$$c\underline{f}(y_1) \geqq y^2 f(y_1/y) = y^2 f(y_2/(yy_2/y_1)) \geqq (y_2/y_1)^{-2}\underline{f}(y_2)\,.$$

This gives the desired contradiction.

Using (2.10) and the fact that $\underline{f}$ satisfies (2.9), we see that $y^{-2}\underline{f}(y) \uparrow C < \infty$ as $y \downarrow 0$. Thus, $\underline{f}(x) \leqq Cx^2$ for all x, and condition (2.2) is satisfied. □

Note that by change of variables, (2.2) is equivalent to

$$\int_{1/\varepsilon}^{\infty} y^{-\frac{1}{2}}\underline{f}^{\frac{1}{2}}(1/y)\,dy < \infty\,. \tag{2.11}$$

3. Properties of $\underline{f}$. In order that Theorem 1 be best possible it is necessary that $\underline{f}$ distills essentially all of the information regarding $E(X(t) - X(s))^2$ given by f. The obvious candidate is not $\underline{f}$, but the function f^* defined by

$$f^*(y) = \inf\{n \textstyle\sum_{j=1}^{n} f(t_j) : \sum_{j=1}^{n} t_j = y,\ t_j \geqq 0,\ n \geqq 1\}$$

because f^* is the smallest function determined by f which is guaranteed to upper bound $E(X(t) - X(s))^2$. To see this note that for any $y > 0$ and $0 = t_0 \leqq t_1 \leqq \cdots \leqq t_n$ such that $\sum_{j=1}^{n} t_j = y$,

$$\begin{aligned}(X(s+y) - X(s))^2 &= (\textstyle\sum_{j=1}^{n}(X(s + \sum_{i=0}^{j} t_i) - X(s + \sum_{i=0}^{j-1} t_i)))^2 \\ &\leqq n \textstyle\sum_{j=1}^{n}(X(s + \sum_{i=1}^{j} t_i) - X(s + \sum_{i=0}^{j-1} t_i))^2\,.\end{aligned}$$

Taking expectations, $E(X(s+y) - X(s))^2 \leqq n \sum_{j=1}^{n} f(t_j)$. Hence taking the infimum over all n and all appropriate t_j's,

$$E(X(s+y) - X(s))^2 \leqq f^*(y)\,.$$

In order that f^* result in an improvement over condition (2.2) it is necessary that $\{y : f^*(y) < \underline{f}(y)\}$ be nonempty. This is not the case, however, as the following argument indicates: Take any y and write it as $y = \sum_{j=1}^{n} t_j$, $t_j \geqq 0$.

$$\begin{aligned} n \textstyle\sum_{j=1}^{n} f(t_j) &\geqq n \textstyle\sum_{j=1}^{n} \underline{f}(t_j) \\ &\geqq n \textstyle\sum_{j=1}^{n} (t_j/y)^2 \underline{f}(y) \quad \text{by (2.10)} \\ &= ny^{-2}\underline{f}(y) \textstyle\sum_{j=1}^{n} \left(t_j - \frac{y}{n} + \frac{y}{n}\right)^2 \\ &= ny^{-2}\underline{f}(y) \textstyle\sum_{j=1}^{n} \left(\left(t_j - \frac{y}{n}\right)^2 + 2\left(t_j - \frac{y}{n}\right)\frac{y}{n} + \left(\frac{y}{n}\right)^2\right) \\ &= ny^{-2}\underline{f}(y) \left(y^2 n^{-1} + \textstyle\sum_{j=1}^{n} \left(t_j - \frac{y}{n}\right)^2\right) \\ &\geqq \underline{f}(y)\,. \end{aligned}$$

Moreover, $\underline{f}$ itself cannot be improved by again applying _ or *. To see that $\underline{f}$ is stable under these operations let $g = \underline{f}$. It is obvious that $\underline{g} = g$, and we also have the inequalities $g = \underline{g} \leqq g^* \leqq g$.

The main reason Theorem 1 improves (1.2) is because $\underline{f}$ is more consistent than f in reflecting the magnitude of $E(X(t) - X(s))^2$ as t and s vary. For example, we know that $E(X(t) - X(s))^2 \leqq f(|t - s|)$. This gives us information regarding the magnitude of $d = E(X(2t) - X(2s))^2$. Specifically, $d \leqq 4f(|t-s|)$. Thus if $f(2|t-s|) > 4f(|t-s|)$ then f is forgetting some of the information it has previously given us regarding how large d could be. Observe that $\underline{f}$ does not suffer from this flaw since $\underline{f}(2|t-s|) \leqq 4\underline{f}(|t-s|)$.

Since $\underline{f}$ will be used to construct the examples of Section 4 we list here three properties that will be needed:

$$\begin{array}{ll} & \text{If } f \text{ is a nonnegative, nondecreasing function on } [0, 1] \\ & \text{which is continuous at } 0 \text{ with value } 0 \text{ then} \\ (3.1) & (1)\ \ H(x) \equiv x^2\underline{f}(1/x) \text{ is nondecreasing on } [1, \infty]; \\ & (2)\ \ H(x) \text{ is continuous } [1, \infty); \\ & (3)\ \ \underline{f}(x) \text{ is continuous on } [0, 1]. \end{array}$$

In (2.10) it was shown that $y^{-2}\underline{f}(y)$ is nonincreasing on $(0, 1]$ so the change of variables $x = y^{-1}$ implies that $x^2\underline{f}(1/x)$ is nondecreasing on $[1, \infty)$.

Since $H(x)$ is nondecreasing its discontinuities must consist solely of upward jumps. Being the product of two functions neither of which has upward jumps, $H(x)$ is continuous.

$\underline{f}(x) = x^2H(1/x)$ being the product of continuous functions is thus continuous on $[0, 1]$. By construction $\underline{f}(0) = f(0) = 0$. Since $0 \leqq \underline{f}(x) \leqq f(x)$ which tends to 0 as x goes to 0, $\underline{f}$ is continuous at 0.

4. Construction of the examples. Let f be a nonnegative function which is nondecreasing on $[0, \varepsilon]$ and such that f does not satisfy (2.2). The aim of this section is to construct a discontinuous process $X(t, \omega)$ on $[0, 1] \times [0, 1]$ such that

$$E(X(t) - X(s))^2 \leqq f(|t-s|)\,, \qquad |t-s| \leqq \varepsilon\,.$$

Since the behavior of $f(x)$, hence $\underline{f}(x)$, does not matter for $x > \varepsilon$, we may assume that f is nondecreasing on $[0, 1]$ and that $\underline{f}(1) = 1$. We assert that it suffices to assume f is continuous and has value 0 at $x = 0$; whence, of course, $\underline{f}$ has the same property. For if this is not the case there exists $a > 0$ such that $\lim_{x\to 0^+} f(x) = a$. Therefore $f(x) \geqq a$ for $x \in (0, \varepsilon]$. Hence it suffices to construct a process for $h(x) = ax/\varepsilon$.

Let $X(t, \omega)$ be a real-valued stochastic process defined on $[0, 1] \times [0, 1]$ such that for each t

$$(4.1)\qquad \begin{aligned} X(t, \omega) &= (2(2)^{\frac{1}{2}}\pi)^{-1} \textstyle\sum_{k\geqq 1} b_k \cos 2\pi k(t-\omega)\,, && 0 < |t-\omega| < 1 \\ &= 0\,, && |t-\omega| = 0 \text{ or } 1 \end{aligned}$$

where the sequence $\{b_k\}$ will be constructed to have the following properties:

$$\begin{array}{ll} & (1)\ \ b_k \geqq b_{k+1}\,; \\ (4.2) & (2)\ \ \sum_{k\geqq 1} b_k = \infty\,; \\ & (3)\ \ \sum_{k=1}^{j} k^2b_k^2 + j^2 \sum_{k\geqq j+1} b_k^2 \leqq j^2\underline{f}(1/j)\,; \\ & (4)\ \ kb_k \text{ is bounded.} \end{array}$$

For a monotonically decreasing sequence $\{b_k\}$ the series in (4.1) converges and is continuous for $0 < |t - \omega| < 1$ (Zygmund (1959), pages 4, 184), insuring the existence of $X(t, \omega)$.

Properties (1), (2) and (4), together with the following lemma show that $X(t, \omega)$ is discontinuous at $t = \omega$ and possesses no finite-valued continuous version.

LEMMA 3. *Let $h(x) = \sum_{k=1}^{\infty} c_k \cos kx$, where c_k decreases to 0. Then for $1/(n+1) \leqq |x| \leqq 1/n$,*

$$h(x) = \alpha_x \sum_{k=1}^{n} c_k + \beta_x n c_n \tag{4.3}$$

where $\frac{1}{2} \leqq \alpha_x \leqq 1$ and $|\beta_x| \leqq 10$. Hence, if nc_n is bounded and $\sum_{k \geq 1} c_k = \infty$, then $\lim_{x \to 0} h(x) = \infty$.

PROOF. Since $h(x)$ is an even function we may suppose $1/(n+1) \leqq x \leqq 1/n$. Let $D_0 = 0$ and for $k \geqq 1$ let $D_k = \sum_{j=1}^{k} \cos jx$. Classical arguments show that

$$|D_k| \leqq \min\left\{k, \frac{2 \cdot 2^{\frac{1}{2}}}{x}\right\} \leqq \min\{k, 5n\}.$$

Now choosing $N \gg n$ and summing by parts,

$$\sum_{k=1}^{N} c_k \cos kx = \sum_{k=1}^{n} c_k \cos kx + c_N D_N + \sum_{k=n+1}^{N-1} (c_k - c_{k+1}) D_k - c_{n+1} D_n .$$

The first term equals $\alpha_x \sum_{k=1}^{n} c_k$ where $\frac{1}{2} \leqq \alpha_x \leqq 1$ since $\frac{1}{2} \leqq \cos kx \leqq 1$. For $k \geqq n$, $|D_k| < 5n$, so the last two terms are in absolute value each $\leqq 5nc_n$ and the second term goes to 0 as $N \to \infty$. Hence, letting $N \to \infty$,

$$h(x) = \sum_{k=1}^{\infty} c_k \cos kx = a_x \sum_{k=1}^{n} c_k + \beta_x n c_n . \qquad \square$$

The third property will be used to show that $E(X(t) - X(s))^2 \leqq f(|t - s|)$. Notice that $\sin^2 x \leqq \min\{1, x^2\}$. So if $1/j \leqq |t - s| \leqq 1/(j-1)$ for $j \geqq 2$, then

$$\begin{aligned} E(X(t) - X(s))^2 &= (4\pi^2)^{-1} \textstyle\sum_{k\geq 1} b_k^2 \sin^2 \pi k|t - s| \\ &\leqq (4\pi^2)^{-1}\{(t-s)^2\pi^2 \textstyle\sum_{k=1}^{j} k^2 b_k^2 + \sum_{k \geq j+1} b_k^2\} \\ &= (4\pi^2)^{-1}\{j^2(t-s)^2(\pi/j)^2 \textstyle\sum_{k=1}^{j} k^2 b_k^2 + \sum_{k\geq j+1} b_k^2\} \\ &\leqq (4\pi^2)^{-1}(j/(j-1))^2\pi^2\{j^{-2} \textstyle\sum_{k=1}^{j} k^2 b_k^2 + \sum_{k \geq j+1} b_k^2\} \\ &\leqq j^{-2}\{\textstyle\sum_{k=1}^{j} k^2 b_k^2 + j^2 \sum_{k\geq j+1} b_k^2\} \\ &\leqq f(1/j) \quad \text{by property} \quad (3) . \end{aligned} \tag{4.4}$$

All that remains is the construction of a sequence $\{b_k\}$ satisfying the four properties in (4.2). Observe that for any sequence $\{c_k\}$ satisfying (4.2)(1)—(3) we can easily construct a sequence $\{b_k\}$ satisfying (4.2)(1)—(4) by letting

$$b_k = c_1 c_k / \sum_{j=1}^{k} c_j . \tag{4.5}$$

The following lemma contains the basic ideas needed in the construction of the sequence $\{c_k\}$:

LEMMA 4. *If (2.2) does not hold, then there exists a random variable Y with* *values in* $[1, \infty)$ *such that*

(a) $E(Y \wedge y)^2 \leqq y^2 f(1/y)$, *and*
(b) $\int_1^\infty y^{-\frac{1}{2}}(P(Y \geqq y))^{\frac{1}{2}}\, dy = \infty$.

PROOF. Let Z denote a random variable such that for $y \geqq 1$

$$P(Z > y) = f(1/y) . \tag{4.6}$$

This is possible since f is continuous, monotone, $f(0) = 0$ and $f(1) = 1$.

Construct a sequence of real numbers by letting $t_0 = 1$ and for $n \geqq 1$ let

$$t_n = \sup\{x : H(x^2) \equiv x^4 P(Z > x^2) \leqq 4^n\} . \tag{4.7}$$

Since H is a continuous nondecreasing function, (3.1)(1)—(2), $H(t_n^2) = 4^n$.

Moreover, $t_{n+1} \geqq 2^{\frac{1}{2}} t_n$, because if one takes any C such that $t_n < C < 2^{\frac{1}{2}} t_n$

$$\begin{aligned} C^4 P(Z > C^2) &\leqq C^4 P(Z > (C/2^{\frac{1}{2}})^2) \\ &= 4(C/2^{\frac{1}{2}})^4 P(Z > (C/2^{\frac{1}{2}})^2) \\ &\leqq 4H(t_n^2) \quad \text{by (3.1)(1)} \\ &= 4^{n+1} . \end{aligned}$$

Hence, $t_{n+1} \geqq C$. Since C is arbitrary, $t_{n+1} \geqq 2^{\frac{1}{2}} t_n$.

Let Y denote a nonnegative random variable such that

$$\begin{aligned} &(1) \quad t_n^4 P(Y > t_n^2) = 4^{n-2} ; \\ &(2) \quad \text{for} \quad t_n^2 \leqq y \leqq 2t_n^2 \quad \text{let} \\ &\qquad P(Y > y) = P(Y > t_n^2)(2t_n^2 - y)/t_n^2 + P(Y > t_{n+1}^2)(y - t_n^2)/t_n^2 . \end{aligned} \tag{4.8}$$

Notice that for all $2t_n^2 \leqq y \leqq t_{n+1}^2$.

$$P(Y > y) = P(Y > t_{n+1}^2) .$$

In order to prove that Y is a random variable note that $4^{n-2} t_n^{-4} = P(Z > t_n^2)/16$ is nonincreasing and tends to 0 as n increases.

For $t_n^2 \leqq y \leqq t_{n+1}^2$,

$$\begin{aligned} E(Y \wedge y)^2 &= 2 \int_0^y x P(Y > x)\, dx \\ &\leqq 1 + 2 \sum_{j=1}^{n+1} \left(\int_{t_{j-1}^2}^{2t_{j-1}^2} x P(Y > x)\, dx + \int_{2t_{j-1}^2}^{t_j^2} x P(Y > x)\, dx\right) \\ &\leqq 1 + \sum_{j=1}^{n+1} (P(Y > t_{j-1}^2) 3t_{j-1}^4 + t_j^4 P(Y > t_j^2)) \\ &= 1 + 3P(Y > 1) + 4 \sum_{j=1}^{n} t_j^4 P(Y > t_j^2) + t_{n+1}^4 P(Y > t_{n+1}^2) \\ &= 1 + \tfrac{3}{16} + \sum_{j=1}^{n} 4^{j-1} + 4^{n-1} \\ &< 4^n \quad \text{for} \quad n \geqq 1 . \end{aligned} \tag{4.9}$$

So for $t_n^2 \leqq y \leqq t_{n+1}^2$, using (4.9) and (3.1)(1),

$$E(Y \wedge y)^2 < 4^n = t_n^4 P(Z > t_n^2) \leqq y^2 P(Z > y) .$$

Hence, for all $y \geqq 1$,

$$E(Y \wedge y)^2 < y^2 P(Z > y)\,, \tag{4.10}$$

verifying (a).

Continuing with (b),

$$\begin{aligned}
\textstyle\int_1^\infty y^{-\frac{1}{2}}(P(Y \geqq y))^{\frac{1}{2}}\,dy &\geqq \textstyle\int_1^\infty (P(Y \geqq u^2))^{\frac{1}{2}}\,du \\
&= \textstyle\sum_{n=1}^\infty \int_{t_{n-1}}^{t_n} (P(Y > u^2))^{\frac{1}{2}}\,du \\
&\geqq \textstyle\sum_{n=1}^\infty (t_n - t_{n-1})(P(Y > t_n^2))^{\frac{1}{2}} \\
&\geqq (1 - 1/2^{\frac{1}{2}}) \textstyle\sum_{n=1}^\infty t_n(P(Y > t_n^2))^{\frac{1}{2}} \\
&= (1 - 1/2^{\frac{1}{2}}) \textstyle\sum_{n=1}^\infty t_n^{-1}2^{n-2} \\
&= (1 - 1/2^{\frac{1}{2}})2^{-3} \textstyle\sum_{n=1}^\infty t_n^{-1}(t_{n+1}^4 \underline{f}(t_{n+1}^{-2}))^{\frac{1}{2}} \qquad (4.11)\\
&\geqq (1 - 1/2^{\frac{1}{2}})2^{-3} \textstyle\sum_{n=1}^\infty t_{n+1}^2 \underline{f}^{\frac{1}{2}}(t_{n+1}^{-2})(t_n^{-1} - t_{n+1}^{-1}) \\
&= (1 - 1/2^{\frac{1}{2}})2^{-3} \textstyle\sum_{n=1}^\infty \int_{t_n}^{t_{n+1}} \underline{f}^{\frac{1}{2}}(t_{n+1}^{-2})t_{n+1}^2 y^{-2}\,dy \\
&\geqq (1 - 1/2^{\frac{1}{2}})2^{-3} \textstyle\sum_{n=1}^\infty \int_{t_n}^{t_{n+1}} \underline{f}^{\frac{1}{2}}(y^{-2})\,dy \\
&\qquad \text{since} \quad H(x) \quad \text{increases} \\
&= (1 - 1/2^{\frac{1}{2}})2^{-3} \textstyle\int_{t_1}^\infty \underline{f}^{\frac{1}{2}}(y^{-2})\,dy \\
&= (1 - 1/2^{\frac{1}{2}})2^{-4} \textstyle\int_0^{t_1^{-2}} x^{-\frac{3}{2}} \underline{f}^{\frac{1}{2}}(x)\,dx = \infty\,. \qquad \square
\end{aligned}$$

We also need the following inequality whose proof can be found in Jain and Marcus ((1973), page 275).

Boas' inequality. Let $h_n = \sum_{j \geqq n} e_j^2$ where e_j is a nonnegative nonincreasing sequence. Then

$$\textstyle\sum_{n \geqq 1} n^{-\frac{1}{2}} h_n^{\frac{1}{2}} \leqq 2 \sum_{n \geqq 1} e_n\,. \tag{4.12}$$

The two series in Boas' inequality actually converge and diverge together. To see this note that

$$\begin{aligned}
\textstyle\sum_{k \geqq 1} e_k &= \textstyle\sum_{k \geqq 1} \sum_{m \geqq k} m^{-1} e_m \\
&\leqq \textstyle\sum_{k \geqq 1} h_k^{\frac{1}{2}} (\sum_{m \geqq k} m^{-2})^{\frac{1}{2}} \qquad (4.13)\\
&\leqq 2^{\frac{1}{2}} \textstyle\sum_{k \geqq 1} k^{-\frac{1}{2}} h_k^{\frac{1}{2}}\,.
\end{aligned}$$

The sequence $\{c_k\}$ can now be obtained as follows:

Set $a_n = (P(Y \geqq n))^{\frac{1}{2}}$, then a_n decreases and by Lemma 4(b) $\sum a_n/n^{\frac{1}{2}} = \infty$. Let $\{g_n^2\}$ be the largest convex minorant of $\{a_n^2\}$. A simple argument (e.g., Jain and Marcus ((1973), page 294) shows that $\sum g_n/n^{\frac{1}{2}} = \infty$ also. Set $c_n^2 = g_n^2 - g_{n+1}^2$. By convexity of $\{g_n^2\}$, c_n decreases. Using Boas' inequality, $\sum c_n = \infty$. Finally,

$$\begin{aligned}
&\textstyle\sum_{k=1}^j k^2 c_k^2 + j^2 \sum_{k \geqq j+1} c_k^2 \\
&\quad = \textstyle\sum_{k=1}^j (2k-1) g_k^2 \leqq \sum_{k=1}^j (2k-1) a_k^2 \leqq \sum_{k=1}^j 2 \int_{k-1}^k x P(Y \geqq x)\,dx \qquad (4.14)\\
&\quad = E(Y \wedge j)^2 \leqq j^2 \underline{f}(1/j) \quad \text{by Lemma 4(a)}\,,
\end{aligned}$$

thus $\{c_n\}$ satisfies (5.2)(1)—(3) as desired.

Hence we have proven the following:

THEOREM 5. *If f is a nonnegative function which is nondecreasing on $[0, \varepsilon]$ and such that*

$$\int_0 y^{-\frac{3}{2}} \underline{f}^{\frac{1}{2}}(y)\, dy = \infty ,$$

there exists a stochastic process $\{X(t), t \in [0, 1]\}$ with discontinuous sample paths and no real-valued continuous version satisfying

$$E(X(t) - X(s))^2 \leqq f(|t - s|) , \qquad |t - s| \leqq \varepsilon .$$

5. An example to show that Theorem 1 is an improvement over those in Hahn (1975). One of the essential features of the class of counterexamples constructed in Section 4 is the existence of a sequence $\{b_k\}$ which is not in l^1 but such that

$$\sum_{k=1}^{j} k^2 b_k^2 + j^2 \sum_{k \geqq j+1} b_k^2 \leqq j^2 \underline{f}(1/j) \qquad \text{for all} \quad j . \tag{5.1}$$

In particular, it is necessary that

$$\sum_{k=1}^{j} k^2 b_k^2 \leqq j^2 \underline{f}(1/j) \qquad \text{for all} \quad j . \tag{5.2}$$

It is the failure of the existence of such a sequence when $\liminf_{x \to 0} x^{-2} f(x) < \infty$ which originally led to the formulation of condition (2.9) and hence ultimately to Theorem 1.

PROPOSITION 6. *There exists a monotone function f such that*

$$\int_0 y^{-\frac{3}{2}} f^{\frac{1}{2}}(y)\, dy = \infty \tag{5.3}$$

but

$$\int_0 y^{-\frac{3}{2}} \underline{f}^{\frac{1}{2}}(y)\, dy < \infty . \tag{5.4}$$

PROOF. Let $n_1 = 1$, $n_{k+1} = [e^{n_k} n_k] + 1$ for $k \geqq 1$ where [ ] denotes the integral part. Define $a_j = (j n_k)^{-1}$ for $n_k \leqq j < n_{k+1}$. The sequence $\{a_j\}$ is not in l^1 since

$$\begin{aligned} \sum_{j=1}^{\infty} a_j &= \sum_{k=1}^{\infty} n_k^{-1} \sum_{j=n_k}^{n_{k+1}-1} j^{-1} \\ &\geqq \sum_{k=1}^{\infty} n_k^{-1} \int_{n_k}^{n_{k+1}} x^{-1}\, dx \\ &= \sum_{k=1}^{\infty} n_k^{-1} \log (n_{k+1}/n_k) \\ &\geqq \sum_{k=1}^{\infty} n_k^{-1} \log e^{n_k} = \infty . \end{aligned} \tag{5.5}$$

Let

$$f\left(\frac{1}{k}\right) = {\sum}'_{j \geqq k} a_j^2 , \qquad k \geqq 1 \tag{5.6}$$

and define $f(x)$ to be linear in between.

By the computations in (4.13) we obtain

$$\begin{aligned} &\int_0 y^{-\frac{3}{2}} f^{\frac{1}{2}}(y)\, dy \\ &\quad = \int_1^{\infty} v^{-\frac{1}{2}} f^{\frac{1}{2}}(1/v)\, dv \geqq \sum_{k \geqq 2} k^{-\frac{1}{2}} f^{\frac{1}{2}}(1/k) \geqq 2^{-\frac{1}{2}}\left(\sum_{k \geqq 1} a_k - f^{\frac{1}{2}}(1)\right) \\ &\quad = \infty . \end{aligned}$$

In order to verify (5.4) it suffices, by the proof of Corollary 2, to show that $\liminf_{x\to 0} x^{-2}f(x) < \infty$. This is easy to see since

$$\begin{aligned} n_k^2 f(1/n_k) &= n_k^2 \textstyle\sum_{j\geqq n_k} a_j^2 \\ &\leqq n_k^2 \textstyle\sum_{j\geqq n_k} (jn_k)^{-2} \\ &\leqq \textstyle\int_{n_k-1}^{\infty} x^{-2}\,dx \end{aligned}$$

which tends to 0 as $k \to \infty$. □

Since a large number of the functions f which arise in applications are convex, we remark that the function f constructed in Proposition 6 is convex. This shows that condition (2.2) is necessary even under more stringent hypotheses about the function f.

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