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A FUNCTIONAL LAW OF THE ITERATED LOGARITHM FOR EMPIRICAL DISTRIBUTION FUNCTIONS OF WEAKLY DEPENDENT RANDOM VARIABLES[1]

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DEDICATED TO PROFESSOR EDMUND HLAWKA ON HIS 60TH BIRTHDAY.

Let $\{\eta_k, k \geqq 1\}$ be a sequence of random variables uniformly distributed over $\{0, 1\}$ and let $F_N(t)$ be the empirical distribution function at stage N. Put $f_n(t) = N(F_N(t) - t)(N \log \log N)^{-\frac{1}{2}}$, $0 \leqq t \leqq 1$, $N \geqq 3$. For strictly stationary sequences $\{\eta_k\}$ where η_k is a function of random variables satisfying a strong mixing condition or where $\eta_k = n_k x \bmod 1$ with $\{n_k, k \geqq 1\}$ a lacunary sequence of real numbers a fuctional law of the iterated longarithm is proven: The sequence $\{f_N(t), N \geqq 3\}$ is with probability 1 relatively compact in $D[0, 1]$ and the set of its limits is the unit ball in the reproducing kernel Hilbert space associated with the covariance function of the appropriate Gaussian process.

CONTENTS

1. Introduction. The purpose of this paper is to establish functional laws of the iterated logarithm for the empirical distribution functions of functions of

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random variables satisfying a strong mixing condition as well as for the empirical distribution functions of lacunary sequences $\{\langle n_k \omega \rangle, k \geqq 1\}$ $(0 \leqq \omega < 1)$. Let $F_N(t)$ be the empirical distribution function at stage N of a sequence $\{\eta_n, n \geqq 1\}$ of random variables uniformly distributed over $[0, 1]$. Then $F_N \in D[0, 1]$. We give D the topology defined by the supremum norm $\|\cdot\|_\infty$. For $N \geqq 3$ we put

$$f_N(t) = N(F_N(t) - t)(2N \log \log N)^{-\frac{1}{2}} \qquad 0 \leqq t \leqq 1 . \tag{1.1}$$

We shall prove, under the above-mentioned assumptions on the dependence of the random variables η_n, that the sequence $\{f_N(t), N \geqq 3\}$ is with probability 1 relatively compact and that the set of its limit points is the unit ball in the reproducing kernel Hilbert space associated with the covariance function of the appropriate Gaussian limit process.

For independent identically distributed random variables this result is due to Finkelstein (1971). For m-dependent sequences it has been recently obtained by Oodaira (1975), who also obtained partial results for random variables satisfying a strong mixing condition. Furthermore, Oodaira in his paper points out that the most natural way to describe the set of limit points of the sequences $\{f_N(t)\}$ for dependent random variables is in terms of the reproducing kernel Hilbert space (which we shall call kernel space from now on).

In the lacunary case we obtain as a by-product a result in probabilistic number theory on the discrepancy of lacunary sequences. Let $\{n_k, k \geqq 1\}$ be a lacunary sequence of real numbers, i.e., a sequence satisfying

$$n_{k+1}/n_k \geqq q > 1 \tag{1.2}$$

for all $k \geqq 1$. Let $\{[0, 1], \mathscr{F}, \lambda\}$ be the unit interval with Lebesgue measurability and Lebesgue measure λ. Then $\{\langle n_k \omega \rangle, k \geqq 1\}$ can be considered as a sequence of random variables with asymptotically uniform distribution. Here $\langle \varepsilon \rangle$ denotes the fractional part of ε. Let $F_N(t)$ be the empirical distribution function at stage N. Then

$$D_N = D_N(\omega) = \sup_{0 \leqq t \leqq 1} |F_N(t) - t| \tag{1.3}$$

is called the discrepancy of the sequence $\{\langle n_k \omega \rangle, 1 \leqq k \leqq N\}$, a concept important in probability as well as in number theory. Recently I proved (Philipp (1975)) that for lacunary sequences of integers

$$\tfrac{1}{4} \leqq \limsup_{N \to \infty} \frac{N D_N(\omega)}{(N \log \log N)^{\frac{1}{2}}} \leqq C(q) \tag{1.4}$$

with probability 1 where $C(q)$ is a constant depending on q only. The right-hand inequality in (1.4) was conjectured by Erdös and Gál in 1954 (see Erdös (1964), page 56). In Section 4 it is shown that (1.4) continues to hold for lacunary sequences $\{n_k\}$ where the n_k are not necessarily integers.

Except for the value of the constant, the left-hand inequality in (1.4) has been well known since the publication of a result of Erdös and Gál (1955). As a matter of fact, this left inequality was the basis for their conjecture. For a proof of the left inequality and a short history of the conjecture see Philipp (1975).

2. Description of the method and basic theorems. Chover (1967) gave a proof of a weaker version of Strassen's (1964) functional law of the iterated logarithm for sums of independent identically distributed random variables using only classical results such as maximal inequalities and the central limit theorem with remainder. His approach consists of two steps. He first proves that the sequence of bookkeeping functions is with probability 1 uniformly equicontinuous and bounded and thus by the Arzelà–Ascoli theorem is relatively compact. He then identifies the class of limit points by showing that certain polygonal functions defined in terms of these bookkeeping functions converge to the corresponding polygonal functions defined in terms of Strassen's class K.

A modified version of Chover's approach, which at the same time is more general, has been formulated by Oodaira (1975). Let $T = T_m = \{t_1, \cdots, t_m\}$ be a finite subset of $[0, 1]$. Denote by $\phi^T = (\phi(t_1), \cdots, \phi(t_m))$ the restriction of a function ϕ to T and for a class A of functions ϕ on $[0, 1]$ denote by $A^T = \{\phi^T : \phi \in A\}$. Let $\{T_m\}$ be an increasing sequence of subsets T_m such that $\bigcup_{m=1}^\infty T_m$ is dense in $[0, 1]$. The following proposition is due to Oodaira (1975).

PROPOSITION 2.1. *Let $\{g_N(t) = g_N(t, \omega), N \geqq 1\}$ be a sequence of random functions in $C[0, 1]$. Suppose that*

(2.1) $\{g_N(t)\}$ *is with probability* 1 *relatively compact*

and that

(2.2) *for each* $T \in \{T_m\}$, *the set of limit points of random vectors* $\{g_N{}^T\}$ *is* K^T *with probability* 1 *where* K *is a compact set in* $C[0, 1]$.

Then the set of limit points of $\{g_N(t)\}$ is K a.s.

Hence in view of Oodaira's proposition the proof of the functional law of the iterated logarithm may be carried out in two steps, consisting of the proof of (2.1) and (2.2).

We start with an informal discussion of the relative compactness. Let $\{\eta_n, n \geqq 1\}$ be a sequence of random variables with η_n uniformly distributed over $[0, 1]$. As a rule this assumption does not result in any loss of generality when we consider the limit properties of the empirical distribution since the general case can be easily reduced to the case of uniformly distributed random variables. (See Section 3.1.) For fixed s and t with $0 \leqq s < t \leqq 1$ write

$$L = [s, t) , \qquad l = t - s \tag{2.3}$$

and

$$x_n = x_n(s, t) = 1_L(\eta_n) - l . \tag{2.4}$$

Here $1_L(\cdot)$ denotes the indicator function of L. We observe that

$$f_N(t) = (2N \log \log N)^{-\frac{1}{2}} \sum_{n \leq N} x_n(0, t) .$$

In Sections 3 and 4 we shall prove probability estimates of the large deviations of the sums $\sum_{n=H+1}^{H+N} x_n$ for all $H \geqq 0$ and $N \geqq 1$. These estimates will then be used to prove some sort of Lipschitz condition for the bookkeeping functions $f_N(t)$ defined in (1.1). (See (3.1.8) and (4.1.3) below.) At the end of Section 3.1 it is shown that this Lipschitz condition implies the relative compactness of $\{f_N(t), N \geqq 3\}$.

To obtain the probability estimates of the large deviations for the sums $\sum x_n$ we shall approximate them by martingales. This technique is explained at length in the memoir Philipp and Stout (1975). The martingale approximation used here is, in fact, somewhat simpler than the one used in Philipp and Stout (1975) since it consists of centering the "blocks" at conditional expectations. This is particularly useful here since then the approximating martingale is a sequence of bounded random variables.

We also need some notation on kernel spaces in the simplest setup. Let $\Gamma(s, t)$ be a positive definite function on $E \times E$ where $E \subset \mathbb{R}$. Let K_m be the class of functions on E which can be written in the form

$$f(x) = \sum_{i \leq m} \alpha_i \Gamma(x, y_i)$$

where $y_i \in E$ and $\alpha_i \in \mathbb{R}$. If

$$g(x) = \sum_{i \leq m} \beta_i \Gamma(x, y_i)$$

then the inner product (f, g) of f and g is defined by

$$(f, g) = \sum_{j,k \leq m} \alpha_j \beta_k \Gamma(y_j, y_k) .$$

$\Gamma(s, t)$ has the reproducing kernel property on K_m since

$$(f, \Gamma(\cdot, y_k)) = \sum_{j \leq m} \alpha_j \Gamma(y_j, y_k) = f(y_k) .$$

The inner product defines a norm on $K = \bigcup_{m \geqq 1} K_m$. But K is, in general, not complete. The kernel space $H(\Gamma)$ over E associated with $\Gamma(s, t)$ is then defined as the completion of K. Its norm is denoted by $||\cdot||_H$.

Let $T = \{t_1, \cdots, t_m\}$ and let Γ^T denote the restriction of Γ to $T \times T$. Denote by $H(\Gamma^T)$ the kernel space with reproducing kernel Γ^T.

LEMMA 2.1 (Oodaira (1975)). *For each T, the restriction of the unit ball of $H(\Gamma)$ to T is the unit ball of $H(\Gamma^T)$.*

LEMMA 2.2 (Oodaira (1972)). *If $\Gamma(s, t)$ is continuous on the unit square then the unit ball of $H(\Gamma)$ is a compact set in $C[0, 1]$.*

For more details on reproducing kernel Hilbert spaces see Aronszajn (1950) or Meschkowski (1962).

The second step in the proof of the functional law of the iterated logarithm consists of verifying condition (2.2) with $K = H(\Gamma)$. To this end we define random vectors $y_k \in \mathbb{R}^m$ with components $x_k(0, t_j)$ $(1 \leqq j \leqq m)$. Under the assumptions we are going to make the $m \times m$ matrix $\Gamma_m = ((\Gamma(t_i, t_j)))_{i,j=1}^m$ defined by

$$\Gamma(t_i, t_j) = \lim_{N\to\infty} N^{-1} \sum_{l,k\leqq N} E(x_k(0, t_i)x_l(0, t_j))$$

is positive definite. It then turns out that the sequence

$$\left\{\frac{\sum_{k\leqq N} y_k}{(2N \log\log N)^{\frac{1}{2}}}, \ N \geqq 3\right\}$$

of random vectors $\in \mathbb{R}^m$ is bounded almost surely and has the ellipsoid $E_m = \{x \in \mathbb{R}^m : x'\Gamma_m^{-1}x \leqq 1\}$ as its set of limit points. This is proved by basic linear algebra, by means of Lemma 5.1.1, reminiscent of the Cramér–Wold device coupled with almost sure invariance principles for partial sums of weakly dependent random variables. By a simple linear transformation it is then shown that E_m equals the unit ball in the kernel space $H(\Gamma_m)$. An application of Lemma 2.1 will then show that (2.1) holds.

3. Functions of strongly mixing random variables.

3.1. *Introduction.* Let $\{\xi_n, n \geqq 1\}$ be a strictly stationary sequence of random variables satisfying a strong mixing condition

$$|P(AB) - P(A)P(B)| \leqq \rho(n) \downarrow 0 \tag{3.1.1}$$

for all $A \in \mathscr{F}_1^t$ and $B \in \mathscr{F}_{t+n}^\infty$. Here $\mathscr{F}_a^b$ denotes the σ-field generated by ξ_n $(a \leqq n \leqq b)$. Let f be a measurable mapping from the space of infinite sequences $(\alpha_1, \alpha_2, \cdots)$ of real numbers into the real line. Define

$$\eta_n = f(\xi_n, \xi_{n+1}, \cdots), \qquad n \geqq 1 \tag{3.1.2}$$

and

$$\eta_{mn} = E(\eta_n \mid \mathscr{F}_n^{n+m}), \qquad m, n \geqq 1. \tag{3.1.3}$$

As is usual we assume that η_n can be closely approximated by η_{mn} in the form

$$E|\eta_n - \eta_{mn}| \leqq \psi(m) \downarrow 0 \tag{3.1.4}$$

for all $m, n \geqq 1$.

Denote by $F_N(t)$ the empirical distribution function of the sequence $\{\eta_n, n \geqq 1\}$ at stage N. Without loss of generality (see the end of this section) we assume that η_n is uniformly distributed over $[0, 1]$. Write

$$f_N(t) = N(F_N(t) - t)(2N \log\log N)^{-\frac{1}{2}}, \qquad 0 \leqq t \leqq 1. \tag{3.1.5}$$

THEOREM 3.1. *Let $\{\xi_n, n \geqq 1\}$ be a strictly stationary sequence of random variables satisfying a strong mixing condition* (3.1.1) *with*[2]

$$\rho(n) \ll n^{-8}. \tag{3.1.6}$$

[2] Throughout the Vinogradov symbol $\ll$ instead of 0 is used whenever convenient.

Suppose that the random variables η_n *defined by* (3.1.2) *are uniformly distributed over* [0, 1] *and that they satisfy* (3.1.4) *with*

$$\psi(m) \ll m^{-12}\,. \tag{3.1.7}$$

Then for each $\varepsilon > 0$ *there is with probability* 1 *a random index* $N_0 = N_0(\varepsilon)$ *such that*

$$|f_N(t) - f_N(s)| \leqq C|t-s|^{\frac{1}{120}} + \varepsilon \tag{3.1.8}$$

for all $0 \leqq s \leqq t \leqq 1$ *and all* $N \geqq N_0$. *The constant C only depends on the constants implied by* $\ll$ *in* (3.1.6) *and* (3.1.7). *In particular* (3.1.8) *implies that the sequence* $\{f_N(t), N \geqq 3\}$ *is with probability* 1 *relatively compact in* $D[0, 1]$.

In order to identify the limits of the sequence $\{f_N(t)\}$ we need some more notation and an additional hypothesis. Write

$$g_n(t) = 1\{0 \leqq \eta_n < t\} - t = x_n(0, t)\,. \tag{3.1.9}$$

Under the hypothesis of Theorem 3.1 the two series defining the covariance function

$$\Gamma(s, t) = E(g_1(s)g_1(t)) + \sum_{n=2}^{\infty} E(g_1(s)g_n(t)) + \sum_{n=2}^{\infty} E(g_n(s)g_1(t)) \tag{3.1.10}$$

($0 \leqq s, t \leqq 1$) converge absolutely (see Billingsley (1968), Section 22).

Let $\{T_m, m \geqq 1\}$ be an increasing sequence of finite subsets $\{t_1, \cdots, t_m\} \subset [0, 1]$ such that $\bigcup_{m \geqq 1} T_m$ is dense in [0, 1]. Let B_m be the set of all functions f on [0, 1] defined by

$$f(x) = \sum_{j \leqq m} \alpha_j \Gamma(x, t_j)\,, \qquad \alpha_j \in \mathbb{R}$$

satisfying

$$\sum_{j,k \leqq m} \alpha_j \alpha_k \Gamma(t_j, t_k) \leqq 1\,.$$

THEOREM 3.2. *Suppose that in addition to the hypotheses of Theorem* 3.1 *the covariance function* $\Gamma(s, t)$ *is positive definite. Then the sequence* $\{f_N(t), N \geqq 3\}$ *is with probability* 1 *relatively compact and has the unit ball in the kernel space* $H(\Gamma)$ *as its set of limit points. Equivalently, the set of limit points equals* $\overline{\bigcup_{m \geqq 1} B_m}$ *where the closure is in the topology defined by the supremum norm over* [0, 1].

REMARKS. (3.1.8) implies

$$\limsup_{N \to \infty} \sup_{0 \leqq t \leqq 1} |f_N(t)| \leqq C \quad \text{a.s.} \tag{3.1.11}$$

Except for the value of the constant, (3.1.11) can be regarded as a generalization of the Chung–Smirnov law of the iterated logarithm for empirical distribution functions of independent uniformly distributed random variables (see Chung (1949)). For independent random variables Cassels (1951) proved that (3.1.8) holds with the right-hand side replaced by $((t-s)(1-t+s))^{\frac{1}{2}} + \varepsilon$. Hence except for the value of C relation (3.1.8) applied to independent random variables stands somewhere between Cassels' theorem and the Chung–Smirnov theorem.

But actually much more is true. We first observe that Theorem 3.2 contains

Finkelstein's result as a special case since, as is well known, the limit set appearing in Finkelstein's (1971) Theorem 1 is precisely the unit ball in the reproducing kernel Hilbert space of the Brownian bridge.

Second it might be interesting to note that Finkelstein's theorem (and hence Theorem 3.2) implies Cassels' theorem. To prove this we need the following lemma, due to Riesz (1955), page 75.

LEMMA 3.1.1. *Let f be a real-valued function on the unit interval. The following two conditions are equivalent*:

1. *f is absolutely continuous with respect to Lebesgue measure and*

$$\int_0^1 (f'(x))^2\,dx \leqq 1 .$$

2. *For every finite partition* $0 \leqq x_0 < x_1 < \cdots < x_s \leqq 1$ *of* $[0, 1]$

$$\sum_{i=1}^{s} \frac{(f(x_i) - f(x_{i-1}))^2}{x_i - x_{i-1}} \leqq 1 .$$

We now shall prove that Finkelstein's theorem implies Cassels' theorem which in turn obviously implies the Chung–Smirnov law of the iterated logarithm. Indeed, by Lemma 3.1.1 we observe that for each function $f \in K$ we have for $0 \leqq s < t \leqq 1$

$$\frac{f^2(s)}{s} + \frac{(f(t) - f(s))^2}{t - s} + \frac{f^2(t)}{1 - t} \leqq 1 .$$

Since by elementary calculations

$$\frac{f^2(s)}{s} + \frac{f^2(t)}{1 - t} \geqq \frac{(f(t) - f(s))^2}{1 - t + s}$$

we conclude that

$$|f(t) - f(s)| \leqq \{(t - s)(1 - t + s)\}^{\frac{1}{2}} .$$

Using the relative compactness of $\{f_N(t), N \geqq 3\}$ one can now easily deduce Cassels' theorem.

We shall show now that (3.1.8) implies the relative compactness of $\{f_N(t), N \geqq 3\}$ over $[0, 1]$. In order to apply the Arzelà–Ascoli theorem we approximate $f_N(t)$ by a continuous function $h_N(t)$ as follows. Fix $\omega \in \Omega_1$, where Ω_1 is the set on which (3.1.8) holds. Denote by $\alpha_1, \cdots, \alpha_m$ the discontinuities of $f_N(t)$, $0 \leqq t \leqq 1$ and put $\alpha_0 = 0$ and $\alpha_{M+1} = 1$. We define $h_N(t)$ to be a piecewise linear function on $[0, 1]$ with

$$h_N(\alpha_m) = f_N(\alpha_m) \qquad 0 \leqq m \leqq M + 1 . \tag{3.1.12}$$

By comparing the graphs of h_N and f_N we observe that on each interval $(\alpha_m, \alpha_{m+1}]$

$$0 \leqq f_N(t) - h_N(t) \leqq f_N(\alpha_m +) - f_N(\alpha_m) \leqq 2\varepsilon \tag{3.1.13}$$

for $N \geqq N_0$ using (3.1.8). Let $0 \leqq s < t \leqq 1$ with $C|t - s|^{\frac{1}{120}} < \varepsilon$. Then by (3.1.13) and (3.1.8)

$$|h_N(s) - h_N(t)| < 5\varepsilon$$

for $N \geqq N_0$. Hence $\{h_N(t), N \geqq 3\}$ is equicontinuous over $[0, 1]$. Moreover, it is uniformly bounded since $\{f_N(t), N \geqq 3\}$ is. Thus by the Arzelà–Ascoli theorem $\{h_N(t), N \geqq 3\}$ is relatively compact over $[0, 1]$ and so is $\{f_N(t), N \geqq 3\}$ by (3.1.13).

We shall prove now the claim made earlier that in Theorems 3.1 and 3.2 there is no loss of generality to assume that the random variables η_n are all uniformly distributed over $[0, 1)$. Suppose that the common distribution function G of a sequence $\{\zeta_n, n \geqq 1\}$ is continuous. Denote its empirical distribution function by G_N and put

$$g_N(t) = N(G_N(t) - G(t))(2N \log\log N)^{-\frac{1}{2}}\,.$$

Then $\eta_n = G(\zeta_n)$ has uniform distribution over $[0, 1]$ and the empirical distribution function F_N of $\{\eta_n, n \geqq 1\}$ satisfies

$$G_N(t) = F_N(G(t)) \quad \text{a.s.}$$

for all t.

Suppose now that the conclusion of Theorem 3.2 holds for the sequence $\{\eta_n, n \geqq 1\}$ where each η_n has uniform distribution over $[0, 1]$. Consider the mapping φ from $D[0, 1] \to D[0, 1]$ defined by

$$(\varphi x)(t) = x(G(t))\,.$$

Then φ is continuous in the supremum norm and

$$(\varphi f_N)(t) = f_N(G(t)) = g_N(t) \quad \text{a.s.}$$

Consequently $\{g_N, N \geqq 1\}$ is with probability 1 relatively compact and the set of its limit points is $\varphi(H(\Gamma))$.

Incidentally, the last two arguments show that the proofs of Finkelstein's Theorems 1 and 2 can be somewhat simplified. Indeed, Cassels' (1951) theorem and the argument that (3.1.8) implies relative compactness show that her sequence $\{G_n, n \geqq 3\}$ is with probability 1 relatively compact. Hence by Oodaira's Proposition 2.1 it remains to show that (2.2) holds. But this follows from her Lemma 4 on page 611.

3.2. *Preliminaries.*

LEMMA 3.2.1. *Let X and Y be random variables with*

$$E|X - Y| < \varepsilon\,.$$

Suppose that X is uniformly distributed over $[0, 1]$. *Then for all* $0 \leqq t \leqq 1$

$$E|1\{X \leqq t\} - 1\{Y \leqq t\}| \leqq 4\varepsilon^{\frac{1}{2}}\,.$$

PROOF. By Markov's inequality

$$P\{|X - Y| \geqq \varepsilon^{\frac{1}{2}}\} \leqq \varepsilon^{\frac{1}{2}}\,.$$

Hence, if $Y \leqq t$ then $X \geqq t + \varepsilon^{\frac{1}{2}}$ with probability not exceeding $\varepsilon^{\frac{1}{2}}$. Similarly if $Y > t$ then $X \leqq t - \varepsilon^{\frac{1}{2}}$ with probability not exceeding $\varepsilon^{\frac{1}{2}}$. Consequently,

$$\begin{aligned} E|1\{Y \leqq t\} - 1\{X \leqq t\}| &\leqq \textstyle\int_{\{Y \leqq t\}} |1 - 1\{X \leqq t + \varepsilon^{\frac{1}{2}}\}| + P\{t < X \leqq t + \varepsilon^{\frac{1}{2}}\} \\ &\quad + \textstyle\int_{\{Y > t\}} 1\{X \leqq t - \varepsilon^{\frac{1}{2}}\} + P\{t - \varepsilon^{\frac{1}{2}} < X \leqq t\} \leqq 4\varepsilon^{\frac{1}{2}}\,. \end{aligned}$$

The following lemma is due in part to Volkonskii and Rozanov (1959) and in part to Davydov (1970). For a proof see Deo (1973).

LEMMA 3.2.2. *Let p, q and r be positive numbers with* $p^{-1} + q^{-1} + r^{-1} = 1$. *Let* ξ *and* η *be random variables measurable with respect to* $\mathscr{F}_1^t$ *and* $\mathscr{F}_{t+n}^\infty$. *If*

$$||\xi||_p < \infty \qquad \text{and} \qquad ||\eta||_q < \infty ,$$

then

$$|E(\xi\eta) - E\xi \cdot E\eta| \leqq 10(\rho(n))^{1/r}||\xi||_p||\eta||_q .$$

If, in particular

$$||\xi||_\infty < \infty \qquad \text{and} \qquad ||\eta||_\infty < \infty ,$$

then

$$|E(\xi\eta) - E\xi \cdot E\eta| \leqq 4\rho(n)||\xi||_\infty||\eta||_\infty .$$

LEMMA 3.2.3. *For fixed s and t with* $0 \leqq s < t \leqq 1$

$$E(\textstyle\sum_{n\leqq N} x_n)^2 = N\sigma^2 + O((t-s)^{\frac{1}{2}}) \tag{3.2.1}$$

where

$$\sigma^2 = \sigma^2(s, t) = Ex_1^{\,2} + 2 \textstyle\sum_{n=2}^\infty E(x_1 x_n) \ll (t-s)^{\frac{3}{4}} \tag{3.2.2}$$

and where the constants implied by $\ll$ *and by O only depend on the constants implied by* $\ll$ *in* (3.1.6) *and* (3.1.9).

PROOF. Recall that the x_n's were introduced in (2.4). In a similar fashion write

$$x_{mn} = x_{mn}(s, t) = 1\{s \leqq \eta_{mn} < t\} - l \tag{3.2.3}$$

where $l = t - s$ was introduced in (2.3). Since $|x_n| \leqq 1$ and $|x_{mn}| \leqq 1$ we have by Lemma 3.2.1 and (3.1.7)

$$\begin{aligned} E(x_n - x_{mn})^2 &\leqq 2E|x_n - x_{mn}| \ll m^{-6} , \\ E(x_n - x_{mn})^3 &\ll m^{-6} . \end{aligned} \tag{3.2.4}$$

Since

$$||x_n||_3 \ll l^{\frac{1}{3}} \qquad \text{and} \qquad ||x_n||_2 \ll l^{\frac{1}{2}} \tag{3.2.5}$$

we obtain from (3.2.4) and Minkowski's inequality

$$||x_{mn}||_3 \ll l^{\frac{1}{3}} + m^{-2} \tag{3.2.6}$$

and

$$||x_{mn}||_2 \ll l^{\frac{1}{2}} + m^{-3} . \tag{3.2.7}$$

By Cauchy's inequality

$$|E(x_1 x_n)| \ll ||x_1||_2||x_n||_2 \ll l . \tag{3.2.8}$$

If $n \geqq 3l^{-\frac{1}{4}}$, we put $m = [\frac{1}{3}n]$ and obtain by Lemma 3.2.2, (3.2.4)—(3.2.7)

$$\begin{aligned} |E(x_1 x_n)| &\leqq |E(x_n(x_1 - x_{m1}))| + |E((x_n - x_{mn})x_{m1}) + |E(x_{m1} x_{mn})| \\ &\leqq ||x_n||_2||x_1 - x_{m1}||_2 + ||x_n - x_{mn}||_2||x_{m1}||_2 + 10||x_{m1}||_3||x_{mn}||_3\rho^{\frac{1}{3}}(m) \\ &\ll l^{\frac{1}{2}}m^{-3} + l^{\frac{2}{3}} \cdot m^{-\frac{8}{3}} \ll l^{\frac{1}{2}}n^{-\frac{8}{3}} . \end{aligned} \tag{3.2.9}$$

Hence by (3.2.8) and (3.2.2)

$$\sigma^2 \ll l^{-\frac{1}{4}} \cdot l + l^{\frac{1}{2}} \cdot l^{\frac{5}{12}} \ll l^{\frac{3}{4}} .$$

Now by stationarity

$$\begin{aligned} E(\textstyle\sum_{n \leqq N} x_n)^2 &= NEx_1^2 + 2 \textstyle\sum_{n<N} (N-n)E(x_1 x_{n+1}) \\ &= N\sigma^2 - 2N \textstyle\sum_{n=N}^{\infty} E(x_1 x_{n+1}) - 2 \textstyle\sum_{n<N} nE(x_1 x_{n+1}) . \end{aligned} \tag{3.2.10}$$

Suppose first that $N \leqq 3l^{-\frac{1}{4}}$. Then by (3.2.8) and (3.2.9)

$$\textstyle\sum_{n<N} nE(x_1 x_{n+1}) \ll N^2 \cdot l \ll l^{\frac{1}{2}}$$

and

$$\textstyle\sum_{n \geqq N} E(x_1 x_{n+1}) \ll \sum_{n=N}^{3l^{-\frac{1}{4}}} l + \sum_{n>3l^{-\frac{1}{4}}} n^{-\frac{8}{3}} l^{\frac{1}{2}} \ll l^{\frac{3}{4}} + l^{\frac{1}{2}} \cdot l^{\frac{5}{12}} \ll l^{\frac{3}{4}} \ll l^{\frac{1}{2}} N^{-1} .$$

Hence by (3.2.10)

$$E(\textstyle\sum_{n \leqq N} x_n)^2 - N\sigma^2 \ll l^{\frac{1}{2}} .$$

If, on the other hand, $N > 3l^{-\frac{1}{4}}$, then as before

$$\textstyle\sum_{n \geqq N} E(x_1 x_{n+1}) \ll l^{\frac{1}{2}} \sum_{n \geqq N} n^{-\frac{8}{3}} \ll l^{\frac{1}{2}} N^{-\frac{5}{3}}$$

and

$$\textstyle\sum_{n<N} nE(x_1 x_{n+1}) \ll \sum_{n<3l^{-\frac{1}{4}}} nl + \sum_{n=3l^{-\frac{1}{4}}}^{N} n \cdot n^{-\frac{8}{3}} \cdot l^{\frac{1}{2}} \ll l^{\frac{1}{2}} .$$

Consequently, by (3.2.10)

$$E(\textstyle\sum_{n \leqq N} x_n)^2 - N\sigma^2 \ll l^{\frac{1}{2}} .$$

LEMMA 3.2.4. *Let $g_n(t)$ be defined by* (3.1.9) *and* $\Gamma(s, t)$ *by* (3.1.10). *Then*

$$\lim_{N\to\infty} \frac{1}{N} E(\textstyle\sum_{m,n \leqq N} g_m(s) g_n(t)) = \Gamma(s, t)$$

for $0 \leqq s, t \leqq 1$. *Moreover,* $\Gamma(s, t)$ *is continuous on the unit square and*

$$\sigma^2(s, t) = \sigma^2(0, t) + \sigma^2(0, s) - 2\Gamma(s, t) .$$

PROOF. By (3.1.9), (3.1.10) and (3.2.9), $\Gamma(s, t)$ is a uniformly convergent series of continuous functions on the unit square. The remainder of the lemma follows from Lemma 3.2.3 and the following identity, valid for all $0 \leqq s < t \leqq 1$

$$\begin{aligned} E(\textstyle\sum_{n \leqq N} x_n(s, t))^2 &= E(\textstyle\sum_{n \leqq N} (x_n(0, t) - x_n(0, s)))^2 \\ &= E(\textstyle\sum_{n \leqq N} x_n(0, t))^2 - 2E(\textstyle\sum_{n,m \leqq N} x_n(0, t) x_m(0, s)) \\ &\quad + E(\textstyle\sum_{n \leqq N} x_n(0, s))^2 . \end{aligned} \tag{3.2.11}$$

We also need the following lemma due to Stout (1974, page 299).

LEMMA 3.2.5. *Let* $\{U_n, \mathscr{F}_n\}_{n=1}^{\infty}$ *be a supermartingale with* $EU_1 = 0$. *Put*

$$U_0 = 0 \quad \textit{and} \quad Y_j = U_j - U_{j-1} \qquad j \geqq 1 .$$

Suppose that

$$Y_j \leqq c \quad \text{a.s.}$$

for some constant $c > 0$ and for all $j \geqq 1$. For $\lambda > 0$ define

$$T_n = \exp\{\lambda U_n - \tfrac{1}{2}\lambda^2(1 + \tfrac{1}{2}\lambda c) \sum_{j \leq n} E(Y_j^2 \mid \mathscr{F}_{j-1})\}, \quad n \geqq 1$$

and $T_0 = 1$ a.s. Then for each λ with $\lambda c \leqq 1$ the sequence $\{T_n, \mathscr{F}_n\}_{n=1}^{\infty}$ is a non-negative supermartingale satisfying

$$P\{\sup_{n \geqq 0} T_n > \alpha\} \leqq 1/\alpha$$

for each $\alpha > 0$.

3.3. *Relative compactness.* In this section we shall prove Theorem 3.1. This will imply relative compactness of $\{f_N(t)\}$ as was shown at the end of Section 3.1. The proof will be carried out in two steps. In the first step we prove exponential bounds and in the second step we conclude the proof of Theorem 3.1.

3.3.1. *Exponential bounds.* The following proposition is fundamental for the proof of Theorem 3.1.

PROPOSITION 3.3.1. *Let $H \geqq 0$, $N \geqq 1$ be integers and let $R \geqq 1$. Suppose that $l \geqq N^{-2}$ and that the hypotheses of Theorem* 3.1 *are satisfied. Then as $N \to \infty$*

$$P\{|\sum_{n=H+1}^{H+N} x_n| \geqq ARl^{\frac{1}{120}}(N \log \log N)^{\frac{1}{2}}\} \ll \exp(-6Rl^{-\frac{1}{120}} \log \log N) + R^{-2}N^{-1.03}$$

where both $A\,(> 1)$ and the constant implied by $\ll$ only depend on the constants implied by (3.1.6) *and* (3.1.7).

Since the sequence $\{x_n, n \geqq 1\}$ is strictly stationary it is enough to prove the proposition for $H = 0$ only.

For simplicity of notation put

$$\alpha = \tfrac{1}{120}\,. \tag{3.3.1}$$

We define now blocks H_j and I_j of consecutive integers inductively as follows. H_j consists of $[j^{100\alpha}]$ and I_j also consists of $[j^{100\alpha}]$ consecutive integers respectively. We leave no gaps between the blocks. The order is $H_1, I_1, H_2, I_2, \cdots$. We define random variables y_j and z_j by

$$y_j = \sum_{n \in H_j} x_{mn}$$
$$z_j = \sum_{n \in I_j} x_{mn}$$

where we put $m = [j^{99\alpha}]$. Recall that x_n and x_{mn} were defined in (2.4) and (3.2.3) respectively.

Let $M = M_N$ be the index j of the block H_j or I_j containing N and let h_j be the smallest member of H_j. Then

$$h_M \leqq N < h_{M+1}$$

and

$$\operatorname{card}(H_M \cup I_M) = [M^{100\alpha}] + [M^{100\alpha}] \ll M^{100\alpha} \ll N^{100\alpha/(100\alpha+1)}$$

since

$$M^{100\alpha+1} \ll \sum_{j \leq M} j^{100\alpha} \ll N\,. \tag{3.3.2}$$

The proof of the proposition requires a series of lemmas. We are going to decompose $\sum_{n\leqq N} x_n$ in the form (3.3.8) below. This will motivate all of the lemmas to follow. The first lemma shows that the sum of the x_n's is closely approximated by the sum over the y_j's plus the z_j's.

LEMMA 3.3.1. *As* $N \to \infty$

$$P\{|\sum_{n<h_{M+1}} x_n - \sum_{j\leqq M} (y_j + z_j)| \geqq Rl^\alpha N^{\frac{1}{2}}\} \ll R^{-3}N^{-1.1} .$$

PROOF. Since $l^\alpha N^{\frac{1}{2}} \geqq N^{58\alpha}$ it is enough to estimate

$$\begin{aligned} P\{\textstyle\sum_{n<h_{M+1}} |x_n - x_{mn}| \geqq RN^{58\alpha}\} &\leqq R^{-3}N^{-174\alpha}(\textstyle\sum_{n<h_{M+1}} ||x_n - x_{mn}||_3)^3 \\ &\ll R^{-3}N^{-174\alpha}(\textstyle\sum_{j\leqq M} j^{100\alpha} \cdot (j^{99\alpha})^{-2})^3 \\ &\ll R^{-3}N^{-174\alpha} \cdot (M^{1-98\alpha})^3 \\ &\ll R^{-3}N^{-1.1} \end{aligned}$$

by (3.2.4). We have also used the fact that by (3.3.1) and (3.3.2)

$$-174\alpha + \frac{3(1-98\alpha)}{100\alpha + 1} = -\frac{174}{120} + \frac{3.22}{220} < -1.1 . \tag{3.3.3}$$

LEMMA 3.3.2. *As* $N \to \infty$

$$\sum_{n=h_M}^{h_{M+1}-1} |x_n| \ll l^\alpha N^{\frac{1}{2}} .$$

PROOF. The sum in question does not exceed

$$h_{M+1} - h_M \ll M^{100\alpha} \ll N^{\frac{100}{220}} \ll l^\alpha N^{\frac{1}{2}} . \qquad \square$$

The next lemma is used in Lemma 3.3.6 below.

LEMMA 3.3.3. *As* $N \to \infty$

$$\sum_{j\leqq M} Ey_j^2 \ll l^{\frac{1}{2}} N .$$

PROOF. By stationarity, Lemma 3.2.3 and (3.2.4) we have with $m = [j^{99\alpha}]$

$$\begin{aligned} ||y_j||_2 = ||\textstyle\sum_{n\in H_j} x_{mn}||_2 &\leqq ||\textstyle\sum_{n\in H_j} x_n||_2 + \textstyle\sum_{n\in H_j} ||x_n - x_{mn}||_2 \\ &\ll j^{50\alpha} l^{\frac{1}{4}} + j^{100\alpha}(j^{99\alpha})^{-3} \\ &\ll j^{50\alpha} l^{\frac{1}{4}} + j^{-197\alpha} . \end{aligned}$$

Thus

$$Ey_j^2 \ll j^{100\alpha} l^{\frac{1}{2}} + j^{-394\alpha}$$

and

$$\sum_{j\leqq M} Ey_j^2 \ll l^{\frac{1}{2}} N . \qquad \square$$

Let $\mathscr{L}_j$ be the σ-field generated by $y_1, \cdots, y_j$.

LEMMA 3.3.4. *The random variables* y_j *can be represented in the form*

$$y_j = Y_j + v_j$$

where $(Y_j, \mathscr{L}_j)$ *is a martingale difference sequence and* $v_j = E(y_j | \mathscr{L}_{j-1})$ *satisfies*

$$||v_j||_4 \ll j^{-100\alpha} .$$

PROOF. Put $Y_j = y_j - E(y_j \mid \mathscr{L}_{j-1})$. Then $(Y_j, \mathscr{L}_j)$ is a martingale difference sequence and the 4th moment of v_j can be estimated as follows. For simplicity we drop the subscripts in y_j and $\mathscr{L}_{j-1}$. Then by Lemma 3.2.2 with $p = \infty$, $q = \frac{4}{3}$ and $r = 4$

$$E\{(E(y \mid \mathscr{L}))^4\} = E\{E(y \mid \mathscr{L})(E(y \mid \mathscr{L}))^3\} = E\{y \cdot E(y \mid \mathscr{L})^3\} \ll ||y||_\infty \cdot E^{\frac{3}{4}}\{E(y \mid \mathscr{L})^4\} \cdot \rho^{\frac{1}{4}}(j^{100\alpha}) .$$

We divide by $E^{\frac{3}{4}}\{\cdots\}$ and obtain

$$||E(y \mid \mathscr{L})||_4 \ll ||y||_\infty \rho^{\frac{1}{4}}(j^{100\alpha}) \ll j^{100\alpha}(j^{100\alpha})^{-2} \ll j^{-100\alpha} .$$

LEMMA 3.3.5. *As* $N \to \infty$

$$P\{\textstyle\sum_{j \leq M} |v_j| \geqq R l^\alpha N^{\frac{1}{2}}\} \ll R^{-4} N^{-\frac{3}{2}} .$$

PROOF. Since $l^\alpha N^{\frac{1}{2}} \geqq N^{58\alpha}$ it is enough to estimate

$$\begin{aligned} P\{\textstyle\sum_{j \leq M} |v_j| \geqq R N^{58\alpha}\} &\leqq R^{-4} N^{-232\alpha} (\textstyle\sum_{j \leq M} ||v_j||_4)^4 \\ &\ll R^{-4} N^{-232\alpha} (\textstyle\sum_{j \leq M} j^{-100\alpha})^4 \\ &\ll R^{-4} N^{-232\alpha} (M^{1-100\alpha})^4 \ll R^{-4} N^{-\frac{3}{2}} \end{aligned}$$

by a calculation similar to (3.3.3).

LEMMA 3.3.6. *Let* $B \geqq 1$ *be the constant implied by* $\ll$ *in Lemma* 3.3.3. *Then*

$$P\{\textstyle\sum_{j \leq M} E(Y_j^2 \mid \mathscr{L}_{j-1}) \geqq 2RBl^{3\alpha}N\} \ll R^{-2} N^{-1.03} .$$

PROOF. By Lemma 3.3.4 and Minkowski's inequality

$$\begin{aligned} ||E(Y_j^2 \mid \mathscr{L}_{j-1}) - Ey_j^2||_2 \ll\ & ||E(y_j^2 \mid \mathscr{L}_{j-1}) - Ey_j^2||_2 + ||E(y_j v_j \mid \mathscr{L}_{j-1})||_2 \\ & + ||E(v_j^2 \mid \mathscr{L}_{j-1})||_2 . \end{aligned} \tag{3.3.4}$$

To estimate the first term in (3.3.4) we put $u = y_j^2 - Ey_j^2$ and drop the index in $\mathscr{L}_{j-1}$. As in the proof of Lemma 3.3.4 we obtain

$$E\{(E(u \mid \mathscr{L}))^2\} = E\{E(u \mid \mathscr{L})E(u \mid \mathscr{L})\} = E\{uE(u \mid \mathscr{L})\} \ll ||u||_\infty ||E(u \mid \mathscr{L})||_2 \rho^{\frac{1}{2}}(j^{100\alpha}) .$$

Thus

$$||E(y_j^2 \mid \mathscr{L}_{j-1}) - Ey_j^2||_2 \ll j^{200\alpha}(j^{100\alpha})^{-4} \ll j^{-200\alpha} . \tag{3.3.5}$$

Next

$$||E(v_j^2 \mid \mathscr{L}_{j-1})||_2 \ll ||v_j||_4^2 \ll j^{-200\alpha} . \tag{3.3.6}$$

Finally, by Jensen's inequality and Lemma 3.3.4

$$\begin{aligned} ||E(y_j v_j \mid \mathscr{L}_{j-1})||_2 &\leqq E^{\frac{1}{2}}\{E((y_j v_j)^2 \mid \mathscr{L}_{j-1})\} = E^{\frac{1}{2}}\{y_j^2 v_j^2\} \leqq ||y_j||_4 ||v_j||_4 \\ &\ll j^{75\alpha} \cdot j^{-100\alpha} \ll j^{-25\alpha} \end{aligned} \tag{3.3.7}$$

since $Ey_j^4 \leqq ||y_j^2||_\infty Ey_j^2 \ll j^{200\alpha} \cdot j^{100\alpha} \ll j^{300\alpha}$. Hence by (3.3.4)—(3.3.7) and

Chebyshev's and Minkowski's inequalities

$$P\{\sum_{j\leq M}|E(Y_j^2|\mathscr{L}_{j-1}) - Ey_j^2| \geqq RNl^{3\alpha}\}$$
$$\ll R^{-2}N^{-2+12\alpha}\cdot(\sum_{j\leq M} j^{-25\alpha})^2 \ll R^{-2}N^{-2+12\alpha}(M^{1-25\alpha})^2 \ll R^{-2}N^{-1.03}$$

since $-\frac{228}{120}+\frac{190}{220} < -1.03$. Thus by Lemma 3.3.3

$$P\{\sum_{j\leq M} E(Y_j^2|\mathscr{L}_{j-1}) \geqq 2RBNl^{3\alpha}\} \ll R^{-2}N^{-1.03}.$$

LEMMA 3.3.7. *As* $N\to\infty$

$$P\{|\sum_{j\leq M} Y_j| \geqq 8RBl^{\alpha}(N\log\log N)^{\frac{1}{2}}\} \ll \exp(-6Rl^{-\alpha}\log\log N) + R^{-2}N^{-1.03}.$$

PROOF. We prove the inequality without the absolute value signs since the remaining inequality follows then by replacing Y_j by $-Y_j$. For simplicity we introduce the following notation:

$$\begin{aligned} U_n &= \sum_{j\leq n} Y_j &&\text{for } n\leqq M\\ &= U_M &&\text{for } n> M\\ s_n^2 &= \sum_{j\leq n} E(Y_j^2|\mathscr{L}_{j-1}) &&\text{for } n\leqq M\\ &= s_M^2 &&\text{for } n>M\end{aligned}$$

$c = 2M^{100\alpha}$, $\lambda = 2l^{-2\alpha}(\log\log M)^{\frac{1}{2}}M^{-\frac{1}{2}-50\alpha}$, $K = 4RBl^{3\alpha}M^{1+100\alpha}$ and

$$T_n = \exp(\lambda U_n - \tfrac{1}{2}\lambda^2(1+\tfrac{1}{2}\lambda c)s_n^2).$$

Then $\{U_n, n\geqq 1\}$ defines a martingale. Moreover, by Lemma 3.3.4

$$Y_j = U_j - U_{j-1} \leqq 2j^{100\alpha} \leqq c$$

and

$$\lambda c \leqq 1.$$

Hence Lemma 3.2.5 applies and thus the desired probability does not exceed by Lemma 3.3.6

$$\begin{aligned} &P\{\sup_{n\geq 0} U_n > 8RBl^{\alpha}(M^{1+100\alpha}\log\log M)^{\frac{1}{2}}\}\\ &\quad = P\{\sup_{n\geq 0} U_n > \lambda K\}\\ &\quad = P\{\sup_{n\geq 0}\exp\lambda U_n > \exp(\lambda^2 K)\}\\ &\quad \leqq P\{\sup_{n\geq 0} T_n > \exp(\lambda^2 K - \tfrac{1}{2}\lambda^2(1+\tfrac{1}{2}\lambda c)s_M^2\}\\ &\quad \leqq P\{\sup_{n\geq 0} T_n > \exp(\lambda^2 K - 2RB\lambda^2 M^{1+100\alpha}l^{3\alpha}\} + R^{-2}N^{-1.03}\\ &\quad \leqq \exp(-8RB\log\log M\cdot l^{-\alpha}) + R^{-2}N^{-1.03}. \end{aligned}$$ □

Obviously Lemmas 3.3.3—3.3.7 remain valid if the y_j's are replaced by z_j's. We denote the corresponding martingale difference sequence by $\{Z_j, j\geqq 1\}$.

Finally we can complete the proof of Proposition 3.3.1. We have

$$\begin{aligned} |\sum_{n\leq N} x_n| &\leqq |\sum_{n<h_{M+1}} x_n - \sum_{j\leq M}(y_j+z_j)| + \sum_{n=h_M}^{h_{M+1}-1}|x_n|\\ &\quad + \sum_{j\leq M}|y_j - Y_j| + \sum_{j\leq M}|z_j - Z_j| + |\sum_{j\leq M} Y_j|\\ &\quad + |\sum_{j\leq M} Z_j|. \end{aligned} \tag{3.3.8}$$

By Lemmas 3.3.1, 3.3.2, 3.3.4, 3.3.5 and 3.3.7 we conclude that the RHS of (3.3.8) does not exceed

$$10^3 RBl^\alpha (N \log\log N)^{\frac{1}{2}}$$

with probability

$$\ll \exp(-6Rl^{-\alpha} \log\log N) + R^{-2}N^{-1.03}.$$

3.3.2. *Proof of Theorem* 3.1. As was proved in Section 3.1 relation (3.1.8) implies relative compactness. Now (3.1.8) follows at once from Proposition 3.3.1 and the following proposition which we prove in full generality since it is needed in the next section.

PROPOSITION 3.3.2. *Let* $A \geqq 1$, $\alpha > 0$ *and* $0 < \beta \leqq 1$ *be constants. Let* $x_n = x_n(s, t)$ *be defined by* (2.4) *for some sequence of random variables* η_n. *Suppose that*

$$P\{|\textstyle\sum_{n=H+1}^{H+N} x_n(s, t)| \geqq ARl^\alpha (N \log\log N)^{\frac{1}{2}}\} \ll \exp(-3Rl^{-\alpha} \log\log N) + R^{-2}N^{-1-\beta} \tag{3.3.9}$$

uniformly for all $H \geqq 0$, $N \geqq 1$, $R \geqq 1$ *and* (s, t) *with* $0 \leqq s < t \leqq 1$ *and* $l \geqq N^{-\frac{1}{2}}$. *Then for each* $\varepsilon > 0$ *there exists with probability* 1 *a* $N_0(\varepsilon)$ *such that*

$$|\textstyle\sum_{n \leqq N} x_n(s, t)| \leqq C(A, \alpha, \beta)((t-s)^\alpha + \varepsilon)(N \log\log N)^{\frac{1}{2}}$$

for all $N \geqq N_0$ *and all* $0 \leqq s < t \leqq 1$. *Here the constant* $C(A, \alpha, \beta)$ *depends on* A, α *and* β *only.*

For the proof of Proposition 3.3.2 we use a triple dyadic expansion of $\sum_{n \leqq N} x_n(s, t)$ and then we sum over all the parameters. This method is a combination of techniques due to Cassels (1951) and Erdös and Gál (1954). We write for $0 \leqq s < t \leqq 1$ and integers $P \geqq 0$, $Q \geqq 1$

$$Z(P, Q, s, t) = |\textstyle\sum_{n=P+1}^{P+Q} x_n(s, t)|.$$

We observe that for $s < r < t$

$$\begin{aligned} Z(P, Q, s, t) &\leqq Z(P, Q, s, r) + Z(P, Q, r, t) \\ Z(P, Q, r, t) &\leqq Z(P, Q, s, r) + Z(P, Q, s, t). \end{aligned} \tag{3.3.10}$$

Let m, M be integers with $1 \leqq m \leqq M$ to be chosen suitably later. We write s and t in dyadic expansion:

$$\begin{aligned} s &= \textstyle\sum_{i=1}^\infty \sigma_i 2^{-i} \quad \sigma_i = 0, 1, \\ t &= \textstyle\sum_{i=1}^\infty \tau_i 2^{-i} \quad \tau_i = 0, 1. \end{aligned}$$

Then

$$\begin{aligned} s &= a2^{-m} + \textstyle\sum_{i=m+1}^M \sigma_i 2^{-i} + \theta_1 2^{-M} \\ t &= b2^{-m} + \textstyle\sum_{i=m+1}^M \tau_i 2^{-i} + \theta_2 2^{-M} \end{aligned} \tag{3.3.11}$$

where a and b are integers with $0 \leqq a, b \leqq 2^m$ and $0 \leqq \theta_1, \theta_2 \leqq 1$. We note that

$$Z(P, Q, h2^{-M}, (h+\theta)2^{-M}) \leqq Z(P, Q, h2^{-M}, (h+1)2^{-M}) + Q2^{-M}$$

for $0 \leqq h < 2^M$. Hence by a repeated application of (3.3.10) and by (3.3.11) we obtain for all $P \geqq 0$, $Q \geqq 1$ and $0 \leqq s < t \leqq 1$

$$\begin{aligned} Z(P, Q, s, t) \leqq\ & Z(P, Q, a2^{-m}, b2^{-m}) \\ & + \textstyle\sum_{i=m+1}^{M} Z(P, Q, a_i 2^{-i}, (a_i + 1)2^{-i}) \\ & + \textstyle\sum_{i=m+1}^{M} Z(P, Q, b_i 2^{-i}, (b_i + 1)2^{-i}) \\ & + Z(P, Q, a_{M+1}2^{-M}, (a_{M+1} + 1)2^{-M}) \\ & + Z(P, Q, b_{M+1}2^{-M}, (b_{M+1} + 1)2^{-M}) + 2Q2^{-M} \end{aligned} \tag{3.3.12}$$

where a, b, a_i, b_i $(m < i \leqq M + 1)$ are integers with $0 \leqq a < b \leqq 2^m$, $0 \leqq a_i, b_i < 2^i$ $(m < i \leqq M + 1)$.

We also observe that for integers $P \geqq 0$ and $1 \leqq Q < R$ and $0 \leqq s < t \leqq 1$

$$Z(P, R, s, t) \leqq Z(P, Q, s, t) + Z(P + Q, R - Q, s, t)\,. \tag{3.3.13}$$

Let $N \geqq 1$ be sufficiently large. Put

$$n = [\log N/\log 2]\,. \tag{3.3.14}$$

We write N in dyadic expansion:

$$\begin{aligned} N &= 2^n + \textstyle\sum_{j=1}^{n} \varepsilon_j 2^{j-1} \\ &= 2^n + \textstyle\sum_{j=[\frac{1}{2}n]+1}^{n} \varepsilon_j 2^{j-1} + \theta N^{\frac{1}{2}}\,, \end{aligned}$$

where $\varepsilon_j = 0, 1$ $(1 \leqq j \leqq n)$ and $0 \leqq \theta < 1$. Hence for each $0 \leqq s < t \leqq 1$ we obtain applying (3.3.13) repeatedly

$$Z(0, N, s, t) \leqq Z(0, 2^n, s, t) + \textstyle\sum_{\frac{1}{2}n \leqq j \leqq n} Z(2^n + h_j 2^j, 2^{j-1}, s, t) + N^{\frac{1}{2}} \tag{3.3.15}$$

where h_j are integers with $0 \leqq h_j < 2^{n-j}$ $(j \leqq n)$.

Our goal is, of course, to show that $Z(0, N, s, t)$ is almost surely uniformly small for all $0 \leqq s < t \leqq 1$ and all sufficiently large N. This will follow from the next lemma. To simplify the notation we write

$$\phi(k) = 2A(k \log\log k)^{\frac{1}{2}} \qquad k \geqq 3\,. \tag{3.3.16}$$

Put

$$m = [(\log n)^{\frac{1}{2}}]\,. \tag{3.3.17}$$

We define the following events

$$\begin{aligned} E_n(a, b) &= \{Z(0, 2^n, a2^{-m}, b2^{-m}) \geqq ((b - a)2^{-m})^{\alpha}\phi(2^n)\} \\ E_n &= \textstyle\bigcup_{0 \leqq a, b < 2^m} E_n(a, b) \\ F_n(i, a) &= \{Z(0, 2^n, a2^{-i}, (a + 1)2^{-i}) \geqq 2^{-\alpha i}\phi(2^n)\} \\ F_n &= \textstyle\bigcup_{m < i \leqq \frac{1}{2}n} \bigcup_{0 \leqq a < 2^i} F_n(i, a) \\ G_n(a, b, j, h) &= \{Z(2^n + h2^j, 2^{j-1}, a2^{-m}, b2^{-m}) \geqq ((b - a)2^{-m})^{\alpha} 2^{\frac{1}{2}(j-n)\beta}\phi(2^n)\} \\ G_n &= \textstyle\bigcup_{0 \leqq a, b < 2^m} \bigcup_{\frac{1}{2}n \leqq j \leqq n} \bigcup_{0 \leqq h < 2^{n-j}} G_n(a, b, j, h) \\ H_n(i, a, j, h) &= \{Z(2^n + h2^j, 2^{j-1}, a2^{-i}, (a + 1)2^{-i}) \geqq 2^{-\alpha i} 2^{\frac{1}{2}(j-n)\beta}\phi(2^n)\} \\ H_n &= \textstyle\bigcup_{\frac{1}{2}n \leqq j \leqq n} \bigcup_{m < i \leqq \frac{1}{2}j} \bigcup_{0 \leqq a < 2^i} \bigcup_{0 \leqq h < 2^{n-j}} H_n(i, a, j, h)\,. \end{aligned}$$

LEMMA 3.3.8. *With probability* 1 *only a finite number of the events* E_n, F_n, G_n *and* H_n *occur.*

PROOF. We estimate the probabilities of these events and apply the Borel–Cantelli lemma. We first estimate $P(E_n(a, b))$. We apply (3.3.9) with $H = 0$, $N = 2^n$, $R = 1$ and obtain for fixed a and b

$$P\{E_n(a, b)\} \ll \exp(-2 \cdot 2^{m\alpha} \log n) + 2^{-n} .$$

Hence

$$P(E_n) \ll 2^{2m} \exp(-2 \cdot 2^{m\alpha} \log n) + 2^{2m-n} \ll n^{-2} \tag{3.3.18}$$

by (3.3.17). Similarly, putting $H = 0$, $N = 2^n$, $R = 1$ we obtain for fixed i and a

$$P\{F_n(a, i)\} \ll \exp(-2 \cdot 2^{\alpha i} \log n) + 2^{-n} .$$

Hence

$$\begin{aligned} P(F_n) &\ll \textstyle\sum_{m<i\leq\frac{1}{2}n} 2^i \exp(-2 \cdot 2^{\alpha i} \log n) + \sum_{m<i\leq\frac{1}{2}n} 2^i 2^{-n} \\ &\ll n^{-2} . \end{aligned} \tag{3.3.19}$$

Similarly, putting $H = 2^n + h2^j$, $N = 2^{j-1}$, $R = 2^{\frac{1}{4}(n-j)(2-\beta)}$ we obtain for fixed a, b, j, h

$$P\{G_n(a, b, j, h)\} \ll \exp(-2 \cdot 2^{m\alpha} \cdot 2^{\frac{1}{4}(n-j)(2-\beta)} \log j) + 2^{-\frac{1}{2}(n-j)(2-\beta)} 2^{-j(1+\beta)} .$$

Hence

$$\begin{aligned} P(G_n) \ll\ & 2^{2m} \textstyle\sum_{\frac{1}{2}n\leq j\leq n} 2^{n-j} \exp(-2 \cdot 2^{m\alpha} 2^{\frac{1}{4}(n-j)(2-\beta)} \log j) \\ &+ 2^{2m} \textstyle\sum_{\frac{1}{2}n\leq j\leq n} 2^{n-j} 2^{-\frac{1}{2}(n-j)(2-\beta)} 2^{-j(1+\beta)} \ll n^{-2} . \end{aligned} \tag{3.3.20}$$

Finally, with the same choice of H, N and R as before we obtain for fixed i, a, j, h

$$P\{H_n(i, a, j, h)\} \ll \exp(-2 \cdot 2^{\alpha i} 2^{\frac{1}{4}(n-j)(2-\beta)} \log j) + 2^{-\frac{1}{2}(n-j)(2-\beta)} \cdot 2^{-j(1+\beta)} .$$

Hence

$$\begin{aligned} P(H_n) \ll\ & \textstyle\sum_{\frac{1}{2}n\leq j\leq n} \sum_{m<i\leq\frac{1}{2}j} 2^i \cdot 2^{n-j} \exp(-2 \cdot 2^{\alpha i} 2^{\frac{1}{4}(n-j)(2-\beta)} \log j) \\ &+ \textstyle\sum_{\frac{1}{2}n\leq j<n} \sum_{m<i\leq\frac{1}{2}j} 2^i 2^{n-j} 2^{-\frac{1}{2}(n-j)(2-\beta)} 2^{-j(1+\beta)} \\ \ll\ & n^{-2} . \end{aligned} \tag{3.3.21}$$

Lemma 3.3.8 follows now from (3.3.18)—(3.3.21). □

Finally, we can finish the proof of Proposition 3.3.2. We put in (3.3.12) $P = 0$, $Q = 2^n$ and $M = [\frac{1}{2}n]$ and obtain that with probability 1

$$\begin{aligned} Z(0, 2^n, s, t) &\ll ((b - a)2^{-m})^\alpha \phi(2^n) + \textstyle\sum_{i=m+1}^{M} 2^{-\alpha i} \phi(2^n) + 2^{\frac{1}{2}n} \\ &\ll ((t - s)^\alpha + \tfrac{1}{2}\varepsilon)\phi(2^n) \end{aligned}$$

using Lemma 3.3.8 and (3.3.17). Similarly with $P = 2^n + h_j 2^j$, $Q = 2^{j-1}$ and $M = [\frac{1}{2}j]$ we obtain

$$\begin{aligned} &Z(2^n + h_j 2^j, 2^{j-1}, s, t) \\ &\quad\ll ((b - a)2^{-m})^\alpha 2^{\frac{1}{4}(j-n)\beta} \phi(2^n) + \textstyle\sum_{i=m+1}^{M} 2^{-\alpha i} 2^{\frac{1}{4}(j-n)\beta} \phi(2^n) + 2^{j-\frac{1}{2}j} \\ &\quad\ll ((t - s)^\alpha + \tfrac{1}{2}\varepsilon) 2^{\frac{1}{4}(j-n)\beta} \phi(2^n) + 2^{\frac{1}{2}j} . \end{aligned}$$

Hence by (3.3.15)

$$Z(0, N, s, t) \ll ((t-s)^\alpha + \tfrac{1}{2}\varepsilon)\phi(N)(1 + \textstyle\sum_{\frac{1}{2}n \leq j \leq n} 2^{\frac{1}{4}(j-n)\beta}) + \sum_{\frac{1}{2}n \leq j \leq n} 2^{\frac{1}{2}j} + N^{\frac{1}{2}}$$
$$\ll ((t-s)^\alpha + \varepsilon)\phi(N)$$

for all $N \geqq N_0(\varepsilon)$ and all $0 \leqq s < t \leqq 1$.

4. Lacunary sequences.

4.1. *Introduction.* Let $\{n_k, k \geqq 1\}$ be a sequence of real numbers satisfying

$$n_{k+1}/n_k \geqq q > 1 \qquad k = 1, 2, \cdots$$

for some $q > 1$. For fixed s and t with $0 \leqq s < t \leqq 1$ write $L = [s, t)$, $l = t - s$ and

(4.1.1) $$x_k = x_k(s, t) = 1\{s \leqq n_k \omega < t\} - (t - s) = 1_L(n_k \omega) - l$$

where $1\{\cdots\} = 1_L\{\cdot\}$ is extended with period 1. In other words we are investigating the sequence $\{\langle n_k \omega\rangle, k \geqq 1\} = \{\eta_k, k \geqq 1\}$ (say) of random variables as described in Section 1. Denote by $F_N(t)$ the empirical distribution function of $\{\langle n_k \omega\rangle, k \geqq 1\}$ at stage N. Define

(4.1.2) $$f_N(t) = N(F_N(t) - t)(2N \log\log N)^{-\frac{1}{2}} \qquad 0 \leqq t \leqq 1 .$$

In this section we shall prove the following theorem.

THEOREM 4.1. *Let $\{n_k, k \geqq 1\}$ be a lacunary sequence of real numbers. Then for each $\varepsilon > 0$ there exists with probability 1 a $N_0(\varepsilon)$ such that*

(4.1.3) $$|f_N(t) - f_N(s)| \leqq C|t-s|^{\frac{1}{8}} + \varepsilon$$

for all $N \geqq N_0$ and all $0 \leqq s < t \leqq 1$. The constant C only depends on q. In particular, (4.1.3) *implies that the sequence $\{f_N(t), N \geqq 3\}$ is with probability* 1 *relatively compact in* $D[0, 1]$.

The statement about the relative compactness can be shown as in Section 3.1.

As pointed out in Section 1 Theorem 4.1 also implies a law of the iterated logarithm of the form (1.4). Indeed, we have with probability 1

$$\frac{N|F_N(t) - t|}{(N \log\log N)^{\frac{1}{2}}} \ll 1$$

for all $N \geqq N_0$ and for all $0 \leqq t \leqq 1$. Hence taking first the supremum over all t with $0 \leqq t \leqq 1$ and then the limit superior as $N \to \infty$ we obtain the right-hand side of (1.4).

We can identify the limit points of $\{f_N(t), N \geqq 3\}$ only if we make some further assumptions. We assume that for all step functions f with period 1 and $\int_0^1 f(x)\,dx = 0$ and for all $k \geqq 1$, $0 \leqq i < 2^k$ and $M, N \geqq 1$ we have for some $\sigma > 0$ depending on f only

(4.1.4) $$2^k \int_{i \cdot 2^{-k}}^{(i+1)2^{-k}} \Big(\sum_{j=M}^{M+N-1} f(n_j \omega)\Big)^2 d\omega = \sigma^2 N(1 + o(1))$$

where the constant implied by o depends on q and on f only. In particular, (4.1.4) and (3.2.11) imply that

$$\text{(4.1.5)} \qquad \lim_{N\to\infty} \frac{1}{N} E(\textstyle\sum_{j,k\leqq N} x_j(0, s)x_k(0, t)) \\ = \Gamma(s, t) = \frac{1}{2}(\sigma^2(0, t) + \sigma^2(0, s) - \sigma^2(s, t)) .$$

(4.1.4) says that the conditional variances of the partial sums given the σ-field generated by the dyadic intervals of order k equal asymptotically the variances of these sums which in turn equal asymptotically a constant multiple of the length of these partial sums. Under these conditions Berkes (1976) proved an almost sure invariance principle for the sums $\sum_{k\leqq N} f(n_k \omega)$.

An almost sure invariance principle for these sums under somewhat simpler conditions has been recently established by Berkes and Philipp (1977). Let us say that a sequence of integers $\{m_k, k \geqq 1\}$ satisfies condition B_2 if there is a constant C such that the number of solutions of the equation $m_k \pm m_l = \nu$ does not exceed C for any integer ν. We replace (4.1.4) by the following two conditions

$$\text{(4.1.4)*} \qquad \int_0^1 (\textstyle\sum_{j=M}^{M+N-1} f(n_j \omega))^2 \, d\omega = \sigma^2 N(1 + o(1))$$

and

(4.1.6) for any integer $m \geqq 1$ the set-theoretic union of the sequences $\{[n_k], k \geqq 1\}, \{[2n_k], k \geqq 1\}, \cdots, \{[mn_k], k \geqq 1\}$, arranged in increasing order and considered as a new sequence, satisfies condition B_2.

THEOREM 4.2. *Let $\{n_k, k \geqq 1\}$ be a lacunary sequence of real numbers such that either* (4.1.4) *or both* (4.1.4)* *and* (4.1.6) *hold for all step functions f with period* 1 *and $\int_0^1 f(x)\,dx = 0$. Suppose that $\Gamma(s, t)$ defined in* (4.1.5) *is positive definite. Then the sequence $\{f_N(t), N \geqq 3\}$ is with probability* 1 *relatively compact in $D[0, 1]$ and has the unit ball in the kernel space $H(\Gamma)$ as the set of its limit points.*

REMARK. Equivalently, the set of limits points equals $\overline{\bigcup_{m\geqq 1} B_m}$ where the closure is in the topology defined by the supremum norm over $[0, 1]$. Here B_m is defined in the same way as in Section 3.1.

An example of a lacunary sequence satisfying the hypotheses of Theorem 4.2 is given in the following corollary.

COROLLARY 4.1. *Let $\{n_k, k \geqq 1\}$ be a sequence of real numbers with $n_{k+1}/n_k \to \infty$. Then the sequence $\{f_N(t), N \geqq 3\}$ is with probability* 1 *relatively compact in $D[0, 1]$ and has the class $K = \{h$ absolutely continuous on $[0, 1]$, $h(0) = h(1) = 0$, $\int_0^1 (h'(t))^2\,dt \leqq 1\}$ as its set of limit points.*

PROOF. By Lemma 3.3 of Berkes (1975, Part I) condition (4.1.4)* is satisfied with $\sigma^2 = \int_0^1 f^2(x)\,dx$. It is well known (see Gaposhkin (1966) or Berkes (1975)) that $\{n_k, k \geqq 1\}$ satisfies (4.1.6). Hence by (4.1.5) $\Gamma(s, t) = s(1 - t)$

for $0 \leqq s \leqq t \leqq 1$ which is the covariance function of the Brownian bridge. Moreover, as is well known, the limit set appearing in Corollary 4.1 is precisely the unit ball in the kernel space $H(\Gamma)$.

4.2. *Relative compactness.* In view of Proposition 3.3.2 for the proof of Theorem 4.1 it is enough to show the following exponential bound.

PROPOSITION 4.2.1. *Let $H \geqq 0$, $N \geqq 1$ be integers and let $R \geqq 1$. Suppose that $l \geqq N^{-\frac{4}{3}}$. Then as $N \to \infty$*

$$P\{|\textstyle\sum_{k=H+1}^{H+M} x_k| \geqq ARl^{\frac{1}{8}}(N \log\log N)^{\frac{1}{2}}\} \ll \exp(-10Rl^{-\frac{1}{8}} \log\log N) + R^{-6}N^{-\frac{3}{2}} .$$

where both A and the constant implied by $\ll$ only depend on q.

The proof of Proposition 4.2.1 is by and large parallel to that of Proposition 3.3.1. We start with two simple observations.

Since the sequence $\{n_{k+H}\}_{k=1}^{\infty}$ is lacunary with the same ratio q it is enough to prove the proposition with $H = 0$.

Next let r be the smallest integer with

$$q^r \geqq 2 ; \qquad \text{i.e.,} \qquad r = \left[\frac{\log 2}{\log q}\right] + 1 .$$

Since each sequence $\{n_{a+kr}\}_{k=1}^{\infty}$ is lacunary with ratio $\geqq 2$ ($a = 0, 1, \cdots, r-1$) there is no loss of generality if we prove the proposition under the additional assumption $q \geqq 2$.

For the proof of the proposition we need a series of simple facts which we state as lemmas.

LEMMA 4.2.1. *For $0 \leqq a < b \leqq 1$ we have*

$$\int_a^b 1_L(n_k \omega)\, d\omega = l(b - a + 4\theta n_k^{-1})$$

where θ is a constant with $|\theta| \leqq 1$.

PROOF. The integral equals

$$\begin{aligned} n_k^{-1} \int_{an_k}^{bn_k} 1_L(\omega)\, d\omega &= n_k^{-1}\{([bn_k] - [an_k])l - \int_{[an_k]}^{an_k} 1_L(\omega)\, d\omega + \int_{[bn_k]}^{bn_k} 1_L(\omega)\, d\omega\} \\ &= n_k^{-1}\{n_k(b-a)l + 4\theta l\} = l(b - a + 4\theta n_k^{-1}) . \qquad \square \end{aligned}$$

Let r_k be the largest integer r with

$$2^r \leqq n_k k^{12} \tag{4.2.1}$$

and let $\mathscr{F}_k$ be the σ-field generated by the intervals

$$U_{\nu k} = [\nu 2^{-r_k}, (\nu + 1)2^{-r_k}) \qquad 0 \leqq \nu < 2^{r_k} .$$

LEMMA 4.2.2. *We have for $k \geqq 0$ and $j \geqq 1$*

$$E(x_{j+k} \,|\, \mathscr{F}_j) \ll lj^{12}2^{-k} \quad \text{a.s.}$$

where the constant implied by $\ll$ is absolute.

PROOF. We first observe using Lemma 4.2.1 and (4.2.1) that for $0 \leqq \nu < 2^{r_j}$ we have

$$\textstyle\int_{U_{\nu j}} x_{j+k}(\omega)\, d\omega \ll l n_{j+k}^{-1} \ll l 2^{-r_j} j^{12} n_j n_{j+k}^{-1} \ll l 2^{-r_j} j^{12} 2^{-k} \quad \text{a.s.}$$

Hence

$$E(x_{j+k} \mid \mathscr{F}_j) = \textstyle\sum_{\nu=0}^{2^{r_j}-1} 1_{U_{\nu j}}(\cdot) 2^{r_j} \int_{U_{\nu j}} x_{j+k}(\omega)\, d\omega \ll l j^{12} 2^{-k} \quad \text{a.s.} \qquad \square$$

We define now blocks I_j and H_j of consecutive integers inductively as follows. H_j consists of $2[j^{\frac{1}{2}}]$ and I_j consists of $2[j^{\frac{1}{2}}]$ consecutive integers respectively. We leave no gaps between the blocks. The order is $H_1, I_1, H_2, I_2, \cdots$. Thus $H_1 = \{1, 2\}, I_1 = \{3, 4\}, \cdots, H_4 = \{13, 14, 15, 16\}, I_4 = \{17, 18, 19, 20\}, \cdots$. Let $M = M_N$ be the index of the block I_j or H_j containing N and let h_j be the largest number of H_j. Then

$$h_{M-1} < N \leqq h_M$$

and

$$\text{card } H_M \cup I_M = 4[M^{\frac{1}{2}}] \ll N^{\frac{1}{3}}$$

since

(4.2.2) $$M^{\frac{3}{2}} \ll \textstyle\sum_{j \leqq M} j^{\frac{1}{2}} \ll N .$$

Define

(4.2.3) $$w_j = \textstyle\sum_{\nu \in H_j} x_\nu ,$$

(4.2.4) $$y_j = E(w_j \mid \mathscr{F}_{h_j})$$

and

(4.2.5) $$\xi_\nu = E(x_\nu \mid \mathscr{F}_{h_j}) \qquad \text{if} \quad \nu \in H_j ,$$

so that

(4.2.6) $$y_j = \textstyle\sum_{\nu \in H_j} \xi_\nu .$$

LEMMA 4.2.3. *We have*

$$\|x_k - \xi_k\|_2 \ll k^{-6}$$

where the constant implied by $\ll$ *is absolute.*

PROOF. The random variables x_k assume only two values, namely $1 - t + s$ and $-t + s$. Thus $\xi_k = x_k$ throughout all but at most $2n_k$ intervals $U_{\nu h_j}$. These exceptional intervals are these where $1\{s \leqq n_k \omega \leqq t\}$ has a jump. Hence if $k \in H_j$

$$E(x_k - \xi_k)^2 \leqq 2n_k \cdot 2^{-r_{h_j}} \ll n_{h_j} \cdot 2^{-r_{h_j}} \ll h_j^{-12} \ll k^{-12} .$$

LEMMA 4.2.4. *As* $N \to \infty$

$$P\{\textstyle\sum_{j \leqq M} |y_j - w_j| \geqq R l^{\frac{1}{3}} N^{\frac{1}{2}}\} \ll R^{-6} N^{-\frac{3}{2}} .$$

PROOF. We first estimate

$$\begin{aligned} E(y_j - w_j)^2 &= E(\textstyle\sum_{\nu \in H_j} x_\nu - \xi_\nu)^2 \\ &\ll \textstyle\sum_{\nu \in H_j} E(x_\nu - \xi_\nu)^2 + \sum_{\mu < \nu \in H_j} |E(x_\mu - \xi_\mu)(x_\nu - \xi_\nu)| \\ &\ll \textstyle\sum_{\nu \in H_j} E(x_\nu - \xi_\nu)^2 + \sum_{\mu < \nu \in H_j} |E\{(x_\nu - \xi_\nu) x_\mu\}| \end{aligned}$$

since for $\mu, \nu \in H_j$ by (4.2.5)

$$E(\xi_\mu \xi_\nu) = E(\xi_\mu x_\nu) = E(\xi_\nu x_\mu) .$$

Hence by Lemma 4.2.3 we have

$$(4.2.7) \qquad \begin{aligned} E(y_j - w_j)^2 &\ll \textstyle\sum_{\nu \in H_j} \nu^{-12} + \sum_{\mu < \nu \in H_j} \nu^{-6} \\ &\ll h_j^{-12} j^{\frac{1}{2}} + h_j^{-6} j \ll (j^{\frac{3}{2}})^{-6} j \ll j^{-8} . \end{aligned}$$

Since $l^{\frac{1}{2}} N^{\frac{1}{2}} \gg N^{-\frac{1}{6}} N^{\frac{1}{2}} \geqq N^{\frac{1}{3}}$, the probability in question does not exceed

$$\begin{aligned} P\{\textstyle\sum_{j \leq M} |y_j - w_j| \geqq RN^{\frac{1}{3}}\} &\ll R^{-6} N^{-2} (\textstyle\sum_{j \leq M} \|y_j - w_j\|_6)^6 \\ &\ll R^{-6} N^{-2} (\textstyle\sum_{j \leq M} (j^2 E(y_j - w_j)^2)^{\frac{1}{6}})^6 \\ &\ll R^{-6} N^{-2} (\textstyle\sum_{j \leq M} j^{-1})^6 \ll R^{-6} N^{-\frac{3}{2}} \end{aligned}$$

by (4.2.7), (4.2.3) and (4.2.4).

LEMMA 4.2.5. *We have*

$$E(w_j^2 \,|\, \mathscr{F}_{h_{j-1}}) \ll l j^{\frac{1}{2}} \quad \text{a.s.}$$

where the constant implied by $\ll$ *is absolute.*

PROOF. For simplicity we write $r = r_{h_{j-1}}$ and $U_{\nu h_{j-1}} = U$. We first observe that for $k \in H_j$

$$(4.2.8) \qquad n_k 2^{-r} \geqq n_k h_{j-1}^{-12} n_{h_{j-1}}^{-1} \gg (j^{\frac{3}{2}})^{-12} 2^{2j^{\frac{1}{2}}} \gg 2^{j^{\frac{1}{2}}} .$$

Hence by Lemma 4.2.1 we have for $k \in H_j$

$$(4.2.9) \qquad \begin{aligned} \textstyle\int_U 1_L(n_k \omega)\, d\omega &= l(2^{-r} + 4\theta n_k^{-1}) \\ &= l 2^{-r}(1 + 4\theta 2^{-j^{\frac{1}{2}}}) \end{aligned}$$

where θ denotes a constant with $|\theta| \leqq 1$, but not necessarily the same at each occurrence. Thus

$$(4.2.10) \qquad \begin{aligned} \textstyle\int_U x_k^2\, d\omega &= (1 - 2l) \textstyle\int_U 1_L(n_k \omega)\, d\omega + l^2 2^{-r} \\ &\ll l \cdot 2^{-r} . \end{aligned}$$

Next we note that

$$1_L(n_i \omega) = \textstyle\sum_{\nu=0}^{n_i - 1} 1\{(s + \nu) n_i^{-1} \leqq \omega < (t + \nu) n_i^{-1}\} .$$

Hence, and since by (4.2.8) $n_i 2^{-r}$ is large, the integrals

$$(4.2.11) \qquad \textstyle\int_U 1_L(n_i \omega) 1_L(n_k \omega)\, d\omega \qquad\qquad i < k \in H_j$$

can be written as the sum of $n_i 2^{-r} + 2\theta'$ integrals of the form

$$\textstyle\int_b^a 1_L(n_k \omega)\, d\omega$$

with $b - a = l n_i^{-1}$ except for at most two such integrals for which $b - a < l n_i^{-1}$. By Lemma 4.2.1 such an integral equals

$$l(l n_i^{-1} + 4\theta n_k^{-1})$$

or is

$$\leqq 5 l n_i^{-1}$$

respectively. Hence by (4.2.8) the integral in (4.2.11) equals

$$(n_i 2^{-r} + 2\theta)l(ln_i^{-1} + 4\theta n_k^{-1}) + 10\theta ln_i^{-1} = l^2 2^{-r} + 20\theta ln_i^{-1} + 4\theta 2^{-r} n_i n_k^{-1} l$$
$$= 2^{-r}(l^2 + 20\theta l 2^{-j^{\frac{1}{2}}} + 4\theta 2^{-k+i} l) .$$

For $i < k \in H_j$, we thus obtain using (4.2.9)

$$\int_U x_i x_k \, d\omega = \int_U 1_L(n_i \omega) 1_L(n_k \omega) - l \int_U 1_L(n_i \omega) - l \int_U 1_L(n_k \omega) + l^2 2^{-r}$$
$$\ll 2^{-r} l (2^{-j^{\frac{1}{2}}} + 2^{-k+i}) .$$

Consequently, we obtain writing $U_{\nu h_{j-1}} = U_\nu$ and using (4.2.10)

$$E(w_j^2 \mid \mathscr{F}_{h_{j-1}}) = \sum_{\nu=0}^{2^r-1} 1_{U_\nu}(\cdot) 2^r \int_{U_\nu} w_j^2$$
$$\ll \sum_{\nu=0}^{2^r-1} 1_{U_\nu}(\cdot) 2^r (\sum_{k \in H_j} \int_U x_k^2 + \sum_{i<k \in H_j} |\int_U x_i x_k|)$$
$$\ll \sum_{\nu=0}^{2^r-1} 1_{U_\nu}(\cdot)(l \cdot j^{\frac{1}{2}} + l \cdot j \cdot 2^{-j^{\frac{1}{2}}} + l \sum_{i<k \in H_j} 2^{-k+i})$$
$$\ll l j^{\frac{1}{2}} \quad \text{a.s.}$$

LEMMA 4.2.6. *As $N \to \infty$*

$$\sum_{n=N+1}^{h_M} x_n \ll l^{\frac{1}{2}} N^{\frac{1}{2}}$$

where the constant implied by $\ll$ is absolute.

PROOF. We have

$$|\sum_{n=N+1}^{h_M} x_n| \leqq h_M - N \ll M^{\frac{1}{2}} \ll N^{\frac{1}{3}} \ll l^{\frac{1}{2}} N^{\frac{1}{2}} .$$

LEMMA 4.2.7. *The random variables y_j can be represented in the form*

$$y_j = Y_j + v_j$$

where $(Y_j, \mathscr{L}_j)$ is a martingale difference sequence, $\mathscr{L}_j$ is the σ-field generated by $y_1, \cdots, y_j$ and $v_j = E(y_j \mid \mathscr{L}_{j-1})$ satisfies

$$v_j \ll l \cdot 2^{-j^{\frac{1}{2}}} \quad \text{a.s.}$$

with an absolute constant implied by $\ll$.

PROOF. Put $Y_j = y_j - E(y_j \mid \mathscr{L}_{j-1})$. Then $(Y_j, \mathscr{L}_j)$ is a martingale difference sequence and

$$v_j = y_j - Y_j = E(y_j \mid \mathscr{L}_{j-1}) = E(E(w_j \mid \mathscr{F}_{h_j}) \mid \mathscr{L}_{j-1}) = E(w_j \mid \mathscr{L}_{j-1})$$
$$= E(E(w_j \mid \mathscr{F}_{h_{j-1}}) \mid \mathscr{L}_{j-1}) .$$

But by Lemma 4.2.2

$$E(w_j \mid \mathscr{F}_{h_{j-1}}) = \sum_{\nu \in H_j} E(x_\nu \mid \mathscr{F}_{h_{j-1}})$$
$$\ll j^{\frac{1}{2}} l h_j^{12} 2^{-2j^{\frac{1}{2}}} \ll l 2^{-j^{\frac{1}{2}}} \quad \text{a.s.}$$

This proves the lemma.

LEMMA 4.2.8. *As $N \to \infty$*

$$\sum_{j \leqq M} E(Y_j^2 \mid \mathscr{L}_{j-1}) \ll lN \quad \text{a.s.}$$

PROOF. By Lemma 4.2.5 and Jensen's inequality

$$\begin{aligned} E(y_j^2 \mid \mathscr{L}_{j-1}) &= E\{(E(w_j \mid \mathscr{F}_{h_j}))^2 \mid \mathscr{L}_{j-1}\} \\ &\leqq E\{E(w_j^2 \mid \mathscr{F}_{h_j}) \mid \mathscr{L}_{j-1}\} = E(w_j^2 \mid \mathscr{L}_{j-1}) \\ &= E\{E(w_j^2 \mid \mathscr{F}_{h_{j-1}}) \mid \mathscr{L}_{j-1}\} \\ &\ll l j^{\frac{1}{2}} . \end{aligned}$$

Hence by Lemma 4.2.6

$$E(Y_j^2 \mid \mathscr{L}_{j-1}) \ll E(y_j^2 \mid \mathscr{L}_{j-1}) + E(v_j^2 \mid \mathscr{L}_{j-1}) \ll l j^{\frac{1}{2}} + l^2 j^{-2j^{\frac{1}{2}}} \ll l j^{\frac{1}{2}} \quad \text{a.s.}$$

We sum the last inequality over $j \leqq M$ and obtain the result.

LEMMA 4.2.9. *Let $B \geqq 1$ be the constant implied by $\ll$ in Lemma* 4.2.8. *Then as $N \to \infty$*

$$P\{|\textstyle\sum_{j \leqq M} Y_j| > 8RBl^{\frac{1}{8}}(N \log\log N)^{\frac{1}{2}}\} \ll \exp(-10Rl^{-\frac{1}{8}} \log\log N) .$$

PROOF. We prove the inequality without the absolute value signs. The remaining inequality follows then by replacing Y_j by $-Y_j$. For simplicity we introduce the following notation:

$$\begin{aligned} U_n &= \textstyle\sum_{j \leqq n} Y_j && \text{for} \quad n \leqq M , \\ &= U_M && \text{for} \quad n > M ; \\ s_n^2 &= \textstyle\sum_{j \leqq n} E(Y_j^2 \mid \mathscr{L}_{j-1}) && \text{for} \quad n \leqq M , \\ &= s_M^2 && \text{for} \quad n > M ; \end{aligned}$$

$$c = 2M^{\frac{1}{2}} , \qquad \lambda = 2l^{-\frac{1}{4}}(\log\log M)^{\frac{1}{2}} M^{-\frac{3}{4}} , \qquad K = 4RBl^{\frac{3}{8}} M^{\frac{3}{2}}$$

and

$$T_n = \exp(\lambda U_n - \tfrac{1}{2}\lambda^2(1 + \tfrac{1}{2}\lambda c)s_n^2) .$$

Then $\{U_n\}_{n=1}^{\infty}$ defines a martingale. Moreover,

$$Y_j = U_j - U_{j-1} \leqq 2j^{\frac{1}{2}} \leqq 2M^{\frac{1}{2}} = c$$

and

$$\lambda c \leqq 1 .$$

Hence Lemma 3.2.5 applies and thus the desired probability does not exceed

$$\begin{aligned} P\{\sup\nolimits_{n \geqq 0} U_n > 8RBl^{\frac{1}{8}}(M^{\frac{3}{2}} \log\log M)^{\frac{1}{2}}\} & \\ &= P\{\sup\nolimits_{n \geqq 0} U_n > \lambda K\} \\ &= P\{\sup\nolimits_{n \geqq 0} \exp(\lambda U_n) > \exp(\lambda^2 K)\} \\ &\leqq P\{\sup\nolimits_{n \geqq 0} T_n > \exp(\lambda^2 K - \tfrac{1}{2}\lambda^2(1 + \tfrac{1}{2}\lambda c)s_M^2\} \\ &\leqq P\{\sup\nolimits_{n \geqq 0} T_n > \exp(\lambda^2 K - \lambda^2 B l M^{\frac{3}{2}})\} \\ &\leqq \exp(-12RBl^{-\frac{1}{8}} \log\log M) \\ &\ll \exp(-10RBl^{-\frac{1}{8}} \log\log N) . \end{aligned}$$

□

Let

$$z_j = \textstyle\sum_{\nu \in I_j} \xi_\nu .$$

Again, as in Section 3.3.1, we can say that Lemmas 4.2.4—4.2.9 remain valid if the y_j's are replaced by the z_j's. The remainder of the proof of Proposition 4.2.1 is similar to Section 3.3.1. This concludes the proof of Proposition 4.2.1 and hence of Theorem 4.1.

4.3. *Two more lemmas.* The following lemmas are needed in Section 5.2.

LEMMA 4.3.1. *We have for* $0 \leqq s < t \leqq 1$

$$E(\textstyle\sum_{k \leqq N} x_k(s, t))^2 \ll N(t - s) ,$$

where the constant implied by $\ll$ *depends on* q *only.*

PROOF. We define a new lacunary sequence

$$n_1 q^{-H}, n_1 q^{-H+1}, \cdots, n_1 q^{-1}, n_1, n_2, \cdots .$$

We choose H so that the jth block H_j defined for this new sequence contains exactly the N elements corresponding to $n_1, n_2, \cdots, n_N$. Then $[j^{\frac{1}{2}}] = N$ and $H \sim 2N^3$. But then $w_j{}^*$ defined for this new sequence is just $\sum_{k \leqq N} x_k(s, t)$, the sum whose variance we are to estimate. Hence by Lemma 4.2.5

$$E(\textstyle\sum_{k \leqq N} x_k(s, t))^2 = Ew_j{}^{*2} = E(E(w_j{}^{*2} \,|\, \mathscr{F}^*_{h_j-1})) \ll lj^{\frac{1}{2}} \ll lN .$$

LEMMA 4.3.2. $\Gamma(s, t)$ *is continuous on the unit square.*

PROOF. Let $0 \leqq s < t_0 < t \leqq 1$. Then

$$\begin{aligned} N^{-1}E(\textstyle\sum_{n \leqq N} x_n(s, t))^2 &- N^{-1}E(\textstyle\sum_{n \leqq N} x_n(s, t_0))^2 \\ &\leqq N^{-1}E\{(\textstyle\sum x_n(s, t) - x_n(s, t_0))(\sum x_n(s, t) + \sum x_n(s, t_0))\} \\ &\leqq N^{-\frac{1}{2}}\|\textstyle\sum x_n(t_0, t)\|(N^{-\frac{1}{2}}\|\sum x_n(s, t)\| + N^{-\frac{1}{2}}\|\sum x_n(s, t_0)\|) \\ &\ll |t - t_0|^{\frac{1}{2}} \end{aligned}$$

by Lemma 4.3.1. In general, we obtain using the same argument

$$N^{-1}E(\textstyle\sum_{n \leqq N} x_n(s, t))^2 - N^{-1}E(\sum_{n \leqq N} x_n(s_0 \cdot t_0))^2 \ll |s - s_0|^{\frac{1}{2}} + |t - t_0|^{\frac{1}{2}}$$

where the constant implied by $\ll$ depends on q only. From (4.1.4) and (4.1.4)* respectively we conclude that

$$|\sigma^2(s, t) - \sigma^2(s_0, t_0)| \ll |s - s_0|^{\frac{1}{2}} + |t - t_0|^{\frac{1}{2}} .$$

The lemma follows now from (4.1.5).

5. Identification of the limits. In this section we prove Theorems 3.2 and 4.2 by verifying relation (2.2) with K being the kernel space $H(\Gamma)$ where $\Gamma(s, t)$ is the appropriate covariance function. For this purpose we require the following two theorems for sums of strongly mixing and lacunary sequences of random variables which are special cases of known results.

THEOREM 5.1. *Let* $\{\xi_n, n \geqq 1\}$ *be a strictly stationary sequence of random variables satisfying a strong mixing condition of the form* (3.1.1) *with*

(5.1) $$\rho(n) \ll n^{-3} .$$

Let f be as in Section 3.1 *and let η_n and η_{nm} be defined by* (3.1.2) *and* (3.1.3) *respectively. Suppose that the function f is bounded. Moreover, assume that f is such that the η_n's are centered at expectations and satisfy* (3.1.4) *with*

$$\phi(m) \ll m^{-5}\,. \tag{5.2}$$

Then

$$\sigma^2 = E\eta_1^2 + 2 \textstyle\sum_{n=2}^{\infty} E(\eta_1\eta_n)$$

is absolutely convergent. Moreover, if $\sigma^2 > 0$, then

$$\limsup_{N\to\infty} (2N\sigma^2 \log\log N)^{-\frac{1}{2}} \textstyle\sum_{n\leq N} \eta_n = 1 \quad \text{a.s.}$$

This follows from Theorem 2.3 of Reznik (1968).

We also need an almost sure invariance principle due to Berkes (1975) and Berkes and Philipp (1977). I quote only a special case.

THEOREM 5.2. *Let $\{n_k, k \geqq 1\}$ be a lacunary sequence of real numbers and let f be a measurable bounded function with period* 1 *and $\int_0^1 f(x)\,dx = 0$. Let s_n denote the* nth *partial sum of its Fourier series. Suppose that for some $\alpha > 0$ and $A > 0$*

$$\|f - s_n\|_2 \leqq An^{-\alpha} \qquad n = 1, 2, \cdots .$$

Suppose that either (4.1.4) *or that both* (4.1.4)* *and* (4.1.6) *hold. Define a continuous parameter process $\{S(t), t \geqq 0\}$ by setting*

$$S(t) = \textstyle\sum_{j\leq t} f(n_j\omega)\,.$$

Then, without changing its distribution, we can redefine the process $\{S(t), t \geqq 0\}$ on a richer probability space together with standard Brownian motion $\{X(t), t \geqq 0\}$ such that

$$X(\tau_N) - S(N) \ll N^{\frac{1}{2}-\lambda} \quad \text{a.s.}$$

Here $\lambda > 0$ is an absolute constant and $\{\tau_N, N \geqq 1\}$ is an increasing sequence of positive random variables with

$$\lim_{N\to\infty} N^{-1}\tau_N = \sigma^2 \quad \text{a.s.}$$

REMARK. Theorem 5.2 implies that

$$\limsup_{N\to\infty} (2\sigma^2 N \log\log N)^{-\frac{1}{2}} S(N) = 1 \quad \text{a.s.}$$

A proof of this statement can be easily modeled after the proof of Strassen's (1964) Theorem 2 or after the proof of Theorem 13.26 in Breiman (1968).

5.1. *An almost sure analogue of the Cramér–Wold device.* Let a be an m-column vector with components a_i $(1 \leqq i \leqq m)$ and let $a' = (a_1, \cdots, a_m)$ be its transpose. We denote by ab the inner product ab' of the vectors a and b and by $|a|$ the length of a. The following lemma can be regarded as an almost sure analogue of the Cramér–Wold device. For its proof we use ideas of Finkelstein (1971).

LEMMA 5.1.1. *Let $\{v_n, n \geqq 1\}$ be a sequence of random vectors in $\mathbb{R}^m$. Suppose that for each vector $s \in \mathbb{R}^m$ with $|s| = 1$ we have*

$$\limsup_{n\to\infty} sv_n = 1 \quad \text{a.s.}$$

Then the sequence $\{v_n, n \geqq 1\}$ *is bounded almost surely and its set* V *of limit points satisfies*

$$\{x \in \mathbb{R}^m : |x| = 1\} \subset V \subset \{x \in \mathbb{R}^m : |x| \leqq 1\}.$$

PROOF. By choosing $s' = (1, 0, \cdots, 0), \cdots, (0, 0, \cdots, 1)$ we observe that each of the sequences $\{v_{nk}, n \geqq 1\}$ for $1 \leqq k \leqq m$ is bounded a.s. and so is $\{v_n, n \geqq 1\}$.

Since $\mathbb{R}^m$ is separable there is a set Ω_1 with $P(\Omega_1) = 1$ such that for each $\omega \in \Omega_1$

$$\limsup_{n\to\infty} sv_n(\omega) = 1 \tag{5.1.1}$$

for all $s \in \mathbb{R}^m$ with $|s| = 1$. Hence by Cauchy's inequality,

$$\limsup_{n\to\infty} |v_n(\omega)| \geqq 1 \tag{5.1.2}$$

for all $\omega \in \Omega_1$. Fix such an $\omega \in \Omega_1$. Suppose that for some subsequence $\{n_j, j \geqq 1\}$

$$|v_{n_j}(\omega)| \geqq a > 1 \qquad j \geqq j_0 . \tag{5.1.3}$$

The vectors $s_{n_j} = v_{n_j}/|v_{n_j}|$ have length 1 and satisfy

$$s_{n_j} v_{n_j} = |v_{n_j}(\omega)| \geqq a \tag{5.1.4}$$

for all $j \geqq j_0$. But $\{s_{n_j}, j \geqq 1\}$ has a limit point s with $|s| = 1$. Thus by (5.2.4)

$$\limsup_{j\to\infty} sv_{n_j}(\omega) \geqq a$$

in violation of (5.1.1). Hence (5.1.3) cannot hold and thus by (5.1.2)

$$\limsup_{n\to\infty} |v_n| = 1 \quad \text{a.s.} \tag{5.1.5}$$

Consequently,

$$V \subset \{x \in \mathbb{R}^m : |x| \leqq 1\}.$$

We finish the proof of the lemma by showing that

$$V \supset \{x \in \mathbb{R}^m : |x| = 1\}.$$

Let $\varepsilon > 0$ and let $|x| \in \mathbb{R}^m$ with $|x| = 1$. Fix $\omega \in \Omega_1$. By hypothesis and by (5.1.5) there is a subsequence $\{n_j, j \geqq 1\}$ such that for all $j \geqq 1$

$$xv_{n_j}(\omega) > 1 - \varepsilon$$

and

$$|v_{n_j}(\omega)| < 1 + \varepsilon .$$

Hence

$$|x - v_{n_j}|^2 = |x|^2 + |v_{n_j}|^2 - 2xv_{n_j} < 5\varepsilon \qquad j \geqq 1 .$$

5.2. *Proof of Theorems* 3.2 *and* 4.2. Let $\{\eta_k, k \geqq 1\}$ satisfy the hypotheses of either Theorems 3.2 or 4.2. Let $T = \{t_1, \cdots, t_m\}$ be a set of m points t_j with $0 < t_j < 1$. Define a sequence of random vectors $y_k = (y_{k1}, \cdots, y_{km})$ by setting

$$y_{kj} = x_k(0, t_j) \qquad 1 \leqq j \leqq m . \tag{5.2.1}$$

Denote the $m \times m$ matrix $((\Gamma(t_i, t_j)))_{i,j=1}^m$ by Γ_m so that by Lemma 3.2.4 and (4.1.5)

$$\Gamma_m = \lim_{N\to\infty} N^{-1} \sum_{l,k \leqq N} E(y_k y_l') . \tag{5.2.2}$$

Put

$$z_N = \frac{\sum_{k \leqq N} y_k}{(2N \log\log N)^{\frac{1}{2}}}. \tag{5.2.3}$$

PROPOSITION 5.2.1. *The sequence $\{z_N, N \geqq 3\}$ is bounded almost surely and has the ellipsoid $E_m = \{x \in \mathbb{R}^m : x'\Gamma_m^{-1}x \leqq 1\}$ as its set of limit points. Moreover, E_m is the unit ball in the kernel space $H(\Gamma_m)$ with kernel Γ_m.*

PROOF. Since by hypothesis Γ is positive definite all eigenvalues of Γ_m are positive. We write

$$\Gamma_m = U\Delta U^{-1}$$

where U is unitary and Δ is diagonal.

We define $\Delta^{-\frac{1}{2}}$ in the obvious way. Put

$$\begin{aligned} u_k &= \Delta^{-\frac{1}{2}}U^{-1}y_k, \quad k \geqq 1 \\ v_N &= \Delta^{-\frac{1}{2}}U^{-1}z_N, \quad N \geqq 1. \end{aligned} \tag{5.2.4}$$

Our goal is to apply Lemma 5.2.1 to $\{v_N, N \geqq 3\}$.

We first observe that

$$Eu_k = \Delta^{-\frac{1}{2}}U^{-1}Ey_k = 0. \tag{5.2.5}$$

Moreover, by (5.2.2)

$$\begin{aligned} \lim_{N\to\infty} \frac{1}{N} \sum_{k,l \leqq N} E(u_k u_l') &= \lim_{N\to\infty} \frac{1}{N} \sum_{k,l \leqq N} \Delta^{-\frac{1}{2}}U^{-1}E(y_k y_l')U\Delta^{-\frac{1}{2}} \\ &= \Delta^{-\frac{1}{2}}U^{-1}\Gamma_m U\Delta^{-\frac{1}{2}} = I_m, \end{aligned} \tag{5.2.6}$$

the identity matrix of order m. Now for each vector $s \in \mathbb{R}^m$ with $|s| = 1$

$$E(s \cdot u_k) = sEu_k = 0 \tag{5.2.7}$$

by (5.2.5) and

$$\begin{aligned} \lim \frac{1}{N} E\{\textstyle\sum_{k \leqq N} su_k\}^2 &= \lim \frac{1}{N} \sum_{k,l \leqq N} E((su_k) \cdot (su_l)) \\ &= \lim \frac{1}{N} \sum_{k,l \leqq N} E(\textstyle\sum_{i,j \leqq m} s_i u_{ki} s_j u_{lj}) \\ &= \sum_{i,j \leqq m} s_i s_j \lim \frac{1}{N} \sum_{k,l \leqq N} E(u_{ki} u_{lj}) \\ &= \sum_{j \leqq m} s_j^2 = 1 \end{aligned} \tag{5.2.8}$$

by (5.2.6). We observe that

$$su_k = s\Delta^{-\frac{1}{2}}U^{-1}y_k = F(\eta_k),$$

where F is the step function

$$F(x) = \sum_{j \leqq m} \alpha_j (1\{0 \leqq x \leqq t_j\} - t_j) \qquad 0 \leqq x \leqq 1 \tag{5.2.9}$$

extended with period 1. By (5.3.7) and (5.3.8),

$$E(F(\eta_k)) = 0, \qquad \sigma^2 = 1.$$

We first check that in the mixing case $\{F(\eta_k)\}$ satisfies the hypotheses of Theorem 5.1. Of course, (5.1) holds. Next, we show that $F(\eta_k) = F(f(\xi_k, \xi_{k+1}, \cdots))$ can be approximated by $F(\eta_{kl}) = F(f(\xi_k, \xi_{k+1}, \cdots, \xi_{k+l-1}))$. We obtain using (5.2.9), (2.4), (3.2.3) and (3.2.4)

$$E|F(\eta_k) - F(\eta_{kl})| \leqq \sum_{j\leqq m} |\alpha_j| E|x_n - x_{kl}| \ll l^{-6} .$$

Hence by Jensen's inequality for conditional expectations (see Billingsley (1968), page 183) Theorem 5.1 applies in the mixing case.

In the lacunary case the hypotheses of Theorem 5.2 are satisfied because of (5.2.8) and since we assumed (4.1.4) or (4.1.4)* and (4.1.6) respectively.

Because of (5.2.3) and (5.2.4) we can write

$$sv_N = (2N \log\log N)^{-\frac{1}{2}} \sum_{k\leqq N} su_k = (2N \log\log N)^{-\frac{1}{2}} \sum_{k\leqq N} F(\eta_k) .$$

By Theorems 5.1 and 5.2 the hypothesis of Lemma 5.1.1 is satisfied and thus the sequence $\{v_N, N \geqq 3\}$ is bounded almost surely and its set of limit points contains the unit sphere and is contained in the unit ball. But then by (5.2.4) $\{z_N, N \geqq 3\}$ is bounded almost surely and its set V_m of limit points is contained in

$$\begin{aligned}\{x \in \mathbb{R}^m : |\Delta^{-\frac{1}{2}} U^{-1} x| \leqq 1\} &= \{x \in \mathbb{R}^m : x' U \Delta^{-\frac{1}{2}} \Delta^{-\frac{1}{2}} U^{-1} x \leqq 1\} \\ &= \{x \in \mathbb{R}^m : x' \Gamma_m^{-1} x \leqq 1\} = E_m\end{aligned}$$

i.e.,

$$V_m \subset E_m . \tag{5.2.10}$$

Similarly,

$$V_m \supset \partial E_m . \tag{5.2.11}$$

This holds for each $m \geqq 1$. Let $t_{m+1} \neq t_j$ $(1 \leqq j \leqq m)$ and let $T_{m+1} = \{t_1, \cdots, t_m, t_{m+1}\}$. Let π be the mapping from $\mathbb{R}^{m+1}$ onto $\mathbb{R}^m$ defined by $\pi(\alpha_1, \cdots, \alpha_m, \alpha_{m+1}) = (\alpha_1, \cdots, \alpha_m)$, for each $(\alpha_1, \cdots, \alpha_{m+1}) \in \mathbb{R}^{m+1}$. Then

$$\pi(z_{N1}, \cdots, z_{Nm}, z_{Nm+1}) = (z_{N1}, \cdots, z_{Nm})$$

and thus

$$\pi V_{m+1} = V_m . \tag{5.2.12}$$

We observe that

$$\pi E_{m+1} = \pi(\partial E_{m+1}) = \hat{E}_m$$

where $\hat{E}_m$ is also an ellipsoid. Thus by (5.2.10)—(5.2.12) applied to $\mathbb{R}^{m+1}$

$$V_m = \pi V_{m+1} \subset \pi E_{m+1} = \hat{E}_m = \pi(\partial E_{m+1}) \subset \pi V_{m+1} = V_m$$

or

$$\hat{E}_m = V_m .$$

Hence by (5.2.10) and (5.2.11)

$$\partial E_m \subset \hat{E}_m \subset E_m .$$

Consequently,

$$E_m = \hat{E}_m = V_m .$$

This proves the first half of the proposition. To prove the second half we observe that the unit ball of the kernel space $H(\Gamma_m)$ on $T = \{t_1, \cdots, t_m\}$ with kernel Γ_m consists of all functions f on T with

$$f(t) = \sum_{j \leq m} \alpha_j \Gamma(t, t_j) \quad t \in T \tag{5.2.13}$$

and

$$1 \geqq \|f\|_H^2 = \sum_{j,k \leq m} \alpha_j \alpha_k \Gamma(t_j, t_k) . \tag{5.2.14}$$

If we write $f = (f(t_1), \cdots, f(t_m))'$ and $\alpha = (\alpha_1, \cdots, \alpha_m)'$ then (5.2.13) and (5.2.14) can be rewritten as

$$f = \Gamma_m \alpha \qquad \text{or} \qquad \alpha = \Gamma_m^{-1} f$$

and

$$1 \geqq \|f\|_H^2 = \alpha' \Gamma_m \alpha = f' \Gamma_m^{-1} f .$$

This shows that E_m is the unit ball in $H(\Gamma_m)$. □

We shall now prove Theorems 3.2 and 4.2. Let f_N be defined by (3.1.5) and (4.1.2) respectively and let h_N be defined by (3.1.12). In view of (3.1.13) it is enough to check (2.1) and (2.2) for $(h_N, N \geqq 3)$. First, $\{h_N, N \geqq 3\}$ is with probability 1 relatively compact. This fact was proved in Section 3.1 to establish the relative compactness of $\{f_N, N \geqq 3\}$. Second, by (5.2.3), $z_N = f_N^T$ and by (5.2.2) $\Gamma_m = \Gamma^T$. Hence by Proposition 5.2.1, $\{f_N^T, N \geqq 3\}$ is bounded almost surely and has the unit ball in the kernel space $H(\Gamma)$ as its set of limit points. By (3.1.13) the same holds true for $\{h_N^T, N \geqq 3\}$. By Lemma 2.1 the unit ball of $H(\Gamma^T)$ is just the restriction of the unit ball of $H(\Gamma)$ to T. Since by Lemmas 3.2.4 and 4.3.2 $\Gamma(s, t)$ is continuous on the unit square the unit ball of $H(\Gamma)$ is compact in $C[0, 1]$ by Lemma 2.2. Hence (2.2) holds for $\{h_N^T, N \geqq 3\}$ with $K =$ unit ball of $H(\Gamma)$. This concludes the proofs of Theorems 3.2 and 4.2.

It remains to show that the unit ball B of $H(\Gamma)$ equals $\overline{\bigcup_{m \geq 1} B_m}$. We first observe that by Lemmas 3.2.3 and 4.3.1,

$$\Gamma(t, t) \ll t \ll 1 \tag{5.2.15}$$

uniformly in $0 \leqq t \leqq 1$. Let $f \in B$. Then given $\varepsilon > 0$ there exists $f^* \in H(\Gamma)$ with

$$\|f^*\|_H \leqq 1 - \tfrac{1}{2}\varepsilon \tag{5.2.16}$$

and

$$\|f - f^*\|_H < \varepsilon . \tag{5.2.17}$$

Simply put $f^* = (1 - \frac{1}{2}\varepsilon)f$. Moreover, since $\bigcup_{m \geq 1} T_m$ is dense in $[0, 1]$, and since by Lemmas 3.2.4 and 4.3.2 $\Gamma(s, t)$ is continuous on the unit square, it follows from the definition of $H(\Gamma)$ (see Section 2.3) that there exists a $g \in K_m = K_m(t_1, \cdots, t_m)$ for some m such that

$$\|f^* - g\|_H < \tfrac{1}{2}\varepsilon . \tag{5.2.18}$$

Since

$$\big| \|f^*\|_H - \|g\|_H \big| \leqq \|f^* - g\|_H < \tfrac{1}{2}\varepsilon$$

we have

$$||g||_H < ||f^*||_H + \tfrac{1}{2}\varepsilon \leqq 1$$

by (5.2.16). Thus $g \in B_m$ and

$$||f - g||_H < 2\varepsilon \tag{5.2.19}$$

by (5.2.17) and (5.2.18). By the reproducing kernel property of Γ, and since by (5.2.15) $\Gamma(t, t)$ is uniformly bounded on $[0, 1]$ by M^2, say, we conclude from (5.2.19) that

$$|f(t) - g(t)| = |(f - g, \Gamma(\cdot, t))|$$
$$\leqq ||f - g||_H \Gamma(t, t)^{\frac{1}{2}} < 2M\varepsilon$$

uniformly in $0 \leqq t \leqq 1$. Hence, $f \in \overline{\bigcup_{m\geqq 1} B_m}$ where the closure is in the topology defined by the supremum norm. This shows $B \subset \overline{\bigcup_{m\geqq 1} B_m}$.

The opposite inclusion follows from the definition of B_m, B, $H(\Gamma)$ and the fact that B is a closed subset of $C[0, 1]$ with uniform norm.

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