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POINTWISE CONVERGENCE THEOREMS FOR SELF-ADJOINT AND UNITARY CONTRACTIONS

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Some conditions are introduced which imply pointwise convergence theorems for increasing sequences of orthogonal projections on $L^2(\mu)$, μ finite, as well as a pointwise ergodic theorem for self-adjoint and unitary contractions. These results generalize to the case of nonpositive operators some theorems of E. M. Stein.

Let $(\Omega, \mathscr{A}, \mu)$ be a finite measure space and let $\{T_k\}_{k\geq 1}$ be a sequence of bounded operators on $L^p(\mu)$, $1 < p < \infty$. A well-known principle of Banach (cf. [3]) states that if $\sup_k |T_k f| < \infty$ a.e. for all $f \in L^p(\mu)$ then the set of f for which $\{T_k f\}$ converges a.e. is closed in $L^p(\mu)$. However, in applying the principle to problems in the theory of a.e. convergence one usually proves a maximal inequality of the form $\|Mf\|_p \leqq A_p \|f\|_p$ where $A_p < \infty$ and $Mf = \sup_k |T_k f|$ and thereby brings the geometry of the space $L^p(\mu)$ into play. The purpose of this note is to show that for many problems a slightly weaker maximal property is more useful and can be applied particularly well to nonpositive operators.

DEFINITION 1. We say that the sequence $\{T_k\}$ of bounded operators on $L^p(\mu)$, $1 \leqq p < \infty$, has the dual maximal property ($(DM)_p$ property) if there is a positive constant $C_p < \infty$ such that for all sequences $\{B_k\} \subseteqq \mathscr{A}$ of disjoint sets and for all positive integers $n \geqq 1$ the functions $g_n = \sum_{k=1}^n T_k^* I_{B_K}$ satisfy $\|g_n\|_q \leqq C_p$. Here I_B is the indicator function of $B \in \mathscr{A}$ and $T_k^* : L^q(\mu) \to L^q(\mu)$ is the adjoint operator of $T_k (1/p + 1/q = 1)$.

The link between $(DM)_p$ and the maximal function Mf is now established by the following

PROPOSITION. *Let $\{T_k\}$ be a sequence of bounded operators on $L^p(\mu)$, $1 \leqq p < \infty$. Then $\{T_k\}$ satisfies $(DM)_p$ if and only if* $\sup_{\|f\|_p \leqq 1} \int Mf < \infty$.

PROOF. Assume first that $\sup_{\|f\|_p \leqq 1} \int Mf < \infty$ and let $\{B_k\} \subseteqq \mathscr{A}$ be any sequence of disjoint sets. If $g_n = \sum_{k=1}^n T_k^* I_{B_k}$ and $f \in L^p(\mu)$ we have

$$|\int fg_n| = |\int f \sum_{k=1}^n T_k^* I_{B_k}| = |\sum_{k=1}^n \int_{B_k} T_k f| \leqq \sum_{k=1}^n \int_{B_k} |T_k f|$$
$$\leqq \sum_{k=1}^n \int_{B_k} Mf \leqq \int Mf \,.$$

Hence

$$\|g_n\|_q = \sup_{\|f\|_p \leqq 1} |\int fg_n| \leqq \sup_{\|f\|_p \leqq 1} \int Mf \equiv C_p$$

independent of $\{B_k\}$ and therefore $\{T_k\}$ has property $(DM)_p$. Conversely, suppose

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that $\{T_k\}$ has property $(DM)_p$, and let $f \in L^p(\mu)$. There exist sequences of disjoint sets $\{B_{k,n}\}_{k=1}^n$ such that $\int Mf = \sup_n \sum_{k=1}^n \int_{B_{k,n}} |T_k f|$. For $k \geqq 1$ let $A_k^+ = \{T_k f \geqq 0\}$, $A_k^- = \{T_k f < 0\}$ and set $B_{k,n}^+ = A_k^+ \cap B_{k,n}$, $B_{k,n}^- = A_k^- \cap B_{k,n}$. Then for each $n \geqq 1$,

$$\begin{aligned}\sum_{k=1}^n \int_{B_{k,n}} |T_k f| &= \sum_{k=1}^n \left(\int_{B_{k,n}^+} T_k f - \int_{B_{k,n}^-} T_k f\right)\\ &= \int f\left(\sum_{k=1}^n T_k^* I_{B_{k,n}^+}\right) - \int f\left(\sum_{k=1}^n T_k^* I_{B_{k,n}^-}\right)\\ &\leqq \|f\|_p \|g_n^+\|_q + \|f\|_p \|g_n^-\|_q\end{aligned}$$

where we have used Hölder's inequality with $g_n^\pm = \sum_{k=1}^n T_k^* I_{B_{k,n}^\pm}$. Now the sequences $\{B_{k,n}^+\}_{k=1}^n$ and $\{B_{k,n}^-\}_{k=1}^n$ consist of disjoint sets and therefore $\|g_n^\pm\|_q \leqq C_p$ for all n from property $(DM)_p$. It follows that $\int Mf \leqq 2C_p \|f\|_p$; hence $\sup_{\|f\|_p \leqq 1} \int Mf \leqq 2C_p < \infty$ and the proof is complete.

It follows now from Banach's principle that if $\{T_k\}$ has property $(DM)_p$ then the set of functions f for which $\{T_k f\}$ converges a.e. is closed in $L^p(\mu)$.

As an application of the above concept we consider sequences of operators $\{T_k\}$ on $L^2(\mu)$ which are uniformly bounded in norm, i.e., $\|T_k\|_2^2 \leqq M < \infty$ for all $k \geqq 1$, and we seek to verify $(DM)_2$ in some special cases. Using the fact that $\mu(\Omega) < \infty$, a simple calculation shows that the sequence $\{T_k\}$ satisfies $(DM)_2$ if and only if the following condition is satisfied:

(C) There is a positive constant $K < \infty$ such that for all sequences $\{B_k\} \subset \mathscr{A}$ of disjoint sets and for all $N \geqq 1$

$$\sum_{n=1}^N \int g_n T_{n+1}^* I_{B_{n+1}} = \sum_{n=1}^N \int_{B_{n+1}} T_{n+1} g_n \leqq K \qquad \text{where} \quad g_n = \sum_{k=1}^n T_k^* I_{B_k}.$$

We now consider classes of operators on $L^2(\mu)$ which have property $(DM)_2$.

DEFINITION 2.

(i) The sequence $\{T_k\}$ of bounded operators on $L^p(\mu)$, $1 \leqq p < \infty$, is said to be uniformly absolutely continuous (u.a.c.) if there is a nonnegative function $g \in L^1(\mu)$ such that $\int |T_k f| \leqq \int g |f|$ for all $f \in L^\infty(\mu)$, $k \geqq 1$.

(ii) A bounded operator T on $L^p(\mu)$, $1 \leqq p < \infty$, is said to be u.a.c. if the sequence $\{T^k\}_{k \geqq 1}$ is u.a.c.

Note that if $\{T_k\}$ is a sequence of operators on $L^p(\mu)$ such that the sequence $\{T_k^*\}$ is uniformly bounded on $L^\infty(\mu)$, i.e., $\|T_k^*\|_\infty \leqq M_1 < \infty$ for all $k \geqq 1$, then $\{T_k\}$ is u.a.c. Indeed, in this case

$$\begin{aligned}\int |T_k f| &= \int_{A_k^+} T_k f - \int_{A_k^-} T_k f = \int f T_k^*(I_{A_k^+} - I_{A_k^-})\\ &\leqq \int |f| \|T_k^*\|_\infty \leqq M_1 \int |f|\end{aligned}$$

for all $f \in L^\infty(\mu)$, $k \geqq 1$. The same argument shows that if the operators $\{T_k\}$ are positive then $\{T_k\}$ is u.a.c. if $\sup_k T_k^* 1 \in L^1(\mu)$.

We now state and prove our main result.

THEOREM 1. *Let* $\{P_k\}$ *be an increasing sequence of orthogonal projections on* $L^2(\mu)$ *with limit* P. *If the sequence* $\{P_k\}$ *is* u.a.c., *then* $\{P_k f\}$ *converges* a.e. *to* Pf *for all* $f \in L^2(\mu)$.

PROOF. Set $T_k = P - P_k$, $k \geqq 1$. Then $\{T_k\}$ is a decreasing sequence of orthogonal projections converging to zero, and to prove a.e. convergence it suffices from Banach's principle to verify condition (C) for $\{T_k\}$. Note first that

$$\begin{aligned} T_{n+1}g_n &= T_{n+1} \textstyle\sum_{k=1}^n T_k I_{B_k} = \sum_{k=1}^n T_{n+1} T_k I_{B_k} = \sum_{k=1}^n T_{n+1} I_{B_k} \\ &= T_{n+1} \textstyle\sum_{k=1}^n I_{B_k} = T_{n+1} I_{A_n} \end{aligned}$$

since $T_{n+1}T_k = T_{n+1}$ for $k \leqq n+1$ and we have put $A_n = \sum_{k=1}^n B_k$. Hence $\sum_{n=1}^N \int_{B_{n+1}} T_{n+1} g_n = \sum_{n=1}^N \int_{B_{n+1}} T_{n+1} I_{A_n}$ and therefore to verify condition (C) it suffices to prove that there is a positive constant $K < \infty$ such that

$$\textstyle\sum_{n=1}^N \int_{B_{n+1}} T_{n+1} I_{A_n} = \sum_{n=1}^N \int_{B_{n+1}} P I_{A_n} - \sum_{n=1}^N \int_{B_{n+1}} P_{n+1} I_{A_n} \leqq K .$$

Now if the sequence $\{P_k\}$ is u.a.c. then

$$\begin{aligned} \left|\textstyle\sum_{n=1}^N \int_{B_{n+1}} P_{n+1} I_{A_n}\right| &\leqq \textstyle\sum_{n=1}^N \left|\int_{A_n} P_{n+1} I_{B_{n+1}}\right| \leqq \sum_{n=1}^N \int |P_{n+1} I_{B_{n+1}}| \\ &\leqq \textstyle\sum_{n=1}^N \int_{B_{n+1}} g \leqq \int g . \end{aligned}$$

Moreover, $\int |P_n I_B| \to_n \int |P I_B|$ for each $B \in \mathscr{A}$ and therefore from $\int |P_n I_B| \leqq \int_B g$ for all $n \geqq 1$ it follows that $\int |P I_B| \leqq \int_B g$ for all $B \in \mathscr{A}$. Hence $\sum_{n=1}^N \int |P I_{B_{n+1}}| \leqq \int g$ so that $|\sum_{n=1}^N \int_{B_{n+1}} T_{n+1} I_{A_n}| \leqq 2 \int g \equiv K$ and the proof is complete.

REMARK. Call an increasing sequence of orthogonal projections $\{P_k\}$ quasi-positive if $\inf_k \inf_{A \supseteq B; \mu(B)>0} 1/\mu(B) \int_B P_k I_A > -\infty$. Clearly if the operators P_k are positive then the sequence $\{P_k\}$ is quasi-positive. A slight modification of the above proof shows that if the sequence $\{P_k\}$ is quasi-positive, then $\{P_k f\}$ converges a.e. for each $f \in L^2(\mu)$.

As an application of Theorem 1 we consider an orthonormal sequence of functions $\{\phi_k\}_{k \geqq 1}$ on $(\Omega, \mathscr{A}, \mu)$ and we set $P_n f = \sum_{k=1}^n \alpha_k(f)\phi_k$ for $n \geqq 1$ where $\alpha_k(f) = \int f\phi_k$ is the kth Fourier coefficient of f. Then $\{P_k\}$ is an increasing sequence of orthogonal projections on $L^2(\mu)$ and we can write for $f \in L^2(\mu)$

$$P_n f(x) = \int f(t)\left(\textstyle\sum_{k=1}^n \phi_k(x)\phi_k(t)\right) d\mu(t) \qquad \text{so that}$$
$$|P_n f(x)| \leqq \int |f(t)| F_n(x, t)\, d\mu(t) \qquad \text{where} \quad F_n(x, t) = \left|\textstyle\sum_{k=1}^n \phi_k(x)\phi_k(t)\right| .$$

If $f \in L^\infty(\mu)$ we have

$$\int |P_n f(x)|\, d\mu(x) \leqq \int |f(t)| L_n(t)\, d\mu(t)$$

where $L_n(t) = \int F_n(x, t)\, d\mu(x)$ is the nth Lebesgue function for the sequence $\{\phi_k\}$. It follows that if $g(t) \equiv \sup_n L_n(t) \in L^1(\mu)$ then $\int |P_n f| \leqq \int |f| g$ for all $f \in L^\infty(\mu)$ and therefore $\{P_n\}$ is u.a.c. so that $\{P_n f\}$ converges a.e. for all $f \in L^2(\mu)$. This is a result of S. Kaczmarz [4]. The real content of Theorem 1 is that the result remains true for arbitrary increasing sequences $\{P_n\}$ of orthogonal projections which are u.a.c.

We turn now to a study of the pointwise ergodic theorem for operators on $L^2(\mu)$. More precisely, let T be an $L^2(\mu)$-contraction, i.e., $\int (Tf)^2 \leqq \int f^2$ for all $f \in L^2(\mu)$, and set $S_k f = 1/k \sum_{j=0}^{k-1} T^j f$ for $k \geqq 1$, $f \in L^2(\mu)$. We are concerned here with conditions on the operator T which imply the a.e. convergence of the sequence $\{S_k f\}$ for each $f \in L^2(\mu)$. If T is positive then a.e. convergence holds as has recently been shown by M. A. Akcoglu [1]. The case where T is positive and self-adjoint or unitary was proven earlier by E. M. Stein [5, page 82]. However, for arbitrary T a.e. convergence may fail, even if T is self-adjoint [2]. Note that if T is positive and self-adjoint then T is u.a.c. Indeed, in this case [5, page 82], $\sup_k T^k 1 \in L^2(\mu)$ and T satisfies the stronger condition $\int |T^k f| \leqq \int |f| g$ for all $f \in L^2(\mu)$, $k \geqq 0$, with $g = \sup_k T^k 1$. Also, if T is self-adjoint and $\sup_k \|T^k\|_\infty < \infty$ then T is u.a.c.

THEOREM 2. *Let T be an $L^2(\mu)$-contraction. Suppose that either* (a) *T is self-adjoint and* u.a.c., *or* (b) *T is unitary and $T^* = T^{-1}$ is* u.a.c. *Then $\{S_k f\}$ converges* a.e. *for all $f \in L^2(\mu)$.*

PROOF. It is well known that the sequence $\{S_k f\}$ converges a.e. for a dense set of $f \in L^2(\mu)$ and hence it suffices to again verify condition (C) for the sequence $\{S_k\}$. Assume first that T is self-adjoint and u.a.c. and let $\{B_k\}$ be a sequence of disjoint sets in $\mathscr{A}$. If $g_n = \sum_{k=1}^n S_k(I_{B_k})$, then

$$\int g_n S_{n+1} I_{B_{n+1}} = \sum_{k=1}^n \int S_k(I_{B_k}) S_{n+1}(I_{B_{n+1}}) = \sum_{k=1}^n \int_{B_k} S_k S_{n+1} I_{B_{n+1}}$$
$$\leqq \int \max_{1 \leqq k \leqq n} |S_k S_{n+1} I_{B_{n+1}}| \,.$$

But

$$S_k S_{n+1} I_{B_{n+1}} = \frac{1}{k(n+1)} \sum_{i=0}^{k-1} T^i \sum_{j=0}^{n} T^j I_{B_{n+1}} = \frac{1}{k(n+1)} \sum_{r=0}^{k+n-1} A_r T^r I_{B_{n+1}}$$

where A_r is the number of ways of writing $r = i + j$ with $0 \leqq i \leqq k-1$ and $0 \leqq j \leqq n$. Since $0 \leqq A_r \leqq k$ for $k \leqq n$ it follows that

$$|S_k S_{n+1} I_{B_{n+1}}| \leqq \frac{1}{k(n+1)} \sum_{r=0}^{k+n-1} A_r |T^r I_{B_{n+1}}| \leqq \frac{1}{(n+1)} \sum_{r=0}^{k+n-1} |T^r I_{B_{n+1}}|$$
$$\leqq \frac{1}{(n+1)} \sum_{r=0}^{2n} |T^r I_{B_{n+1}}| \equiv F_n(I_{B_{n+1}}) \qquad \text{if} \quad k \leqq n \,.$$

Hence

$$\max_{1 \leqq k \leqq n} |S_k S_{n+1} I_{B_{n+1}}| \leqq F_n(I_{B_{n+1}}) \qquad \text{so that}$$
$$\int \max_{1 \leqq k \leqq n} |S_k S_{n+1} I_{B_{n+1}}| \leqq \int F_n(I_{B_{n+1}}) \,.$$

Now

$$\int F_n(I_{B_{n+1}}) = \frac{1}{n+1} \sum_{r=0}^{2n} \int |T^r I_{B_{n+1}}| \leqq \frac{1}{n+1} \sum_{r=0}^{2n} \int_{B_{n+1}} g \leqq 2 \int_{B_{n+1}} g$$

and therefore

$$\sum_{k=1}^N \int g_n S_{n+1} I_{B_{n+1}} \leqq \sum_{k=1}^N \int F_n(I_{B_{n+1}}) \leqq \sum_{k=1}^N 2 \int_{B_{n+1}} g \leqq 2 \int g \equiv K$$

independent of $\{B_k\}$ and N and condition (C) is verified. The case where T is unitary is done in essentially the same way using the fact that $S_k S_{n+1}^* = S_k S_{n+1} T^{-n}$ and we omit the details.

We remark finally that both Theorem 1 and Theorem 2 are consequences of the following fact, whose proof is essentially contained in the proof given above. If $\{T_k\}$ is an arbitrary sequence of operators on $L^2(\mu)$, uniformly bounded in norm, then $\{T_k\}$ has property $(DM)_2$ if there is nonnegative function $g \in L^1(\mu)$ such that $\int \max_{1 \leqq k \leqq n} |T_k T^*_{n+1} I_A| \leqq \int_A g$ for all $A \in \mathscr{A}$, $n \geqq 1$.

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