The Annals of Probability
1977, Vol. 5, No. 4, 571–572

AN INDEPENDENCE IN BROWNIAN MOTION WITH CONSTANT DRIFT

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For Brownian motion with constant drift, when and where the first exit from $(-b, b)$ occur are independent random variables.

Let $X(t)$, $t \geqq 0$ be Brownian motion with constant drift $c > 0$, variance parameter 1 and $X(0) = 0$. Suppose $a < 0 < b$ are given constants. Define the first time the process hits a as $T_a = \inf\{t : X(t) \leqq a\}$ if $X(t) \leqq a$ for some t and as ∞ if $X(t) > a$ for all t, and the first time the process hits b as $T_b = \inf\{t : X(t) \geqq b\}$ if $X(t) \geqq b$ for some t and as ∞ if $X(t) < b$ for all t. Let $T = \min(T_a, T_b)$ be the first time the process leaves the interval (a, b).

THEOREM. $E(\exp[-\alpha T_b] | X(T) = b) = E(\exp[-\alpha T_{-b}] | X(T) = -b)$

PROOF. The Laplace transform for T_b is known [2, page 362] and given by

$$E(\exp[-\alpha T_b]) = \exp[-b((c^2 + 2\alpha)^{\frac{1}{2}} - c)] \tag{1}$$

while the Laplace transform of T_a is given by

$$E(\exp[-\alpha T_a], T_a < \infty) = \exp[a((c^2 + 2\alpha)^{\frac{1}{2}} + c)]\,. \tag{2}$$

By decomposing according to whether the first hit occurs at b or at a and using the strong Markov property, we get [1, pages 29–30].

$$E(\exp[-\alpha T_b]) = E(\exp[-\alpha T_b], X(T) = b) + E(\exp[-\alpha T_a], X(T) = a) \cdot E(\exp[-\alpha T_{b-a}])\,, \tag{3}$$

$$E(\exp[-\alpha T_a], T_a < \infty) = E(\exp[-\alpha T_a], X(T) = a) + E(\exp[-\alpha T_b], X(T) = b) \cdot E(\exp[-\alpha T_{a-b}], T_{a-b} < \infty)\,. \tag{4}$$

Using the results in expression (1) and (2) the solution to these two equations is

$$E(\exp[-\alpha T_b], X(T) = b) = e^{bc} \frac{\sinh[-a(c^2 + 2\alpha)^{\frac{1}{2}}]}{\sinh[(b - a)(c^2 + 2\alpha)^{\frac{1}{2}}]} \tag{5}$$

$$E(\exp[-\alpha T_a], X(T) = a) = e^{ac} \frac{\sinh[-b(c^2 + 2\alpha)^{\frac{1}{2}}]}{\sinh[(a - b)(c^2 + 2\alpha)^{\frac{1}{2}}]}\,. \tag{6}$$

Set $a = -b$ in (5) and (6). For $\alpha = 0$, they give

$$P(X(T) = b) = e^{bc} \sinh[bc]/\sinh[2bc] \tag{7}$$

$$P(X(T) = -b) = e^{-bc} \sinh[-bc]/\sinh[-2bc]\,. \tag{8}$$

Combining (5), (6), (7) and (8) completes the proof.

Received August 13, 1976; revised October 6, 1976.

AMS 1970 *subject classifications.* Primary 60J65; Secondary 60G40.

Key words and phrases. Brownian motion with constant drift, first exit times from intervals.

This theorem might have been derived using the methods of weak convergence and analogous results for the gambler's ruin problem [3], results which suggested the present one. In particular, if a simple random walk on $\{0, 1, 2, \cdots, 2z\}$ begins at the integer z with absorption at either end, the duration of the walk and the point of absorption are independent, whatever the fixed probability of a positive unit step.

Acknowledgment. The author thanks K. L. Chung, D. Iglehart, and the referee for their very helpful comments.

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