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A PROBLEM OF TWO LIFTS

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The following problem is solved: when are we obliged to walk upstairs in a building supplied with two lifts (or elevators), if repairs to the lifts are quickly carried out? Sufficient conditions are given on the distributions of operating- and down-times under which the times when both lifts are inoperative are asymptotically distributed as a Poisson process.

1. The problem. During his visit to Budapest University in October, 1974, A. D. Solovyev referred to the following problem in reliability (there is a reference to a special case of it in (3) of Section 6.2 of the English translation of [4]). A house is supplied by two lifts working independently of one another. The functioning of each lift forms an alternating renewal process with up- or working-time distribution F and down- or repair-time distribution G. Suppose that $G = G^\varepsilon$ depends on the parameter $\varepsilon > 0$ in such a way that the mean $\int_0^\infty x\, dG^\varepsilon(x) \to 0$ as $\varepsilon \to 0$. With both lifts starting in new condition at time 0, let $\tau = \tau^\varepsilon$ denote the time of the first shock of the system, i.e., the first moment after time 0 when both lifts are out of order. Determine the asymptotic distribution of τ^ε under suitable normalization for $\varepsilon \to 0$.

The reason for formulating the problem as above is that there is little hope of determining the exact distribution of τ under general assumptions on F and G. If either the distribution F or G is exponential then one can obtain the Laplace transform of the distribution of τ in an exact form (see [3] and [8]). In practice, repair-times are much shorter than working-times, so an asymptotic result may give a useful approximation. To complete the solution from a practical point of view it would then be necessary to estimate the speed of convergence to the asymptotic distribution but we shall not do so here.

What we do below is to solve the problem under sufficient conditions on F and G: our general result is in the theorem in Section 2, while a simpler version is given in the corollary there. Section 3 contains several lemmas which are used in proving the theorem in Section 4. The theorem we prove in fact answers more than is asked in Solovyev's question, for we give sufficient conditions under which the successive shocks of the system (epochs where one lift becomes out of order with the other lift already out of order) are asymptotically distributed as a Poisson process. This result is not surprising in itself, for we can regard the shocks as being obtained by thinning the sparse down-times of one alternating renewal process, and it is known that thinning of a point process under

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reasonable conditions of independence leads via appropriate normalization to a Poisson limit. Accordingly, the more difficult aspect of the problem is to formulate natural assumptions concerning F and G which will guarantee the existence of this Poisson limit.

Let F and G have means λ and $\varepsilon\mu$ respectively, and assume that F has a non-lattice distribution. Then it is not difficult to show (see (2.6) below) that the mean number of shocks in $(0, t] \sim 2\varepsilon\mu t/(\lambda + \varepsilon\mu)^2$ for large t, indicating that shocks are asymptotically like the points of a Poisson process at rate $2\varepsilon\mu/\lambda^2$. In fact we will prove that if in the process of shocks we take ε^{-1} as unit then the normalized process tends to a Poisson process of parameter $2\mu/\lambda^2$.

2. The theorem. In what follows, some parameters will always have the same meaning and the same range. In particular, t ($t \geqq 0$) means time, i means the label of the lift so $i = 1$ or 2, and ε ($0 < \varepsilon \leqq 1$) is a parameter on which the system depends and which will be supposed to tend to 0.

We agree to denote the point process determined by the random points $\pi_0 \leqq \pi_1 \leqq \cdots \leqq \pi_k \leqq \cdots$ by Π and further let

$$\Pi(S) = \operatorname{card}\{k : \pi_k \in S\}, \qquad \Pi_t = \Pi([0, t]) .$$

Finally denote the expectation measure of Π by $\tilde{\Pi}$, that is, $\tilde{\Pi}(S) = E\Pi(S)$ and $\tilde{\Pi}_t = E\Pi_t$.

Let $\xi_k(i)$ and $\eta_k^\varepsilon(i)$ be independent positive random variables ($k = 1, 2, \cdots$) such that

$$\mathscr{L}(\xi_k(i)) = F , \qquad \mathscr{L}(\eta_k^\varepsilon(i)) = G^\varepsilon \qquad k = 1, 2, \cdots .$$

Define $\zeta_k^\varepsilon(i) = \xi_k(i) + \eta_k^\varepsilon(i)$ and

$$\begin{aligned} \rho_t^\varepsilon(i) &= 1 \qquad \text{if } \textstyle\sum_{l=1}^{L-1} \zeta_l^\varepsilon(i) + \xi_L(i) \leqq t < \sum_{l=1}^{L} \zeta_l^\varepsilon(i) \quad \text{for some} \quad L \geqq 1 \\ &= 0 \qquad \text{otherwise,} \end{aligned}$$

and

$$\rho_t^\varepsilon = \rho_t^\varepsilon(1)\rho_t^\varepsilon(2) .$$

The first shock of the system occurs at the first moment when $\rho_t^\varepsilon = 1$ but we are to consider the whole process ρ_t^ε. Obviously the intervals when $\rho_t^\varepsilon = 1$ coincide with the "walk-up" periods.

We say that at time τ the system is *shocked* if $\rho_\tau^\varepsilon = 1$ but $\lim_{t\to\tau-0} \rho_t^\varepsilon = 0$. Denote by $T^\varepsilon = \{\tau_1^\varepsilon < \tau_2^\varepsilon < \cdots\}$ the sequence of shocks and by W^ε the point process $\{w_1^\varepsilon < w_2^\varepsilon < \cdots\}$ where $w_k^\varepsilon = \varepsilon\tau_k^\varepsilon$.

THEOREM. *Suppose that*

(i) *For some finite integer $n_0 \geqq 1$ the n_0th power of the characteristic function of F is integrable*;

(ii) $$-\textstyle\iint_{|x-y|<1} \log |x - y| \, dF(x) \, dF(y) < \infty ;$$

(iii) *For some $c_1, c_2 > 0$, $F(x) \leqq c_1 x$ whenever $0 \leqq x \leqq c_2$*;

(iv) $$\textstyle\int x \, dF(x) = \lambda \qquad 0 < \lambda < \infty ;$$

(v)

$$\int x\, dG^\varepsilon(x) = \varepsilon \qquad 0 < \varepsilon \leqq 1$$

and the family of measures is relatively compact compact in the weak topology (cf. [1]);

(vi) *There exists for every* ε $(0 < \varepsilon \leqq 1)$ *a distribution function* $N^\varepsilon(x)$ *such that*

$$\frac{1 - G^\varepsilon(x+y)}{1 - G^\varepsilon(y)} \leqq 1 - N^\varepsilon(x)\,, \tag{2.1}$$

whenever $G^\varepsilon(y) < 1$ *and* $\lim_{\varepsilon\to 0} \int x\, dN^\varepsilon(x) = 0$.

Then W^ε *tends to the Poisson process with parameter* $2\lambda^{-2}$, *as* $\varepsilon \to 0$, *in the sense of the weak convergence of finite dimensional distributions and of the weak convergence in space* $D[0, \infty)$ (cf. Lindwall [3]).

The inequality (2.1) in condition (vi) is essentially weaker than the usual condition of possessing increasing failure rate, though it is of similar type.

The conditions of the theorem become much simpler in the important special case $\eta^\varepsilon = \varepsilon\eta$.

COROLLARY. *If* F *satisfies conditions* (i)—(iv) *of the theorem and* G^ε *satisfies conditions*

(vii) $G^\varepsilon(x) = G(\varepsilon^{-1}x)$, *where* G *is a fixed distribution function and* $\int dG(x) = \mu$;

(viii)

$$\frac{1 - G(x+y)}{1 - G(y)} \leqq 1 - N(x)$$

whenever $G(y) < 1$ *with* N *a distribution function having finite expectation*;

then W^ε *tends to a Poisson process of parameter* $2\mu\lambda^{-2}$, *as* $\varepsilon \to 0$, *in the same sense as in the theorem.*

We say that a *shock* τ of the system is *of type* i if $\lim_{t\to\tau-0} \rho_t^\varepsilon(i) = 0$. Under our assumptions all the shocks will be of exactly one type with probability 1. Denote the point processes of shocks of types 1 and 2 by $U^\varepsilon = \{u_1^\varepsilon < u_2^\varepsilon < \cdots\}$ and $V^\varepsilon = \{v_1^\varepsilon < v_2^\varepsilon < \cdots\}$ respectively. Since $W_t^\varepsilon = U^\varepsilon_{\varepsilon^{-1}t} + V^\varepsilon_{\varepsilon^{-1}t}$ and U^ε and V^ε are identically distributed we have $\tilde{W}_t^\varepsilon = 2\tilde{U}^\varepsilon_{\varepsilon^{-1}t}$. Introduce the point processes $R^\varepsilon = \{\zeta_1^\varepsilon(1) + \cdots + \zeta^\varepsilon_{k-1}(1) + \xi_k^\varepsilon(1) : k \geqq 1\}$ and $H^\varepsilon = \{\zeta_1^\varepsilon(1) + \cdots + \zeta_k^\varepsilon(1) : k \geqq 1\}$ and denote $\tilde{\rho}_t^\varepsilon(i) = E\rho_t^\varepsilon(i)$. Finally define the random set B^ε by $B^\varepsilon = \{x : \rho_x^\varepsilon(2) = 1\}$, let $B_t^\varepsilon = B^\varepsilon \cap [0, t]$ and $\tilde{B}_t^\varepsilon = E|B_t^\varepsilon|$ where $|\ |$ denotes Lebesgue measure.

Our approach is based on the reduction of the proof of weak convergence to the verification of the convergence of certain unconditional and conditional expectations. For example, $\tilde{U}^\varepsilon$ will be easily tractable if we transform it as follows:

$$\begin{aligned}\tilde{U}_t^\varepsilon &= ER^\varepsilon(B_t^\varepsilon) = EE(R^\varepsilon(B_t^\varepsilon) \mid B_t^\varepsilon) = E\tilde{R}^\varepsilon(B_t^\varepsilon) = E\int_0^t \rho_x^\varepsilon(2)\, d\tilde{R}^\varepsilon(x) \\ &= \int_0^t \tilde{\rho}_x^\varepsilon(2)\, d\tilde{R}^\varepsilon(x)\,.\end{aligned} \tag{2.2}$$

It is easy to see that

(2.3) $$\tilde{p}_x^\varepsilon(2) = [(1 - G^\varepsilon) * F * \tilde{H}^\varepsilon](x)$$

and

(2.4) $$\tilde{R}^\varepsilon = F \times \tilde{H}^\varepsilon .$$

By applying the key renewal theorem to

$$1 - \tilde{p}_x^\varepsilon(2) = [1 - F) * \tilde{H}^\varepsilon](x)$$

we obtain

(2.5) $$\lim_{x\to\infty} \tilde{p}_x^\varepsilon(2) = \frac{\mu\varepsilon}{\lambda + \mu\varepsilon}$$

where we use the notations of the theorem. Now from (2.2), (2.3) and (2.4),

$$\tilde{U}_t^\varepsilon = \int_0^t \left(\int_0^{t-u} \tilde{p}_{u+x}^\varepsilon(2)\, dF(x)\right) d\tilde{H}^\varepsilon(u) ,$$

and if we use (2.5) then we obtain

(2.6) $$\tilde{U}_t^\varepsilon \sim \frac{\mu\varepsilon t}{(\lambda + \mu\varepsilon)^2}$$

which explains the parameter of the limit process which appears in the theorem.

This analysis already suggests that one might expect a similar result for non-identically distributed lifts. In fact if

$$E\xi_k(i) = \lambda_i \qquad \text{and} \qquad E\eta_k^\varepsilon(i) = \varepsilon\mu_i$$

then the assertion of the theorem will hold and the parameter of the limiting process will be $(\mu_1 + \mu_2)/\lambda_1\lambda_2$.

3. The lemmata. As to notions in connection with point processes we refer to the elegant survey [2]. $\Rightarrow_d$ will denote the weak convergence of random vectors and also that of point processes in the sense of finite dimensional distributions.

LEMMA 1. *Suppose that* $\Pi^\varepsilon = \{\pi_0^\varepsilon \leqq \pi_1^\varepsilon \leqq \cdots \leqq \pi_k^\varepsilon \leqq \cdots\}$ *are point processes on* $R^+ = [0, \infty)$ $(\varepsilon \geqq 0)$. *Then* $\Pi^\varepsilon \Rightarrow_d \Pi^0$ $(\varepsilon \to 0)$ *if and only if for every* $k \geqq 1$

$$(\pi_0^\varepsilon, \cdots, \pi_k^\varepsilon) \Rightarrow_d (\pi_0^0, \cdots, \pi_k^0) \qquad \varepsilon \to 0 .$$

The lemma is obvious.

LEMMA 2. *If* $\Pi^\varepsilon \Rightarrow_d \Pi^0$ $(\varepsilon \to 0)$ *and* Π^0 *is simple* (i.e., *with probability* 1 *the process has only unit jumps*), *then*

$$\Pi_t^\varepsilon \Rightarrow_{D[0,\infty)} \Pi_t^0 .$$

The result has been proved by Jagers [5].

LEMMA 3. *If for every* $t \geqq 0$, $\sup_\varepsilon \tilde{\Pi}_t^\varepsilon < \infty$ *then the family* $\{\Pi^\varepsilon\}$ *is relatively compact in the sense that every sequence* Π^{ε_n} *from the family contains a subsequence* $\Pi^{\varepsilon_{n'}}$ *converging to a limit* Π^0 (*finite dimensional convergence for intervals of continuity of the limit process*).

PROOF. It is enough to prove the sequential compactness, i.e., the tightness (see [1]) of the random variables Π_t^ε for every t and then the proof can be completed by Cantor diagonalization. Tightness follows from the Markov inequality $P\{\Pi_t^\varepsilon \geqq k\} \leqq k^{-1}\tilde{\Pi}_t^\varepsilon$.

LEMMA 4. *There exists an $\varepsilon_0 > 0$ such that all measures $\tilde{R}^\varepsilon$ ($\varepsilon \leqq \varepsilon_0$) possess a decomposition $\tilde{R}^\varepsilon = \tilde{R}_1^\varepsilon + \tilde{R}_2^\varepsilon$, where*

(α) *the $\tilde{R}_1^\varepsilon$'s are finite and weakly compact;*

(β) *the $\tilde{R}_2^\varepsilon$'s are absolutely continuous with densities r_2^ε bounded uniformly in ε and*

$$\lim_{\varepsilon'\to 0, x'\to\infty} \sup_{\varepsilon\leqq\varepsilon'} \sup_{x\geqq x'} \left| r_2^\varepsilon(x) - \frac{1}{\lambda} \right| = 0 .$$

PROOF. In [7] we constructed an analogous decomposition $\tilde{H}^\varepsilon = \tilde{H}_1^\varepsilon + \tilde{H}_2^\varepsilon$ for the renewal measure $\tilde{H}_\varepsilon$. Since $\tilde{R}^\varepsilon = \tilde{H}^\varepsilon * F + F$ we can use the obtained decomposition to have $\tilde{R}^\varepsilon = (\tilde{H}_1^\varepsilon * F + F) + (\tilde{H}_2^\varepsilon * F) = \tilde{R}_1^\varepsilon + \tilde{R}_2^\varepsilon$ which also possesses the desired properties.

LEMMA 5. *Let*

$$\sigma(z) = \int_0^\infty e^{-zx} s(x)\, dx \qquad \operatorname{Re} z > 0$$

and

$$\tau(z) = \int_0^\infty e^{-zx}\, dT(x) \qquad \operatorname{Re} z > 0$$

where $s(x)$ is a continuous bounded function on $[0, \infty)$ and T is a measure on $[0, \infty)$ such that the integral defining $\tau(z)$ exists for Re $z > 0$. *Then the Parseval-formula*

$$\int_0^\infty e^{-zx} s(x)\, dT(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \sigma(w)\tau(z-w)\, dw \tag{3.1}$$

holds whenever Re $z > c > 0$ *and integral on the right-hand side exists.*

PROOF. By standard smoothing. Let

$$q_\alpha(x) = \left[\frac{1}{2\alpha}\left(1 - \frac{|2-x|}{2\alpha}\right)\right]^+$$

where $a^+ = \max\{a, 0\}$ and $0 < \alpha < 1$. Then

$$\hat{q}_\alpha(z) = \int_0^\infty e^{-zx} q_\alpha(x)\, dx = \left(\frac{sh\alpha z}{\alpha z}\right)^2 e^{-2z}$$

is uniformly bounded in α ($0 < \alpha < 1$) and z (Re $z > 0$) and it is also integrable on any vertical line Re $z = c$ ($c > 0$). Thus with $s_\alpha(x) = \int q_\alpha(x-y)s(y)\, dy$ and $t_\alpha(x) = \int q_\alpha(x-y)\, dT(y)$ we have

$$\int_0^\infty e^{-zx} s_\alpha(x) t_\beta(x)\, dx = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \sigma(w)\hat{q}_\alpha(w)\tau(z-w)\hat{q}_\beta(z-w)\, dw \tag{3.2}$$

from the inversion formula known for Laplace transforms (see [4]) for any $0 < c <$ Re z. The functions $s_\alpha(x)$ are uniformly bounded in α and x and for almost every x $\lim_{\alpha\to\infty} s_\alpha(x) = s(x)$. Thus we can first take the limit $\alpha \to 0$ in

(3.2) and then using the conditions about s we can also take limit $\beta \to 0$ to obtain (3.1).

LEMMA 6. *If condition* (ii) *of the theorem holds and* $\varphi(z) = \int_0^\infty e^{-zx}\, dF(x)$ *then for every* $c > 0$

$$\int_{c-i\infty}^{c+i\infty} \frac{|\varphi(z)|^2}{|z|} dz < \infty . \tag{3.3}$$

PROOF. Consider the absolutely convergent integral

$$\begin{aligned}\int_{-\omega}^{\omega} (c^2 + t^2)^{-\frac{1}{2}} \int_0^\infty \int_0^\infty e^{-c(x+y)} e^{-it(x-y)}\, dF(x)\, dF(y)\, dt \\ = \int_0^\infty \int_0^\infty e^{-c(x+y)} 2 \int_0^\omega (c^2 + t^2)^{-\frac{1}{2}} \cos[t(x-y)]\, dt\, dF(x)\, dF(y) .\end{aligned} \tag{3.4}$$

Now

$$\int_0^\omega (c^2 + t^2)^{-\frac{1}{2}} \cos[t(x-y)]\, dt = \int_0^{\omega|x-y|} [c^2(x-y)^2 + u^2]^{-\frac{1}{2}} \cos u\, du . \tag{3.5}$$

However

$$\int_0^{\pi/2} (a^2 + u^2)^{-\frac{1}{2}} \cos du \leqq \int_0^{\pi/2} (a^2 + u^2)^{-\frac{1}{2}}\, du = O\left(\log \frac{1}{a}\right) \qquad a > 0 \tag{3.6}$$

and by the Leibniz convergence criterion

$$\int_{\pi/2}^{\omega} (a^2 + u^2)^{-\frac{1}{2}} \cos u\, du = O(1) \tag{3.7}$$

uniformly in $\omega \geqq \pi/2$ and $0 \leqq a < 1$ and also

$$\int_0^\omega (a^2 + u^2)^{-\frac{1}{2}} \cos u\, du = O(1) \tag{3.8}$$

uniformly in $\omega \geqq 0$ and $a \geqq 1$. Thus if $c|x-y| \geqq 1$ then (3.5) is bounded by (3.8), and if $c|x-y| < 1$ then by (3.6) and (3.7) it is $O(-\log|x-y|)$, uniformly in N. If we apply these remarks to (3.4) then we obtain (3.3).

LEMMA 7. *For* $t \geqq 0$

$$\tilde{W}_t^\varepsilon \to 2\lambda^{-2} t \qquad \varepsilon \to 0 .$$

PROOF. From (2.2) we have

$$\tilde{U}^\varepsilon_{\varepsilon^{-1}t} = E \int_{B^\varepsilon_{\varepsilon^{-1}t}} d\tilde{R}^\varepsilon(x) .$$

Observe that for $\varepsilon \to 0$ the set $B^\varepsilon_{\varepsilon^{-1}t}$ moves out to infinity where the measure $\tilde{R}^\varepsilon$ is nearly stationary by Lemma 4. This observation leads to a further transformation which results in the inequality

$$\begin{aligned}|\tilde{U}^\varepsilon_{\varepsilon^{-1}t} - \lambda^{-2}t| \leqq E \int_{B^\varepsilon_\alpha} d\tilde{R}^\varepsilon(x) + \tilde{R}_1^\varepsilon[\alpha, \infty) \\ + E \int_{B^\varepsilon_{\varepsilon^{-1}t-\alpha}} |r_2^\varepsilon(x) - \lambda^{-1}|\, dx + \lambda^{-1}(\tilde{B}^\varepsilon_{\varepsilon^{-1}t} - \lambda^{-1}t) \\ + \lambda^{-1}\tilde{B}_\alpha^\varepsilon\end{aligned} \tag{3.9}$$

where α ($\leqq \varepsilon^{-1}$) will be fixed later.

We shall prove that for any fixed α

$$\lim_{\varepsilon\to 0} E \int_{B^\varepsilon_\alpha} d\tilde{R}^\varepsilon(x) = 0 , \tag{3.10}$$

and for every fixed t

$$\lim_{\varepsilon\to 0} \tilde{B}^{\varepsilon}_{\varepsilon^{-1}t} = \lambda^{-1}t \,. \tag{3.11}$$

Suppose we have done it. Then all five terms on the right-hand side of (3.9) can be made arbitrarily small.

In fact, if α is large enough then the second term will be uniformly small ($<\gamma$, say) and moreover the integrand in the third term can also be made small ($<2\gamma\lambda^2 t^{-1}$, say). Then fix α and use (3.10) to see the negligibility of the first term and use (3.11) for estimating the third, fourth and fifth terms.

Now for the proof of (3.10). By (2.2)

$$E \int_{B^{\varepsilon}_{\alpha}} d\tilde{R}^{\varepsilon}(x) = \int_0^{\alpha} \tilde{\rho}_x{}^{\varepsilon}(2)\, d\tilde{R}^{\varepsilon}(x) \,.$$

Put

$$\varphi(z) = \int_0^{\infty} e^{-zx}\, dF(x)$$

$$\gamma^{\varepsilon}(z) = \int_0^{\infty} e^{-zx}\, dG^{\varepsilon}(x) \,.$$

An easy calculation yields that for Re $z > 0$

$$\int_0^{\infty} e^{-zx} \tilde{\rho}_x{}^{\varepsilon}(2)\, dx = \frac{1}{z} \cdot \frac{\varphi(1-\gamma^{\varepsilon})}{1-\varphi\gamma^{\varepsilon}}\bigg|_z \tag{3.12}$$

and

$$\int_0^{\infty} e^{-zx}\, d\tilde{R}^{\varepsilon}(x) = \frac{\varphi}{1-\varphi\gamma^{\varepsilon}}\bigg|_z \,. \tag{3.13}$$

Then by Lemma 5 for $0 < c < \text{Re}\, z$

$$\int_0^{\infty} e^{-zx} \tilde{\rho}_x{}^{\varepsilon}(2)\, d\tilde{R}^{\varepsilon}(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{1}{w} \frac{\varphi(1-\gamma^{\varepsilon})}{1-\varphi\gamma^{\varepsilon}}\bigg|_w \frac{\varphi}{1-\varphi\gamma^{\varepsilon}}\bigg|_{z-w} dw \tag{3.14}$$

where the integral on the right-hand side certainly converges by Lemma 6 if $c > 0$ and $z = 2c$. (3.10) will follow if we show that for some $z > 0$, for example, $z = 2c$, the expression (3.14) tends to 0 with $\varepsilon \to 0$. However, the integral on the right-hand side is dominated by $K|\varphi(z)|^2|z|^{-1}$, where K is independent of $z = c + it$ (c is fixed) and ε; thus by the Lebesgue dominated convergence theorem the integral tends to 0, as $\varepsilon \to 0$.

To prove (3.11) we note that

$$\tilde{B}_t{}^{\varepsilon} = \int_0^t \tilde{\rho}_x{}^{\varepsilon}(2)\, dx \,.$$

Consequently by (3.12) we have

$$\int_0^{\infty} e^{-zx}\, \tilde{B}^{\varepsilon}_{\varepsilon^{-1}x}\, dx = \frac{1}{\varepsilon z^2} \cdot \frac{\varphi(1-\gamma^{\varepsilon})}{1-\varphi\gamma^{\varepsilon}}\bigg|_{\varepsilon z} \,. \tag{3.15}$$

Suppose for a moment that

$$\int_0^{\infty} x^2\, dG^{\varepsilon}(x) = o(1) \tag{3.16}$$

if $\varepsilon \to 0$, and use it to obtain

$$1 - \gamma^{\varepsilon}(\varepsilon z) = \varepsilon^2 z + o(\varepsilon^2)$$

and

$$1 - \varphi(\varepsilon z)\gamma^{\varepsilon}(\varepsilon z) = \lambda\varepsilon z + O(\varepsilon^2) \,.$$

These relations help to obtain from (3.15) that

$$\lim_{\varepsilon\to\infty} \int_0^\infty e^{-zx}\tilde{B}^\varepsilon_{\varepsilon^{-1}x}\,dx = (\lambda z^2)^{-1}$$

and this is equivalent with (3.11). Finally, the proof of (3.16) follows from condition (vi) via

$$\begin{aligned}\int_0^\infty x^2\,dG^\varepsilon(x) &= \int_0^\infty x(1 - G^\varepsilon(x))\,dx \\ &\leqq \sum_{k=1}^\infty kh[1 - G^\varepsilon((k-1)h)]h \leqq h^2 + h^2(1 - G^\varepsilon(h))\,\frac{1 + N^\varepsilon(h)}{[N^\varepsilon(h)]^2}\,.\end{aligned}$$

In fact, if for some $\varepsilon_n \to 0$ and $0 < h_0 < 1$

$$\int_0^\infty x^2\,dG^{\varepsilon_n}(x) \geqq h_0$$

then the previous inequality applied to $h = h_0$ yields

$$\int_0^\infty x^2\,dG^{\varepsilon_n}(x) \leqq h_0^2 + h_0\varepsilon_n 2[N^{\varepsilon_n}(h_0)]^{-2} \to h_0^2\,,$$

a contradiction.

LEMMA 8. *For any* $t > 0$

$$E(W_t^\varepsilon \mid w_1^\varepsilon = x) \to 1 + 2\lambda^{-2}(t - x)$$

uniformly in $x \in [0, t]$.

This lemma and the next one as well are understood to be valid for a suitable version of conditional expectations.

PROOF. By the definitions and the symmetry of the processes U and V

$$\begin{aligned}E(W_t^\varepsilon \mid w_1^\varepsilon = x) &= E(W_t^\varepsilon \mid u_1^\varepsilon = \varepsilon^{-1}x,\, v_1^\varepsilon > \varepsilon^{-1}x) \\ &= E(U^\varepsilon_{\varepsilon^{-1}t} \mid u_1^\varepsilon = \varepsilon^{-1}x,\, v_1^\varepsilon > \varepsilon^{-1}x) \\ &\qquad + E(V^\varepsilon_{\varepsilon^{-1}t} \mid u_1^\varepsilon = \varepsilon^{-1}x,\, v_1^\varepsilon > \varepsilon^{-1}x)\,.\end{aligned}$$

We only prove that

$$E(U^\varepsilon_{\varepsilon^{-1}t} \mid u_1^\varepsilon = \varepsilon^{-1}x,\, v_1^\varepsilon > \varepsilon^{-1}x) \to 1 + \lambda^{-2}(t - x) \tag{3.17}$$

since the relation

$$E(V^\varepsilon_{\varepsilon^{-1}t} \mid u_1^\varepsilon = \varepsilon^{-1}x,\, v_1^\varepsilon > \varepsilon^{-1}x) \to \lambda^{-2}(t - x)$$

can be proved analogously. To prove (3.17) we can write

$$\begin{aligned}&E(U^\varepsilon_{\varepsilon^{-1}t} \mid u_1^\varepsilon = \varepsilon^{-1}x,\, v_1^\varepsilon > \varepsilon^{-1}x) \\ &\qquad = 1 + E\Big(\int_{B^\varepsilon\cap[\varepsilon^{-1}x,\varepsilon^{-1}t]} \tilde{H}^\varepsilon(du - \varepsilon^{-1}x) \,\Big|\, u_1^\varepsilon = \varepsilon^{-1}x,\, v_1^\varepsilon > \varepsilon^{-1}x\Big)\,.\end{aligned}$$

Now we make a similar decomposition to (3.9) and conclude that (3.17) will follow if we prove that for any $\alpha > 0$

$$\lim_{\varepsilon\to 0} E\Big(\int_{B^\varepsilon\cap[\varepsilon^{-1}x,\varepsilon^{-1}x+\alpha]} \tilde{H}^\varepsilon(du - \varepsilon^{-1}x) \,\Big|\, u_1^\varepsilon = \varepsilon^{-1}x,\, v_1^\varepsilon > \varepsilon^{-1}x\Big) = 0 \tag{3.18}$$

and for any $t > 0$

$$\lim_{\varepsilon\to 0} E(|B^\varepsilon \cap [\varepsilon^{-1}x, \varepsilon^{-1}t]| \mid u_1^\varepsilon = \varepsilon^{-1}x,\, v_1^\varepsilon > \varepsilon^{-1}x) = \lambda^{-1}(t - x) \tag{3.19}$$

uniformly in $x \in [0, t]$ (these relations correspond to (3.10) and (3.11)).

The proof of (3.18) is similar to that of (3.10) but here we have to use condition (iii). In fact, the difference is in the initial conditions since at $\varepsilon^{-1}x$ the first lift just goes to repair while the second one has already been in repair a period of κ_x^ε (say) and will remain there a period of length ψ_x^ε with some distribution function $\mathscr{J}_x^\varepsilon(y)$. From (vi) it follows that $\mathscr{J}_x^\varepsilon \Rightarrow_d \delta$, the degenerate distribution at 0, as $\varepsilon \to 0$, independently of the length of κ_x^ε, and thus uniformly in x. Let $\vartheta_x^\varepsilon(z) = \int_0^\infty e^{-zy}\, d\mathscr{J}_x^\varepsilon(y)$. It is easy to see that instead of (3.12) now we have

$$\frac{1-\vartheta_x^\varepsilon(z)}{z} + \frac{1}{z} \cdot \frac{\vartheta_x^\varepsilon \varphi(1-\gamma^\varepsilon)}{1-\varphi\gamma^\varepsilon}\bigg|_z$$

and instead of (3.13) we have

$$\frac{\varphi(1-\gamma^\varepsilon)}{1-\varphi\gamma^\varepsilon}\bigg|_z = \varphi(1-\gamma^\varepsilon)|_z + \frac{\varphi^2(1-\gamma^\varepsilon)\gamma^\varepsilon}{1-\varphi\gamma^\varepsilon}\bigg|_z .$$

We can apply our Parseval-formula in a similar way as we did in (3.14) except for the convolution of the first terms since it does not contain the integrable $|\varphi(z)|^2|z|^{-1}$. However, the convolution of the first terms can be estimated probabilistically, because it corresponds to a probability of the form

$$P(\xi + \eta^\varepsilon < \psi_x^\varepsilon) \leqq P(\xi < \psi_x^\varepsilon) = \int_0^\infty (1 - \mathscr{J}_x^\varepsilon(y))\, dF(y) \leqq \int_0^\infty (1 - N^\varepsilon(y))\, dF(y)$$
$$= \int_0^\infty F(y)\, dN^\varepsilon(y) .$$

In fact, by conditions (iii) and (vi)

$$\int_0^\infty F(y)\, dN^\varepsilon(y) \leqq c_1 \int_0^{c_2} y\, dN^\varepsilon(y) + [1 - N^\varepsilon(c_2)] = o(1)$$

and this completes the proof of (3.18).

To prove (3.19) observe first that the convergence (3.11) is uniform in any finite interval. This follows from the fact that the $\tilde{B}^\varepsilon_{\varepsilon^{-1}t}$'s are nondecreasing for every $\varepsilon > 0$ and that their limit is continuous. Moreover, the following estimate is valid:

$$|B^\varepsilon_{\varepsilon^{-1}t+\psi_x^\varepsilon} - B^\varepsilon_{\varepsilon^{-1}x+\psi_x^\varepsilon}| \leqq |B^\varepsilon_{\varepsilon^{-1}t} - B^\varepsilon_{\varepsilon^{-1}x}| \leqq |B^\varepsilon_{\varepsilon^{-1}t+\psi_x^\varepsilon} - B^\varepsilon_{\varepsilon^{-1}x+\psi_x^\varepsilon}| + \psi_x^\varepsilon .$$

Here $B^\varepsilon_{\varepsilon^{-1}t+\psi_x^\varepsilon} - B^\varepsilon_{\varepsilon^{-1}x+\psi_x^\varepsilon}$ has the same distribution as $B^\varepsilon_{\varepsilon^{-1}(t-x)}$ and thus the expectation of its measure tends to $\lambda^{-1}(t-x)$ uniformly in $x \in [0, t]$ by our previous remark. Moreover by condition (vi) $E\psi_x^\varepsilon \to 0$ as $\varepsilon \to 0$, uniformly in x, and thus the proof of (3.19) and of the lemma as well are complete.

We do not go into the proof of the final lemma because it does not require any new ideas.

LEMMA 9. *For every* $t \geqq 0$

$$\lim_{\varepsilon\to 0} E(W_t^\varepsilon - W_y^\varepsilon \mid w_k^\varepsilon = y) = 2\lambda^{-2}(t-y)$$

uniformly in $y \in [0, t]$ *and*

$$\lim_{\varepsilon\to 0} E(W_t^\varepsilon - W_y^\varepsilon \mid w_k^\varepsilon = y,\, w_{k+1}^\varepsilon = x) = 1 + 2\lambda^{-2}(t-x)$$

uniformly in $0 \leqq y \leqq x \leqq t$.

4. Proof of the theorem. From Lemmas 3 and 7 we conclude that the family $\{W^\varepsilon : \varepsilon > 0\}$ is relatively compact. We show that if $W^{\varepsilon_k} \Rightarrow_d \Pi^0$ for $\varepsilon_k \to 0$ ($k \to \infty$) then Π^0 is a Poisson process with parameter $2\lambda^{-2}$.

To obtain the distribution of $\pi_1{}^0$ we start from the identity

$$EW_t{}^{\varepsilon_k} = \int_0^t E(W_t{}^{\varepsilon_k} \mid w_1{}^{\varepsilon_k})\, dP(w_1{}^{\varepsilon_k} < x)\,.$$

By Lemmas 7 and 8 we can take the limit as $\varepsilon_k \to 0$ which results in

$$2\lambda^{-2}t = \int_0^t (1 + 2\lambda^{-2}(t - x))\, dP(\pi_1{}^0 < x)$$

whenever t is a continuity point of the distribution of $\pi_1{}^0$. The Laplace transform technique applied to this equation shows that $\pi_1{}^0$ has a proper probability distribution, namely an exponential distribution with parameter $2\lambda^{-2}$.

Fix y arbitrarily. If $t \geqq y$ then again

$$E(W_t{}^{\varepsilon_k} \mid w_1{}^{\varepsilon_k} = y) = \int_y^t E(W_t{}^{\varepsilon_k} \mid w_1{}^{\varepsilon_k} = y,\, w_2{}^{\varepsilon_k} = x)\, dP(w_2{}^{\varepsilon_k} < x \mid w_1{}^{\varepsilon_k} = y)\,.$$

Take the limit for $\varepsilon_k \to 0$. By Lemma 9 we obtain that

$$2\lambda^{-2}(t - y) = \int_y^t (1 + 2\lambda^{-2}(t - x))\, dP(\pi_2{}^0 < x \mid \pi_1{}^0 = y)$$

and this gives that $\pi_2{}^0 - \pi_1{}^0$ is independent of $\pi_1{}^0$ and has the same distribution.

By induction we can show that all the $\pi_k{}^0 - \pi_{k-1}^0$'s are independent, identically distributed. Thus $W^\varepsilon \Rightarrow_d \Pi^0$, a Poisson process with parameter $2\lambda^{-2}$. Since the Poisson process is simple, $W^\varepsilon \Rightarrow_{D[0,\infty]} \Pi^0$ is also true by Lemma 2. The theorem is proved.

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