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MARTINGALE REPRESENTATIONS AND HOLOMORPHIC PROCESSES

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We give a simple representation of two-parameter martingales in terms of a stochastic integral. This representation leads to the idea of the partial derivate of a martingale and to a generalization of the stochastic Green's theorem of the authors. Green's formula in this generalized form gives us a new and simpler proof of the fact that the derivative of a holomorphic process is holomorphic.

Let $\mathbf{R}_+^2$ be the positive quadrant of the plane, let $(\Omega, \mathscr{F}, P)$ be a complete probability space, and let $W = \{W_z, z \in \mathbf{R}_+^2\}$ be a separable Brownian sheet defined on $(\Omega, \mathscr{F}, P)$, that is, a Gaussian process of mean zero and covariance $E\{W_{uv} W_{st}\} = \min(u, s) \min(v, t)$. Being separable, W has continuous paths.

The two-dimensional parameter set gives a rich structure to the increasing families of σ-fields and there are a number of objects to which one could attach the name martingale. The most natural, perhaps, is this. Give $\mathbf{R}_+^2$ the partial order "$\prec$": $(s, t) \prec (u, v)$ iff $s \leqq u$, $t \leqq v$. For each $z \in \mathbf{R}_+^2$, let $\mathscr{F}_z$ be the σ-field generated by $\{W_\zeta, \zeta \prec z\}$ and the null sets of $\mathscr{F}$. A process $\{M_z, \mathscr{F}_z, z \in \mathbf{R}_+^2\}$ is a *martingale* if M_z is $\mathscr{F}_z$-measurable and $E\{|M_z|\} < \infty$ for all z, and if $z \prec z'$ implies $E\{M_{z'} \mid \mathscr{F}_z\} = M_z$. There are related processes, called 1- and 2-martingales, which will be important in this article. Roughly speaking, a process M is a 1-martingale if it is a martingale in s for each fixed t, and a 2-martingale if it is a martingale in t for each fixed s. More precisely, M is a 1-*martingale* if $\{M_{st}, \mathscr{F}_{s\infty}, s \in \mathbf{R}_+\}$ is a martingale for each t, and a 2-*martingale* if $\{M_{st}, \mathscr{F}_{\infty t}, t \in \mathbf{R}_+\}$ is a martingale for each s.

An i-martingale ($i = 1$ or 2) is said to be *adapted* if it is adapted to the fields $\{\mathscr{F}_z\}$. A process is a martingale iff it is both a 1-martingale and a 2-martingale. (This, by the way, is a consequence of the fact that the fields $\mathscr{F}_{s\infty}$ and $\mathscr{F}_{\infty t}$ are conditionally independent given $\mathscr{F}_{st}$.)

According to a theorem of Wong and Zakai ([4], see also [1]), any square-integrable martingale is the sum of a constant, a stochastic integral, and a double stochastic integral. We remarked in [1] that a double stochastic integral could be rewritten as a single stochastic integral of a weakly adapted integrand, that is, an integrand ϕ_{st} adapted either to $\{\mathscr{F}_{s\infty}\}$ or $\{\mathscr{F}_{\infty t}\}$ but not necessarily to $\{\mathscr{F}_{st}\}$. (We refer readers to [1] for the elementary properties of such integrals.)

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We follow up that remark here, to show that any i-martingale (and in particular, any martingale) can be written as an initial value plus the stochastic integral of a weakly adapted integrand. This representation seems relatively simple and useful, and it is exactly half as deep as that of Wong and Zakai (for, as we show in Section 2, the latter can be obtained directly by applying the representation for i-martingales twice). It leads, in particular, to Theorem 3.1, a natural extension of Green's formula ([1], Theorem 6.3) which relates line and surface integrals of i-martingales.

The main applications we will give of these basic results are to stochastic partial derivatives and holomorphic processes. In particular, Green's formula leads to a straightforward proof of the theorem which establishes that the derivative of a holomorphic process is holomorphic. This theorem was given in [1], but only for processes holomorphic in all of $\mathbf{R}_+^2$. We extend it here to processes holomorphic in arbitrary subdomains of $\mathbf{R}_+^2$.

Our notation is taken from [1]. In particular, if (s, t) and (u, v) are in $\mathbf{R}_+^2$, $(s, t) \prec (u, v)$ means that $s \leqq u$ and $t \leqq v$, $(s, t) \lll (u, v)$, that $s < u$ and $t < v$, whereas $(s, t) \curlywedge (u, v)$ is the complementary order: $s \leqq u$ and $t \geqq v$. We will also use the notation $(s, t) \curlywedge\!\!\curlywedge (u, v)$, which signifies $s < u$ and $t > v$.

If a, b are in $\mathbf{R}_+^2$, then $[a, b]$ and $[a, b)$ denote the rectangles $\{z : a \prec z \prec b\}$ and $\{z : a \prec z \lll b\}$ respectively, while R_a denotes $[0, a]$. Finally, $\mathscr{B}$ and $\mathscr{B}^2$ represent the Borel fields of $\mathbf{R}_+$ and $\mathbf{R}_+^2$ respectively.

1. Representations of 1- and 2-martingales. The basic observation of this section is the following variation on the theorem of Itô to the effect that any square-integrable functional of Brownian motion can be written as a constant plus a stochastic integral.

LEMMA 1.1. *Let X be a square-integrable $\mathscr{F}_{s_0\infty}$-measurable random variable. There exists a measurable process $\{\phi_z, z \in \mathbf{R}_+^2\}$ such that ϕ_{st} is $\mathscr{F}_{s_0t}$-measurable, $\phi_{st} = 0$ if $s > s_0$, and such that*

$$X = E\{X\} + \int_{\mathbf{R}_+^2} \phi_z \, dW_z \,. \tag{1.1}$$

NOTE. Here and below, equalities between random variables are to be interpreted as holding almost surely.

PROOF. Let $0 = u_1 < u_2 < \cdots < u_{n+1} = s_0$. Put $X_v^i = W_{u_{i+1}v} - W_{u_iv}$, and let $\mathscr{G}_v^i = \sigma\{X_{v'}^i, v' \leqq v\}$. Suppose Y^i is a square-integable, $\mathscr{G}_t^i$-measurable random variable of mean 0. By Itô's theorem [2], there exists a measurable $\mathscr{G}_v^i$-adapted process α^i such that

$$Y^i = \int_0^t \alpha_v^i \, dX_v^i \,, \qquad i = 1, \cdots, n \,.$$

For $v \leqq t$, let $Y_v^i = \int_0^v \alpha_{v'}^i \, dX_{v'}^i$, where we take a continuous version of the integral. By Itô's formula,

$$d(Y_v^1 \cdots Y_v^n) = (Y_v^2 \cdots Y_v^n) \, dY_v^1 + \cdots + (Y_v^1 \cdots Y_v^{n-1}) \, dY_v^n \,. \tag{1.2}$$

Let

$$\begin{aligned} \phi_{uv} &= Y_v^2 \cdots Y_v^n \alpha_v^1 && \text{if} \quad 0 \leqq u < u_2\,, \quad v \leqq t\,, \\ &\ \ \vdots \qquad \vdots \\ &= Y_v^1 \cdots Y_v^{n-1} \alpha_v^n && \text{if} \quad u_n \leqq u \leqq s_0\,, \quad v \leqq t\,, \\ &= 0 && \text{otherwise.} \end{aligned}$$

Then ϕ_{uv} is $\mathscr{F}_{s_0 v}$-measurable and $(u, v, \omega) \to \phi_{uv}(\omega)$ is $\mathscr{B}^2 \times \mathscr{F}$-measurable. Furthermore, $E\{\int_{R_{s_0\infty}} \phi_z^2\, dz\} < \infty$. Thus $\int_{R_{s_0\infty}} \phi_z\, dW_z$ is well defined—it is the integral of a weakly-adapted integrand—and (1.2) can be written in the form

$$Y^1 Y^2 \cdots Y^n = \int_{R_{s_0\infty}} \phi_z\, dW_z\,.$$

This proves (1.1) for X of the form $Y^1 \cdots Y^n$. Since sums of products of this form are dense in $L^2(\Omega, \mathscr{F}_{s_0\infty})$, this representation extends to general X in $L^2(\Omega, \mathscr{F}_{s_0\infty})$, and we are done.

Before giving the representation theorem for i-martingales we will address ourselves to a problem of measurability. To handle this, we will do a little more work than is absolutely necessary, and we will construct a family of orthonormal bases for all the Hilbert spaces $L^2(\Omega, \mathscr{F}_{s\infty})$, which vary measurably in s.

Let $\{r_n\}$ be an ordering of the rationals in $(0, \infty)$. For each n, let $\{X_{nj}, j = 1, 2, \cdots\}$ be a sequence of square-integrable mean zero $\mathscr{F}_{r_n\infty}$-measurable random variables which have the property that, together with 1, they span $L^2(\Omega, \mathscr{F}_{r_n\infty})$. By Lemma 1.1, for each n and $j \geqq 1$, there is a process $\{\phi_{nj}(z), z \in \mathbf{R}_+^2\}$ such that $\phi_{nj}(u, v)$ is $\mathscr{F}_{r_n v}$-measurable, $\phi_{nj}(u, v) = 0$ if $u > r_n$, and such that

$$X_{nj} = \int_{\mathbf{R}_+^2} \phi_{nj}(z)\, dW_z\,.$$

Order the X_{nj} in a single sequence, denoted $\{X_n\}$, and order the ϕ_{nj} in the same way into a sequence $\{\phi_n\}$. Define

$$\begin{aligned} N_1(s) &= \inf\{k : X_k = X_{nj} \ \text{ for some } \ n, j \ \text{ such that } \ r_n \leqq s\}\,, \\ N_{i+1}(s) &= \inf\{k > N_i(s) : X_k = X_{nj} \ \text{ for some } \ n, j \ \text{ such that } \ r_n \leqq s\}\,. \end{aligned}$$

The $N_i(s)$ are all finite for each $s > 0$, and if $r_n \leqq s$, the sequence $X_{N_1(s)}$, $X_{N_2(s)}, \cdots$ includes all the X_{nj}, so that, together with the function 1, they span $L^2(\Omega, \mathscr{F}_{s\infty})$. (We are using the fact that $\mathscr{F}_{s\infty} = \bigvee_{r_n \leqq s} \mathscr{F}_{r_n\infty}$ here.) For $s > 0$ and $n \geqq 1$, define

$$Y_n(s; \omega) = X_{N_n(s)}(\omega)$$

and

$$\phi_n(z; s; \omega) = \phi_{N_n(s)}(z; \omega)\,,$$

so that we have

$$Y_n(s) = \int_{\mathbf{R}_+^2} \phi_n(z; s)\, dW_z\,.$$

Now for each n and s, $Y_n(s)$ is $\mathscr{F}_{s\infty}$-measurable and $\phi_n(u, v; s)$ is $\mathscr{F}_{sv}$-measurable. Furthermore, $N_n(s)$ decreases as s increases, so it is certainly Borel measurable. Thus

$$\{(u;\omega): u \leqq s,\ Y_n(u;\omega) \leqq x\} = \bigcup_j \{u \leqq s : N_n(u) = j\} \times \{\omega : X_j(\omega) \leqq x\},$$

which is in $\mathscr{B} \times \mathscr{F}$, so that $Y_n(s)$ is a measurable process. A similar argument shows that $\phi_n(z;s;\omega)$ is $\mathscr{B}^2 \times \mathscr{B} \times \mathscr{F}$-measurable in $(z;s;\omega)$.

Now apply the Gram–Schmidt procedure to the sequence $Y_1(s), Y_2(s), \cdots$ to obtain an orthonormal sequence $Z_1(s), Z_2(s), \cdots$. Then $Z_0(s) \equiv 1, Z_1(s), Z_2(s), \cdots$ will be a basis for $L^2(\Omega, \mathscr{F}_{s\infty})$. The operations performed on the Y_n to get the Z_n being linear, the same operations on the $\phi_n(z;s)$ will give a sequence $\psi_1(z;s)$, $\psi_2(z;s), \cdots$ such that

$$Z_n(s) = \int_{\mathbf{R}_+^2} \psi_n(z;s)\, dW_z, \qquad n \geqq 1.$$

Moreover, Z_n and ψ_n will inherit the measurability properties of Y_n and ϕ_n. To summarize:

LEMMA 1.2. *For each $s > 0$, we can find an orthonormal basis $\{Z_n(s), n \geqq 0\}$ of $L^2(\Omega, \mathscr{F}_{s\infty})$ such that $Z_0(s) \equiv 1$, $(s;\omega) \to Z_n(s;\omega)$ is $\mathscr{B} \times \mathscr{F}$-measurable, and for which there exists a sequence of processes $\{\psi_n(z;s), z \in \mathbf{R}_+^2, s > 0\}$ satisfying, for each $n \geqq 1$,*

(a) *$(z;s;\omega) \to \psi_n(z;s;\omega)$ is $\mathscr{B}^2 \times \mathscr{B} \times \mathscr{F}$-measurable;*
(b) *$\psi_n(u,v;s)$ is $\mathscr{F}_{sv}$-measurable, and $\psi_n(u,v;s) = 0$ if $u > s$;*
(c) *for each $s > 0$,*

$$Z_n(s) = \int_{\mathbf{R}_+^2} \psi_n(z;s)\, dW_z. \tag{1.2}$$

This brings us to the promised representation theorem.

THEOREM 1.3. *Let $M = \{M_z, z \in \mathbf{R}_+^2\}$ be an adapted, measurable, square-integrable 2-martingale. Then there exists a process $\alpha = \{\alpha(z;s), z \in \mathbf{R}_+^2, s \in \mathbf{R}_+\}$ satisfying*

(a) *$(z;s;\omega) \to \alpha(z;s;\omega)$ is $\mathscr{B}^2 \times \mathscr{B} \times \mathscr{F}$-measurable;*
(b) *$\alpha(u,v;s)$ is $\mathscr{F}_{sv}$-measurable if $u \leqq s$, and $= 0$ if $u > s$;*
(c) *for each $(s,t) \in \mathbf{R}_+^2$, $E\{\int_{R_{st}} \alpha^2(z;s)\, dz\} < \infty$;*

and such that

$$M_{st} = M_{s0} + \int_{R_{st}} \alpha(z;s)\, dW_z, \tag{1.3}$$

for each $(s,t) \in \mathbf{R}_+^2$.

REMARKS. 1°. The function α is called the *2-derivate of M*; the corresponding function for a 1-martingale is the *1-derivate*.

2°. We can choose a version of the integral in (1.3) (and consequently of M) which is measurable and a.s. continuous in t for a.e. s.

3°. There is an analogous result for 2-martingales which are not adapted. In this case, $\alpha(u,v;s)$ is adapted to $\mathscr{F}_{\infty v}$ rather than to $\mathscr{F}_{sv}$.

4°. In the case the fields $\mathscr{F}_{st}$ are those generated by the Brownian sheet, $\mathscr{F}_{s0}$ and $\mathscr{F}_{0t}$ are both trivial, so that M_{s0} and M_{0t} in (1.3) are constant in s and t respectively.

PROOF. Let $\{Z_n(s)\}$ and $\{\psi_n(s)\}$ be the processes of Lemma 1.2. Fix $N > 0$ and $s > 0$. Then $M_{sN} \in L^2(\Omega, \mathscr{F}_{s\infty})$, so that if

$$b_n(s) = E\{M_{sN} Z_n(s)\},$$

we have

$$M_{sN} = \sum_{n=0}^{\infty} b_n(s) Z_n(s) = M_{s0} + \sum_{n=1}^{\infty} b_n(s) \int_{\mathbf{R}_+^2} \psi_n(z; s)\, dW_z$$
$$= M_{s0} + \int_{\mathbf{R}_+^2} \left(\sum_{n=1}^{\infty} b_n(s)\psi_n(z; s)\right) dW_z .$$

Now $\sum_{n=1}^{\infty} b_n(s)\psi_n(z; s)$ converges in L^2 for a.e. z, therefore a.s. along Mokobodzki's fast filter [3] to a limit $\alpha_N(z; s)$. Set $\alpha_N(z; s) = 0$ if z is exceptional or if the limit does not exist. Then $\{\alpha_N(z; s), z \in \mathbf{R}_+^2, s > 0\}$ satisfies (a)—(c) of the theorem, and in addition,

$$M_{sN} = M_{s0} + \int_{\mathbf{R}_+^2} \alpha_N(z; s)\, dW_z . \tag{1.4}$$

Condition both sides of (1.4) with respect to $\mathscr{F}_{st}$ to see that, for $t \leqq N$,

$$M_{st} = M_{s0} + \int_{R_{st}} \alpha_N(z; s)\, dW_z . \tag{1.5}$$

Clearly, $\alpha_N(z; s) = \alpha_{N+1}(z; s)$ for a.e. $z \in \mathbf{R}_+ \times [0, N]$. We need only set $\alpha(z; 0) \equiv 0$ and $\alpha(z; s) = \alpha_N(z; s)$ if $s > 0$ and $z \in \mathbf{R}_+ \times [N-1, N)$.

REMARKS. 5°. The definitions, lemmas, and the theorem of this section evidently have their analogues for 1-martingales. We will use these without comment in the future.

6°. If the 2-martingale depends measurably on a real parameter r (that is, if it is of the form M_{st}^r), then a slight modification of the above argument shows that the 2-derivate $\alpha(z; r, s)$ will still exist, and will be measurable in the quadruple $(z; r, s; \omega)$.

2. An application. The aim of this section is to show that Theorem 1.1, applied two times, gives the representation of Wong and Zakai we spoke of in the introduction.

Let $M = \{M_z, z \in \mathbf{R}_+^2\}$ be a square-integrable martingale. Being in particular an adapted 2-martingale, M has a 2-derivate $\alpha = \{\alpha(z; s) : z \in \mathbf{R}_+^2, s \in \mathbf{R}_+\}$, which we can suppose square-integrable. Let $s \leqq s'$ and $t \geqq 0$. Then

$$M_0 + \int_{R_{st}} \alpha(z; s)\, dW_z = M_{st} = E\{M_{s't} \mid \mathscr{F}_{st}\}$$
$$= M_0 + E\left\{\int_{R_{s't}} \alpha(z; s')\, dW_z \mid \mathscr{F}_{st}\right\} .$$

But this least is equal to

$$M_0 + \int_{R_{st}} E\{\alpha(z; s') \mid \mathscr{F}_{st}\}\, dW_z .$$

(This can be seen directly if α is a simple function; it follows in general by approximation by simple functions, cf. Lemma 9.6 of [1].) The equality of the above stochastic integrals implies that, for a.e. $z \in R_{st}$,

$$\alpha(z; s) = E\{\alpha(z; s') \mid \mathscr{F}_{st}\} . \tag{2.1}$$

We can modify α slightly to make (2.1) hold for all z as follows. Choose $s_0 > 0$

and redefine $\alpha(u, v; s)$ for $s < s_0$ by setting

$$\alpha(u, v; s) = E\{\alpha(u, v; s_0) \mid \mathscr{F}_{sv}\} \quad \text{if} \quad u \leqq s,$$
$$= 0 \quad \text{if} \quad u > s,$$

where we take a version of the conditional expectation which is measurable in $(u, v; s; \omega)$. Then α will still be a 2-derivate of M, but now, if $u \leqq s_0$, $\{\alpha(u, v; s), (s, v) \in [u, s_0] \times \mathbf{R}_+\}$ is an adapted square-integrable 1-martingale depending on the real parameter u. By Theorem 1.3 and Remark 6° following its proof, α itself has a 1-derivate $\beta = \{\beta(z'; z) : z', z \in [0, s_0] \times \mathbf{R}_+\}$, so that, if $u \leqq s < s' \leqq s_0$,

$$\alpha(u, v; s') - \alpha(u, v; s) = \int_{R_{s'v} - R_{sv}} \beta(u, v; z)\, dW_z .$$

Thus, if $s \leqq s_0$,

$$\text{(2.2)} \qquad M_{st} = M_0 + \int_{R_{st}} \alpha(u, v; s)\, dW_{uv}$$
$$= M_0 + \int_{R_{st}} [\alpha(u, v; u) + \int_{R_{sv} - R_{uv}} \beta(u, v; z)\, dW_z]\, dW_{uv}$$

Put

$$\phi(u, v) = \alpha(u, v; u) \quad \text{and} \quad \psi(u, v; s, t) = \beta(u, v; s, t) \quad \text{if} \quad (u, v) \curlywedge (s, t),$$
$$= 0 \quad \text{otherwise.}$$

Then (2.2) becomes

$$M_{st} = M_0 + \int_{R_{st}} \phi(z)\, dW_z + \int_{R_{st}} (\int_{R_{st}} \psi(z'; z)\, dW_z)\, dW_{z'}$$
$$= M_0 + \phi \cdot W_{st} + \psi \cdot WW_{st} ,$$

which is Wong and Zakai's representation.

REMARKS. The name "2-derivate" is supposed to suggest that $\alpha(z; s)$ plays the role of $\partial M/\partial_2 W$, so that, in a certain sense, each martingale is differentiable. The above construction of the representation shows that in the same loose sense, each martingale is twice differentiable, for, if $M = \phi \cdot W + \psi \cdot WW$, the process ψ is analogous to $\partial^2 M/\partial_1 W\, \partial_2 W$. One should probably not take these analogies too seriously, but they do suggest some intriguing interpretations. For instance, we could just as well have started with the 1-derivate instead of the 2-derivate to get the process ψ, leading us to conclude that $\partial^2 M/\partial_1 W\, \partial_2 W = \partial^2 M/\partial_2 W\, \partial_1 W$. The characterization in [1], Theorem 8.1, that M is a strong martingale iff $\psi \equiv 0$, can then be rephrased: M is a strong martingale iff $\partial^2 M/\partial_1 W\, \partial_2 W \equiv 0$.

3. Green's formula for 1- and 2-martingales. Let $M = \{M_z, z \in \mathbf{R}_+^2\}$ be an adapted, measurable and square-integrable 2-martingale such that $E\{\int_0^s M_{ut}^2\, du\} < \infty$ for each s and t. Let α be the 2-derivate of M, i.e., α verifies (a)—(c) of Theorem 1.3 and

$$M_{st} = M_{s0} + \int_{R_{st}} \alpha(\zeta; s)\, dW_\zeta .$$

Let

$$\text{(3.1)} \qquad \hat{\alpha}(u, v; s, t) = \alpha(u, v; s) \quad \text{if} \quad u < s, \quad v > t,$$
$$= 0 \quad \text{otherwise.}$$

Then $\hat{\alpha} = \{\hat{\alpha}(\zeta; \xi) : \zeta, \xi \in \mathbf{R}_+^2\}$ is measurable and adapted in the sense that $\hat{\alpha}(\zeta; \xi)$ is $\mathscr{F}_{\zeta \vee \xi}$-measurable. Furthermore, if $z = (s, t)$,

$$(3.2) \qquad E\{\textstyle\int\int_{R_z \times R_z} \hat{\alpha}^2(\zeta; \xi)\, d\zeta\, d\xi\} \leqq tE\{\int_0^s M_{ut}^2\, du\} < \infty .$$

Thus one can define the double stochastic integral $\int\int_{R_z \times R_z} \hat{\alpha}\, dW\, dW$ (cf. [4], see also [1]).

If $A \subset \mathbf{R}_+^2$, the *shadow of* A is the set $\hat{A}$ defined by

$$\hat{A} = \{((u, v), (s, t)) : (u, v) \curlywedge (s, t) \text{ and } (s, v) \in A\} .$$

Note that $\hat{A}$ is in $\mathbf{R}_+^4$, not $\mathbf{R}_+^2$.

THEOREM 3.1. *Let $A \subset \mathbf{R}_+^2$ be a rectangle parallel to the axes, and let $\hat{A}$ be its shadow. Then*

$$(3.3) \qquad \int_{\partial A} M\, \partial_1 W = \int_A M\, dW + \int\int_{\hat{A}} \hat{\alpha}\, dW\, dW ,$$

where the line integral is taken in the clockwise direction.

REMARK. If M has a stochastic partial with respect to (W, t) (see Section 4 for the definition), $\alpha(u, v; s)$ does not depend on u if $(s, v) \in A$, so the last integral reduces to $\int_A \phi\, dJ$, where $\phi(s, v) = \alpha(u, v; s)$. Thus Theorem 3.1 extends Theorem 6.3 of [1].

PROOF. Let $A = [a_1, a_2]$, where $a_i = (\sigma_i, \tau_i)$, $i = 1, 2$, and suppose $A \subset R_{z_0}$, where $z_0 = (s_0, t_0)$. Since $E\{\int_0^{s_0} M_{st_0}^2\, ds\} < \infty$, we can find a sequence of simple functions $\{M_s^n, s \leqq s_0\}$ such that $E\{\int_0^{s_0} (M_{st_0} - M_s^n)^2\, ds\} \to 0$ as $n \to \infty$. Extend $\{M_s^n\}$ to be a 2-martingale in R_{z_0} by $M_{st}^n = E\{M_s^n \mid \mathscr{F}_{st}\}$, and let α_n be the 2-derivate of $M^n = \{M_{st}^n\}$. Let $\hat{\alpha}_n$ be defined from α_n as in (3.1). Suppose (3.3) holds for M^n, i.e.,

$$(3.4) \qquad \int_{\partial A} M^n\, \partial_1 W = \int_A M^n\, dW = \int\int_{\hat{A}} \hat{\alpha}\, dW\, dW .$$

Then (3.3) also holds for M, since

$$E\{\textstyle\int_{\partial A} (M - M^n)^2\, ds\} , \qquad E\{\int_A (M - M^n)^2\, d\zeta\} \qquad \text{and} \qquad E\{\int\int_{\hat{A}} (\hat{\alpha} - \hat{\alpha}^n)^2\, d\zeta\, d\xi\}$$

are all bounded above by $2(t_0 + 1)E\{\int_0^{s_0} (M_{st_0} - M_s^n)^2\, ds\}$, which tends to zero as $n \to \infty$, implying that the three terms of (3.4) tend toward the corresponding terms of (3.3).

It remains to show that (3.3) holds when $\{M_{st_0}, s \leqq s_0\}$ is simple, and for this it is evidently enough to consider the case where $M_{st_0} = YI_{[s_1, s_2]}(s)$, where Y is bounded and $\mathscr{F}_{s_1 t_0}$-measurable, and $\sigma_1 \leqq s_1 < s_2 \leqq \sigma_2$. In this case M_{st} will be constant for $s \in [s_1, s_2]$ for each fixed $t \leqq t_0$, taking on the value $Y_t \equiv E\{Y \mid \mathscr{F}_{s_1 t}\}$ there. Thus, if $X_t = W_{s_2 t} - W_{s_1 t}$,

$$\int_{\partial A} M\, \partial_1 W = Y_{\tau_2} X_{\tau_2} - Y_{\tau_1} X_{\tau_1} \qquad \text{and} \qquad \int_A M\, dW = \int_{\tau_1}^{\tau_2} Y_v\, dX_v .$$

Since M_{st} is constant in s on $[s_1, s_2]$, its 2-derivate $\alpha(u, v; s)$ is, too. Thus

$$\begin{aligned} I_{\hat{A}} \hat{\alpha}(u, v; s, t) &= \alpha(u, v; s_1) \qquad \text{if } u < s_1 ,\ v \in [\tau_1, \tau_2] ,\ s \in [s_1, s_2] \text{ and } t < v , \\ &= 0 \qquad \text{otherwise,} \end{aligned}$$

so that we can regard $\iint_{\hat{A}} \hat{\alpha}\, dW\, dW$ as an iterated integral of the form

$$\int_{\tau_1}^{\tau_2} \int_0^{s_1} (\alpha(u, v; s_1) \int_0^v \int_{s_1}^{s_2} dW_{st})\, dW_{uv} .$$

The integral over (s, t) is exactly X_v, so that

$$\iint_{\hat{A}} \hat{\alpha}\, dW\, dW = \int_{R_{s_1\tau_2} - R_{s_1\tau_1}} \alpha(u, v; s_1) X_v\, dW_{uv} .$$

Thus everything reduces to showing that for each $t \leqq t_0$

$$Y_t X_t = \int_0^t Y_v\, dX_v + \int_{R_{s_1 t}} \alpha(u, v; s_1) X_v\, dW_{uv} . \tag{3.5}$$

We will do this by discrete approximation. Let $0 = t_1 < t_2 < \cdots < t_{n+1} = t$ and write

$$\begin{aligned} Y_t X_t &= \textstyle\sum_{i=1}^n (Y_{t_{i+1}} X_{t_{i+1}} - Y_{t_i} X_{t_i}) \\ &= \textstyle\sum_{i=1}^n Y_{t_i}(X_{t_{i+1}} - X_{t_i}) + \sum_{i=1}^n (Y_{t_{i+1}} - Y_{t_i}) X_{t_i} \\ &\quad + \textstyle\sum_{i=1}^n (Y_{t_{i+1}} - Y_{t_i})(X_{t_{i+1}} - X_{t_i}) . \end{aligned} \tag{3.6}$$

Consider a sequence of subdivisions for which $\delta = \sup_i (t_{i+1} - t_i) \to 0$. Recalling that $Y_v - Y_{t_i} = \int_{R_{s_1 v} - R_{s_1 t_i}} \alpha(\zeta; s_1)\, dW_\zeta$, we can see that

$$E\{\int_0^t (Y_v - \textstyle\sum_{i=1}^n Y_{t_i} I_{(t_i, t_{i+1}]}(v))^2\, dv\} \leqq \delta E\{\int_{R_{s_1 t}} \alpha^2(\zeta; s_1)\, d\zeta\} .$$

This tends to zero, hence the first sum on the right-hand side of (3.6) tends to $\int_0^t Y_v\, dX_v$ in L^2. The second sum can be written

$$\int_{R_{s_1 t}} \alpha(u, v; s_1) X_v\, dW_{uv} + \textstyle\sum_{i=1}^n \int_{\Delta_i} \alpha(u, v; s_1)(X_{t_i} - X_v)\, dW_{uv} ,$$

where $\Delta_i = R_{s_1 t_{i+1}} - R_{s_1 t_i}$. The sum has expected square majorized by

$$\delta(s_2 - s_1) E\{\int_{R_{s_1 t}} \alpha^2(\zeta; s_1)\, d\zeta\} , \tag{3.7}$$

hence the second sum on the right-hand side of (3.6) tends to $\int_{R_{s_1 t}} \alpha(u, v; s_1) X_v\, dW_{uv}$. Finally, the third sum tends to zero since its expected square is also majorized by (3.7), and the theorem is proved.

Using the corresponding theorem for 1-martingales and noting that a martingale is both a 1- and a 2-martingale, we have immediately:

THEOREM 3.2. *Let $A \subset \mathbf{R}_+^2$ be a rectangle and $\hat{A}$ its shadow, and let $M = \{M_z, z \in \mathbf{R}_+^2\}$ be a square-integrable martingale. Then*

$$\int_{\partial A} M\, \partial W = \iint_{\hat{A}} (\hat{\alpha}_2 - \hat{\alpha}_1)\, dW\, dW ,$$

where $\hat{\alpha}_1$ and $\hat{\alpha}_2$ are the processes associated by (3.1) *with the* 1- *and* 2-*derivates, respectively, of M, and where the line integral is taken in the clockwise direction.*

REMARK. As in [1], the theorems above can be extended to regions A whose boundary is a sufficiently smooth curve.

4. The derivative of a holomorphic process is holomorphic. One of the more obscure proofs in [1] showed that the derivative of a process holomorphic in $\mathbf{R}_+^2$ is itself a holomorphic process. Its core was a complicated calculation involving the increasing processes associated with the integrals in Wong and

Zakai's representation. We are now in a position to give a simpler and more transparent—or at least less opaque—proof based on Green's formula and the representation of i-martingales, which will at the same time generalize the theorem to processes defined in arbitrary domains. We do not want to insist too much on the generality, for the most natural domains in which to define these processes are random, not fixed. However, the study of random regions would lead us into a number of areas we cannot go into here. We will enter them in a subsequent article.

Let $\overline{ab}$ be a horizontal line segment in $\mathbf{R}_+^2$ with endpoints a and b, and let Φ be a process defined on $\overline{ab}$. We say Φ has a *stochastic partial derivative with respect to W along $\overline{ab}$* if there exists an adapted measurable process ρ—the partial derivative—defined on $\overline{ab}$ such that

$$(4.1) \qquad E\{\textstyle\int_{\overline{ab}} \rho_\zeta^2 \, d\zeta\} < \infty \qquad \text{and} \qquad \Phi_z = \Phi_a + \textstyle\int_{\overline{az}} \rho \, \partial_1 W \quad \text{for any} \quad z \in \overline{ab} \, .$$

If $A \subset \mathbf{R}_+^2$ is a region and if Φ is a process defined on A, we say that Φ has a *stochastic partial derivative with respect to* (W, s) *in* A if there exists an adapted measurable process ρ—the partial derivative—defined on A which satisfies (4.1) for each horizontal segment $\overline{ab} \subset A$. The stochastic partial derivatives along a vertical segment and with respect to (W, t) are defined analogously. (The stochastic partial derivatives defined in [1] admitted a second integral, $\int \phi \, ds$. This is indeed the correct definition but we will only consider cases where this integral vanishes, as it does, for instance, when Φ is a martingale.)

We say that a process Φ is *holomorphic* in a rectangle $A \subset \mathbf{R}_+^2$ if it is defined in A and if there exists a process $\phi = \{\phi_z, z \in A\}$ which is a partial derivative of Φ with respect to both (W, s) and (W, t). If D is a domain, we say Φ is holomorphic in D if it is holomorphic in each sub-rectangle of D.

REMARKS. 1°. The process ϕ is called the *derivative* of Φ. It is of course possible to define stochastic partial derivatives and holomorphic processes without the strong square-integrability condition of (4.1), but it appears that even a slight relaxation of this condition can lead to some quite wild behavior on the part of holomorphic processes. Thus we will only consider the square-integrable case here.

2°. If Φ is holomorphic with derivative ϕ in a rectangle A, and if Γ is a staircase in A with endpoints a and b, then

$$(4.2) \qquad \Phi_b - \Phi_a = \textstyle\int_\Gamma \phi \, \partial W \, .$$

In particular, the integral of ϕ around a closed staircase vanishes, so that the line integral of ϕ is independent of the path.

3°. A process is holomorphic in a domain D if and only if it has stochastic partial derivatives with respect to both (W, s) and (W, t) in D, and the two partial derivatives are equal.

The existence of a stochastic partial derivative, even along a single line, is a stronger restriction than one might think, and to have two partial derivatives actually implies that a process is holomorphic. Indeed, the promised theorem on holomorphic processes is an easy consequence of the following theorem on stochastic partial derivatives (cf. Theorem 9.10 of [1]).

Let $a_1 = (\sigma_1, \tau_1) \prec a_2 = (\sigma_2, \tau_2)$ be in $\mathbf{R}_+^2$, and let $A = [a_1, a_2]$. Denote the upper-left and lower-right corners of A by $b_1 = (\sigma_1, \tau_2)$ and $b_2 = (\sigma_2, \tau_1)$ respectively.

THEOREM 4.1. *Let M a be square-integrable martingale defined in the closed rectangle $A = [a_1, a_2]$. Suppose that M has a stochastic partial derivative with respect to W along the two lines $\overline{b_1a_2}$ and $\overline{b_2a_2}$. Then M is holomorphic in A and admits a derivative which is holomorphic in $[a_1, a_2)$.*

PROOF. Denote the partial derivatives of M by $\{\rho_z, z \in \overline{b_1a_2}\}$ and $\{\rho_z, z \in \overline{b_2a_2}\}$. Now $\int_{\overline{b_1a_2} \cup \overline{b_2a_2}} E\{\rho_z^2\}\, dz < \infty$, so we may assume, by setting $\rho_z \equiv 0$ if $E\{\rho_z^2\} = \infty$, that the partial derivatives are square-integrable. It is convenient to extend M to all of R_{a_2} by setting $M_z = E\{M_{a_2} \mid \mathscr{F}_z\}$ if $z \in R_{a_2}$. Let B_1 and B_2 be the rectangles $R_{a_2} - R_{b_1}$ and $R_{a_2} - R_{b_2}$ respectively, and define

$$(4.3) \qquad \begin{aligned} \phi_{st} &= E\{\rho_{s\tau_2} \mid \mathscr{F}_{st}\} \quad \text{if} \quad (s, t) \in B_1 , \\ \psi_{st} &= E\{\rho_{\sigma_2 t} \mid \mathscr{F}_{st}\} \quad \text{if} \quad (s, t) \in B_2 , \end{aligned}$$

where we choose measurable versions of the conditional expectations. By Lemma 9.6 of [1], M has the partial derivative ϕ with respect to (W, s) in B_1, and the partial derivative ψ with respect to (W, t) in B_2. Furthermore, ϕ is a square-integrable 2-martingale and ψ is a square-integrable 1-martingale. By Theorem 1.3, ϕ has a 2-derivate α and ψ has a 1-derivate β, so that if $(s, t) \in A$, we have:

$$(4.4) \qquad \begin{aligned} \phi_{st} &= \phi_{s\tau_1} + \int_{R_{st}-R_{s\tau_1}} \alpha(z; s)\, dW_z , \\ \psi_{st} &= \psi_{\sigma_1 t} + \int_{R_{st}-R_{\sigma_1 t}} \beta(t; z)\, dW_z . \end{aligned}$$

Let $\hat\alpha$ and $\hat\beta$ be associated with α and β by (3.1) and the corresponding formula for 1-martingales. By Theorem 3.1, if $\hat A$ is the shadow of A,

$$M_{a_2} - M_{b_1} - M_{b_2} + M_{a_1} = \int_{\partial A} \phi\, \partial_1 W = \int_A \phi\, dW + \iint_{\hat A} \hat\alpha\, dW\, dW$$

and

$$M_{a_2} - M_{b_2} - M_{b_1} + M_{a_1} = -\int_{\partial A} \psi\, \partial_2 W = \int_A \psi\, dW + \iint_{\hat A} \hat\beta\, dW\, dW .$$

But now the stochastic integrals are orthogonal to the double stochastic integrals above, so that we have

$$(4.5) \qquad \int_A \phi\, dW = \int_A \psi\, dW \qquad \text{and} \qquad \iint_{\hat A} \hat\alpha\, dW\, dW = \iint_{\hat A} \hat\beta\, dW\, dW .$$

The first equation implies that $\phi_{st} = \psi_{st}$ for a.e. $(s, t) \in A$. Thus there is a negligible set F such that if $s \in [\sigma_1, \sigma_2] - F$, then $P\{\phi_{st} = \psi_{st}\} = 1$ for a.e. $t \in [\tau_1, \tau_2]$. Now ϕ is a 2-martingale and is thus L^2-continuous in t (i.e., for fixed s, $E\{(\phi_{st'} - \phi_{st})^2\} \to 0$ as $t' \to t$). Since for $s \notin F$, ϕ_{st} equals the 1-martingale ψ_{st} a.s. for a.e. t, it is easy to use this L^2-continuity to conclude that $\{\phi_{st}, (s, t) \in A, s \notin F\}$ is both a 1- and a 2-martingale, and hence a martingale.

We must eliminate the exceptional set F. We can use the conditional expectation to extend ϕ to a martingale M' defined in $A - \overline{b_1 a_2}$. Returning to ψ, an argument similar to the above shows that there exists a negligible set G such that if $t \in [\tau_1, \tau_2] - G$, $P\{M'_{st} = \psi_{st}\} = 1$ for a.e. $s \in [\sigma_1, \sigma_2]$. ψ being a 1-martingale, we can use the conditional expectation once more to extend M', this time to a martingale defined on $A - \{a_2\}$. We again call this martingale M'.

Now with probability one, M' agrees with ϕ a.s. along a given horizontal line, so that M' is a partial derivative of M with respect to (W, s). By symmetry, M' is also a partial derivative of M with respect to (W, t). In particular, M is holomorphic in A with derivative M'.

To show that M' is also holomorphic, we turn to the second of equations (4.5). We have $\hat{\alpha} = \hat{\beta}$ a.e. on $\hat{A}$, which implies that for a.e. $(s, v) \in A$, $\alpha(u, v; s) = \beta(v; s, t)$ for a.e. $u \leqq s$ and a.e. $t \leqq v$. This in turn implies that, for these (s, v), $\alpha(u, v; s)$ and $\beta(v; s, t)$ must be essentially constant in $u \leqq s$ and $t \leqq v$. Call the common value $\chi(s, v)$ and set $\chi(s, v) = 0$ for the other values of (s, v). By (4.4) and the definition of M', we have, for a.e. $s \in [\sigma_1, \sigma_2]$,

$$\int_{\overline{(s,\tau_1)(s,\tau_2)}} E\{\chi_\zeta^2\}\, d\zeta < \infty ,$$

$$M'_{s\tau_2} - M'_{s\tau_1} = \phi_{s\tau_2} - \phi_{s\tau_1} = \int_{R_{s\tau_2} - R_{s\tau_1}} \alpha(\zeta; s)\, dW_\zeta = \int_{\overline{(s,\tau_1)(s,\tau_2)}} \chi\, \partial_2 W ,$$

and for a.e. $t \in [\tau_1, \tau_2]$,

$$\int_{\overline{(\sigma_1,t)(\sigma_2,t)}} E\{\chi_\zeta^2\}\, d\zeta < \infty ,$$

$$M'_{\sigma_2 t} - M'_{\sigma_1 t} = \psi_{\sigma_2 t} - \psi_{\sigma_1 t} = \int_{R_{\sigma_2 t} - R_{\sigma_1 t}} \beta(t; \zeta)\, dW_\zeta = \int_{\overline{(\sigma_1,t)(\sigma_2,t)}} \chi\, \partial_1 W ,$$

i.e., M' has a stochastic partial derivative along both $\overline{(s, \tau_1)(s, \tau_2)}$ and $\overline{(\sigma_1, t)(\sigma_2, t)}$, hence, by the first part of the proof, M' is holomorphic in $[a_1, z]$, where $z = (s, t)$. As z can be chosen arbitrarily close to a_2, M' is holomorphic in $[a_1, a_2)$.

An immediate consequence is the following:

THEOREM 3.2. *Suppose that Φ is a holomorphic process defined on a domain D. Then Φ admits a derivative which is holomorphic in D.*

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