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EQUILIBRIUM MEASURES FOR SEMI-MARKOV PROCESSES[1]

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This paper simplifies and extends previous results on the existence of an equilibrium or stationary measure for the age process associated with a semi-Markov chain:

$$(I_{(t)}, Z_{(t)}) = (\text{last state enterered before time } t, \text{ duration of this last sojourn up to } t).$$

Introduction and definitions. As in [3] we maintain a dichotomy between semi-Markov chains which have sojourn times concentrated on a lattice (for simplicity the integers) and those which do not. In the former case R, (R_+) represents the integers (the nonnegative integers); B, (B_+) is the σ-field of subsets of R, (R_+) and m is counting measure. In the latter case R, (R_+) is $(-\infty, \infty)$, $([0, \infty))$; B, (B_+) is the σ-field of Borel sets on R, (R_+) and m is Lebesgue measure.

Let $(\Pi, \mathscr{G})$ be a measure space. Let $(E, \mathscr{E}) = (\Pi \times R, \mathscr{G} \otimes B)$. A transition kernel $\mathbf{\Pi}$ on $(E, \mathscr{E})$ is called semi-Markovian if

$$\mathbf{\Pi}(\pi, x; d\pi', dx') = \mathbf{\Pi}(\pi, 0; d\pi', dx'),$$

that is the transition is independent of $x((\pi, x) \in E)$. Given an initial probability measure α on $(E, \mathscr{E})$ we may construct a probability space $(\Lambda, \mathscr{A}, \mathbf{\Pi}^\alpha)$ on which a (semi-)Markov chain $(I_n, X_n)_{n=0}^\infty$ is defined having initial distribution α and probability transition kernel $\mathbf{\Pi}$. Moreover, $(I_n)_{n=0}^\infty$ is a Markov chain whose transition kernel is denoted by $\mathscr{R}$—$\mathscr{R}$ is a transition kernel on $(\Pi, \mathscr{G})$. Throughout we denote by the same symbol (e.g. $\mathbf{\Pi}^\alpha$) both the probability measure and the expectation derived from a transition kernel. Also if $\alpha = \delta_{(\pi,x)}$ for $(\pi, x) \in E$ then denote $\mathbf{\Pi}^\alpha$ by $\mathbf{\Pi}^{(\pi,x)}$. The semi-Markov chain is lattice if the $(X_n)_{n=0}^\infty$ take only integer values. Otherwise it is continuous.

Henceforth for any probability measure α on $(E_+, \mathscr{E}_+)$ we assume:

CONDITION (I). $\alpha\mathbf{\Pi}(\Pi \times (0, \infty)) = 1$.

CONDITION (II). $\mathbf{\Pi}^\alpha\{\lim_{n\to\infty} S_n = \infty\} = 1$ where $S_n = \sum_{j=0}^n X_j$.

The first means the sojourn times are strictly positive. The second eliminates "explosions."

With initial measure $\delta_{(\pi,0)}$ we may define a semi-Markov chain $(I_n, X_n)_{n=0}^\infty$ defined on $(\Lambda, \mathscr{A}, \mathbf{\Pi}^{(\pi,0)})$ taking values in $(E_+, \mathscr{E}_+)$. For $t \in R_+$ we may also define the age process:

$$(I_{(t)}, Z_{(t)}) = (I_{n-1}, t - S_{n-1}) \qquad \text{where} \quad S_{n-1} \leqq t < S_n .$$

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With this we may define (as in [3]) the transition kernel on $(E_+, \mathscr{E}_+)$—for $F \in \mathscr{E}_+$:

$$H_t(\pi, x; F) = \Pi^{(\pi,0)}\{(I_{(t+x)}, Z_{(t+x)}) \in F \mid X_1 > x\} \quad \text{if } \Pi^{(\pi,0)}\{X_1 > x\} > 0 ;$$

(1) $$= \chi_F(\pi, x) \quad \text{if } \Pi^{(\pi,0)}\{X_1 > x\} = 0 \text{ and } t + x - [x] < 1 ;$$

(2) $$= \Pi^{(\pi,0)}\{(I_{(t-(1-x+[x]))}, Z_{(t-(1-x+[x]))}) \in F\} \quad \text{if } \Pi^{(\pi,0)}\{X_1 > x\} = 0 \text{ and } t + x - [x] \geqq 1 .$$

($[x]$ is the greatest integer in x.)

H_t is the transition kernel for the age process. If the age process starts out at $(\pi, x) \in E_+$ it is as though the process has already sojourned in the initial state π for a time x. That is, the process really started in π at time $-x$ and has not made a transition by time 0. We say the process has been delayed for a time x. The delay could in practice never exceed the longest sojourn time but if $\Pi^{(\pi,0)}(X_1 > x) = 0$, (1) and (2) are added—the age process stays in π until the next integer time and then jumps to $(\pi, 0)$.

In [1], [2], and [3] conditions are given to ensure that

$$\lim_{t\to\infty} ||\alpha H_t(\cdot) - \beta H_t(\cdot)|| = 0$$

where α and β are probability measures on $(E_+, \mathscr{E}_+)$ and $||\cdot||$ is the total variation on $(E_+, \mathscr{E}_+)$. The existence of an equilibrium measure e on $(E_+, \mathscr{E}_+)$ (that is $eH_t = e$) has been studied in [6], [1], and [2]. If e exists and if $e(E_+) < \infty$ (by normalization take $e(E_+) = 1$) then for any initial measure α

$$\lim_{t\to\infty} ||\alpha H_t(\cdot) - e(\cdot)|| = 0 .$$

We now turn to the existence of equilibrium measures.

Main section. Henceforce we assume:

CONDITION (III). There exists a measure Δ on $(\Pi, \mathscr{G})$ such that $\Delta\mathscr{R} = \Delta$.

Denoting the distribution of the sojourn time in $\pi \in \Pi$ by F^π we define

$$e(d\pi, dx) = \Delta(d\pi) \cdot (1 - F^\pi(x))m(dx) .$$

LEMMA 1.

$$eH_t(F) = \int \Delta(d\pi)\Pi^{(\pi,0)} \int_0^{X_1} \chi_F(I_{(t+y)}, Z_{(t+y)})m(dy)$$

where $F \in \mathscr{E}_+$. *Henceforth denote*

$$\chi_F(I_{(y)}, Z_{(y)}) \text{ by } f_F(y) .$$

PROOF.

$$\begin{aligned} eH_t(F) &= \int \Delta(d\pi) \int_0^\infty H_t(\pi, s; F) \cdot (1 - F^\pi(s))m(ds) \\ &= \int \Delta(d\pi) \int_0^\infty \Pi^{(\pi,0)}\{f_F(t + s) \mid X_1 > s\} \cdot (1 - F^\pi(s))m(ds) \\ &= \int \Delta(d\pi) \int_0^\infty \Pi^{(\pi,0)}\{f_F(t + s) \cdot \chi_{\{X_1>s\}}(s)\}m(ds) \\ &= \int \Delta(d\pi)\Pi^{(\pi,0)} \int_0^{X_1} f_F(t + y)m(dy) . \end{aligned}$$

□

To show $eH_t = e$ we need only show $A \equiv \int \Delta(d\pi)\mathbf{\Pi}^{(\pi,0)} \int_0^{X_1} f_F(y) \cdot m(dy)$ is the same as $B \equiv \int \Delta(d\pi)\mathbf{\Pi}^{(\pi,0)} \int_0^{X_1} f_F(t+y) \cdot m(dy)$.

PROPOSITION 1. *If*

$$\int \Delta(d\pi)\mathbf{\Pi}^{(\pi,0)} \int_0^t f_F(y)m(dy) < \infty \tag{3}$$

then $A = B$.

PROOF.

$$\begin{aligned}
&\int \Delta(d\pi)\mathbf{\Pi}^{(\pi,0)} \int_0^{t+X_1} f_F(y)m(dy) \\
&\qquad = \int \Delta(d\pi)\mathbf{\Pi}^{(\pi,0)} \int_0^t f_F(y)m(dy) + \int \Delta(d\pi)\mathbf{\Pi}^{(\pi,0)} \int_t^{t+X_1} f_F(y)m(dy) \\
&\qquad = \int \Delta(d\pi)\mathbf{\Pi}^{(\pi,0)} \int_0^t f_F(y)m(dy) + B . \qquad \text{Also} \\
&\int \Delta(d\pi)\mathbf{\Pi}^{(\pi,0)} \int_0^{t+X_1} f_F(y)m(dy) \\
&\qquad = \int \Delta(d\pi)\mathbf{\Pi}^{(\pi,0)} \int_0^{X_1} f_F(y)m(dy) + \int \Delta\mathscr{R}(d\pi)\mathbf{\Pi}^{(\pi,0)} \int_0^t f_F(y)m(dy) \\
&\qquad = A + \int \Delta(d\pi)\mathbf{\Pi}^{(\pi,0)} \int_0^t f_F(y)m(dy) .
\end{aligned}$$

Hence $A = B$. □

Note that if $\Delta(\Pi) < \infty$ then (3) holds.

Example 1 shows that (3) may fail even if the chain $(I_n)_{n=0}^\infty$ is recurrent. The following proposition shows that e always exists for recurrent chains.

PROPOSITION 2. *If ϕ is a σ-finite measure on $(\mathbf{\Pi}, \mathscr{G})$ and if $\mathscr{R}$ generates a ϕ-recurrent chain on $(\Pi, \mathscr{G})$ then $A = B$.*

PROOF. Let $\phi(G) > 0$ where $G \in \mathscr{G}$. Also let

$$\begin{aligned}
{}_G\bar{\mathscr{R}}_m(\pi, A) &= \mathscr{R}^{(\pi)}\{I_m \in A,\ I_i \notin G \quad 1 \leqq i \leqq m\} \\
{}_G\mathscr{R}_m(\pi, A) &= \mathscr{R}^{(\pi)}\{I_m \in A,\ I_i \notin G \quad 1 \leqq i < m\} \qquad \text{and} \\
\mathscr{R}_G(\pi, A) &= \textstyle\sum_{m=1}^\infty {}_G\mathscr{R}_m(\pi, A) .
\end{aligned}$$

Note that $\mathscr{R}_G$ defines a transition kernel on G. The resulting chain is called the process on G(see [4], pages 28–29). Let $\tilde{\Delta}_G$ be the invariant probability measure of the process on G. Then

$$\tilde{\Delta}(d\pi) \equiv \int_G \tilde{\Delta}_G(d\rho) \textstyle\sum_{k=0}^\infty {}_G\bar{\mathscr{R}}_k(\rho, d\pi)$$

is an equilibrium measure for $\mathscr{R}$ (see equation 7.2 in [4]). By Theorem 7.2 in [4] $\tilde{\Delta}$ is proportional and we shall assume equal to Δ (and so $\tilde{\Delta}_G$ is Δ restricted to G). Now

$$\begin{aligned}
&\int \Delta(d\pi)\mathbf{\Pi}^{(\pi,0)} \int_0^{X_1} f_F(y)m(dy) \\
&\qquad = \int_G \Delta(d\rho) \textstyle\sum_{k=0}^\infty \int_\pi {}_G\bar{\mathscr{R}}_k(\rho, d\pi)\mathbf{\Pi}^{(\pi,0)} \int_0^{X_1} f_F(y) \cdot m(dy) \\
&\qquad = \int_G \Delta(d\rho)\mathbf{\Pi}^{(\rho,0)} \int_0^\tau f_F(y)m(dy)
\end{aligned}$$

where τ is the time of first return to G. Similarly

$$\begin{aligned}
&\int \Delta(d\pi)\mathbf{\Pi}^{(\pi,0)} \int_0^{X_1} f_F(y+t)m(dy) \\
&\qquad = \int_G \Delta(d\rho)\mathbf{\Pi}^{(\rho,0)} \int_t^{\tau+t} f_F(y) \cdot m(dy) . \qquad \text{Now}
\end{aligned}$$

$$(4) \qquad \int_G \Delta(d\rho)\mathbf{\Pi}^{(\rho,0)} \int_0^{\tau+t} f_F(y)m(dy)$$
$$= \int_G \Delta(d\rho)\mathbf{\Pi}^{(\rho,0)} \int_0^{\tau} f_F(y)m(dy) + \int_G \Delta\mathscr{R}_G(d\rho)\mathbf{\Pi}^{(\rho,0)} \int_0^t f_F(y)m(dy)$$
$$= A + \int_G \Delta(d\rho)\mathbf{\Pi}^{(\rho,0)} \int_0^t f_F(y)m(dy) .$$

Also

$$(4) = \int_G \Delta(d\rho)\mathbf{\Pi}^{(\rho,0)} \int_0^t f_F(y)m(dy) + \int_G \Delta(d\rho)\mathbf{\Pi}^{(\rho,0)} \int_t^{t+\tau} f_F(y)m(dy)$$
$$= \int_G \Delta(d\rho)\mathbf{\Pi}^{(\rho,0)} \int_0^t f_F(y)m(dy) + B .$$

Hence $A = B$. □

Example 2 gives a semi-Markov chain where e does not exist even when Δ does.

As the referee remarked, condition (II), which eliminates explosions, is not essential. If we assume the existence of an age process $(I_{(t)}, Z_{(t)})$ (denoted by (Z_t, U_t) in [6]) satisfying condition A in [5] we may define $\mathscr{R}$ and H_t from this age process and all proofs go through as before.

Examples.

EXAMPLE 1. Let $\Pi = \{0, 1, 2, \cdots\}$. Define

$$\mathscr{R}(n, n-1) = 1 \qquad \text{if} \quad n \in \{1, 2, 3, \cdots\}$$

and

$$\mathscr{R}(0, n) = p_n \qquad \text{where} \quad \sum_{n=0}^{\infty} np_n = \infty .$$

Let the sojourn time in state $n \neq 0$ have a uniform distribution on $[0, 1/2^n]$. Let the sojourn time in 0 be exactly 10 units. Here $\Delta(\{n\}) = \sum_{k=n}^{\infty} p_k$. If we let $t = 5$ and $F = \{0\} \times R_+$ in (1) we see (1) fails. Nevertheless, by Proposition 2, e is the equilibrium measure.

EXAMPLE 2. Let Π be the integers. Define $\mathscr{R}(n, n+1) = 1$ for all $n \in \Pi$. Let the sojourn time at $m \in I^-$ (a negative integer) be uniformly distributed on $[0, 2^m]$. Let the sojourn time for nonnegative integers be uniformly distributed on $[0, 1]$. Here $\Delta(\{n\}) = 1 \ \forall\, n$ and there are no explosions. Nevertheless, letting $F = \{0\} \times R_+$, we see (1) fails. Also $e(I^- \times R_+) < \infty$. However, every unit of time a mass $1 \cdot \int_0^1 (1-x)dx = \frac{1}{2}$ flows into state $\{0\}$. Hence e is not an equilibrium measure.

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