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ERGODIC THEOREMS FOR THE ASYMMETRIC SIMPLE EXCLUSION PROCESS II[1]

BY THOMAS M. LIGGETT

University of California, Los Angeles

Ergodic theorems are proved for the one dimensional translation invariant simple exclusion process, in cases in which the initial distribution is highly nontranslation invariant.

1. Introduction. Let $p(x, y) = p(0, y - x)$ be the transition probabilities for an irreducible random walk on Z^1, and let η_t be the corresponding simple exclusion process. This is the Feller process on $X = \{0, 1\}^{Z^1}$ which describes the evolution of configurations of indistinguishable particles on Z^1 with at most one particle per site, in which particles move in the following way: a particle at x waits an exponential time with parameter one, then it chooses a y according to the probabilities $p(x, y)$, and finally it moves to y if y is vacant at that time, while it remains at x if y is occupied. The generator of the process is the closure in $C(X)$ of the operator Ω which is defined for functions f which depend on only finitely many coordinates by

$$\Omega f(\eta) = \sum_{\eta(x)=1;\eta(y)=0} p(x, y)[f(\eta_{xy}) - f(\eta)],$$

where η_{xy} is the configuration obtained from η by moving the particle at x to y. The simple exclusion process was introduced by Spitzer in [8], and the proof that the closure of the above defined operator Ω is a semigroup generator was given in [3]. In the one-dimensional finite mean case, this result was first proved by Holley in [1]. A survey of results relating to this process, as well as a discussion of conjectures and open problems, can be found in part II of [7]. Basically the current situation is that the behavior of the process is well understood in case $p(x, y)$ is symmetric, but that many open problems remain in the asymmetric case.

In this paper, we extend and simplify the main results of [5]. In [5], it was assumed that $p(x, y) = 0$ for $|y - x| > 1$, and our primary extension is the elimination of this requirement. The main simplification comes from the fact that we avoid using the results and techniques of Section 3 of [5], which is the most difficult and least natural part of that paper, and that part in which the nearest neighbor assumption enters in a crucial way. We will, however, use the results of Section 2 of [5].

In order to state the main result, let $S(t)$ be the semigroup on $C(X)$ which

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corresponds to the process η_t. As usual, the dual semigroup acting on probability measures on X gives the distribution of the process at time t for the given initial distribution. For $0 \leqq \rho \leqq 1$, let ν_ρ be the product measure on X with $\nu_\rho\{\eta: \eta(x) = 1\} = \rho$ for all x.

THEOREM 1.1. *Assume that* $\sum_x |x|p(0, x) < \infty$ *and* $\gamma = \sum_x xp(0, x) > 0$, *and suppose that* μ *is a product probability measure on* X *for which the following limits exist*:

$$(1.2) \qquad \lambda = \lim_{x\to-\infty} \mu\{\eta: \eta(x) = 1\}, \qquad \rho = \lim_{x\to+\infty} \mu\{\eta: \eta(x) = 1\}.$$

(a) *If* $\lambda \geqq \frac{1}{2}$ *and* $\rho \leqq \frac{1}{2}$, *then* $\lim_{t\to\infty} \mu S(t) = \nu_{\frac{1}{2}}$.
(b) *If* $\rho \geqq \frac{1}{2}$ *and* $\lambda + \rho > 1$, *then* $\lim_{t\to\infty} \mu S(t) = \nu_\rho$.
(c) *If* $\lambda \leqq \frac{1}{2}$ *and* $\lambda + \rho < 1$, *then* $\lim_{t\to\infty} \mu S(t) = \nu_\lambda$.

Of course, a similar statement holds in the negative mean case. The case $0 \leqq \lambda < \frac{1}{2}$, $\lambda + \rho = 1$ is omitted from the statement of the theorem for the same reasons as in the nearest neighbor situation, so the reader is referred to [5] for a discussion of that case.

Different techniques apply in the symmetric case, and the result is that $\lim_{t\to\infty} \mu S(t) = \nu_{(\lambda+\rho)/2}$ for all λ and ρ, provided that μ satisfies the assumptions of Theorem 1.1 ([4], [9]). Therefore, for example, if μ is the pointmass on the configuration η for which $\eta(x) = 1$ if $x \leqq 0$ and $\eta(x) = 0$ for $x > 0$, then $\lim_{t\to\infty} \mu S(t) = \nu_{\frac{1}{2}}$ whenever p is symmetric or has positive mean. A natural conjecture is that this is also the case when p has mean zero. As can be seen from [5], however, this result fails if p has negative mean.

Our techniques lead to results for the translation invariant simple exclusion process on Z^d for $d > 1$ as well. We do not give the proofs in the higher dimensional case, since the class of initial distributions covered seems less interesting in this context than in one dimension. A typical result which can be proved, for example, is that the conclusions of Theorem 1.1 hold if $p(0, x)$ has a finite mean with positive first coordinate, and if μ is a product measure which satisfies $\mu\{\eta: \eta(x) = 1\} = \lambda$ if $x^{(1)} \leqq 0$ and $\mu\{\eta: \eta(x) = 1\} = \rho$ if $x^{(1)} > 0$.

The initial distributions dealt with in Theorem 1.1 are very special, and it would of course be of great interest to prove ergodic theorems for this process with more general initial distributions. A first step in this direction, and an important problem in its own right, is to determine completely the class of invariant measures for the process under the assumptions of the theorem. The natural conjecture is that (a) the Markov chain obtained by restricting η_t to the countable set $\{\eta \in X \mid \sum_{x<0} \eta(x) = \sum_{x\geqq 0} [1 - \eta(x)] < \infty\}$ is positive recurrent; and (b) the extremal invariants are given by $\{\nu_\rho, 0 \leqq \rho \leqq 1\} \cup \{\nu_n, -\infty < n < \infty\}$, where $\{\nu_n\}$ are the translates of the stationary distribution of the Markov chain in (a). This conjecture has been proved in [6] in case $p(x, y) = 0$ for $|x - y| > 1$. In the general case, it is only known [6] that the extremal translation invariant, invariant measures are given by $\{\nu_\rho, 0 \leqq \rho \leqq 1\}$. When $\gamma = 0$, it was proved in [6] that $\{\nu_\rho, 0 \leqq \rho \leqq 1\}$ are the only extremal invariants. Another related problem of interest is to prove that $\lim_{t\to\infty} \mu S(t) = \nu_\rho$ when μ is translation invariant and ergodic, where $\rho = \mu\{\eta(x) = 1\}$. This has been proved in the symmetric case [4, 9], but in general it is only known [7] that all weak limits of $\mu S(t)$ are exchangeable and have density ρ.

As in [5], the method of proof of Theorem 1.1 involves making comparisons with related finite systems. The analysis of these finite systems will be carried out in Section 2, and the proof of the theorem will be completed in Section 3. We will assume throughout that p has a finite positive mean.

2. The finite systems. For $\lambda, \rho \in [0, 1]$ and integers $m < n$, consider the Markov chain on $X_{m,n} = \{0, 1\}^{D_m}$ with generator

$$\begin{aligned}\Omega_{m,n}^{\lambda,\rho} f(\eta) = {} & \sum_{\eta(x)=1,\eta(y)=0;m\leqq x,y\leqq n} p(x, y)[f(\eta_{xy}) - f(\eta)] \\ & + \sum_{\eta(x)=1;m\leqq x\leqq n} [f(\eta_x) - f(\eta)]\{(1-\lambda)\sum_{y<m} p(x, y) \\ & + (1-\rho)\sum_{y>n} p(x, y)\} \\ & + \sum_{\eta(y)=0;m\leqq y\leqq n} [f(\eta_y) - f(\eta)]\{\lambda \sum_{x<m} p(x, y) + \rho\sum_{x>n} p(x, y)\},\end{aligned}$$

where $D_{m,n} = \{m, \cdots, n\}$ and η_u is defined by $\eta_u(u) = 1 - \eta(u)$ and $\eta_u(v) = \eta(v)$ for $v \neq u$. This is a simple exclusion process on $D_{m,n}$ with spontaneous creation and destruction of particles at rates which are obtained by imagining that at all times, $\{\eta(y), y \notin D_{m,n}\}$ are independent of $\{\eta(y), y \in D_{m,n}\}$, and $\eta(y)$ has mean λ for $y < m$ and mean ρ for $y > n$. We will denote the corresponding semigroup by $S_{m,n}(t)$, since the values of λ and ρ will always be clear from the context. This chain has a unique invariant measure $\mu_{m,n}(\lambda, \rho)$, since the chain is irreducible unless $\lambda = \rho = 0$ or $\lambda = \rho = 1$, in which case the chain is eventually absorbed by $\eta \equiv 0$ or $\eta \equiv 1$ respectively.

Define a partial order on the set of probability measures on X by $\mu \leqq \nu$ if there exists a probability measure α on $X \times X$ with marginals μ and ν respectively such that $\alpha\{(\eta, \zeta) : \eta \leqq \zeta\} = 1$. In comparing the invariant measures for different values of m and n, it is convenient to have them defined on the same space. Therefore we regard $\mu_{m,n}(\lambda, \rho)$ as a probability measure on X by letting it be the product measure on $X = X_{m,n} \times \prod_{x\notin D_{m,n}} \{0, 1\}$ with

$$\begin{aligned}\mu_{m,n}(\lambda, \rho)\{\eta : \eta(x) = 1\} &= \lambda \qquad \text{if} \quad x < m \\ &= \rho \qquad \text{if} \quad x > n\,.\end{aligned}$$

Theorems 2.4 and 2.13 of [5] give the following monotonicity results in the present context.

PROPOSITION 2.1. (a) $\mu_{m,n}(\lambda, \lambda) = \nu_\lambda$.

(b) *If* $\lambda_1 \leqq \lambda_2$, *then* $\mu_{m,n}(\lambda_1, \rho) \leqq \mu_{m,n}(\lambda_2, \rho)$.

(c) *If* $\rho_1 \leqq \rho_2$, *then* $\mu_{m,n}(\lambda, \rho_1) \leqq \mu_{m,n}(\lambda, \rho_2)$.

(d) *If* $\lambda \leqq \rho$, *then* $\mu_{m-1,n}(\lambda, \rho) \geqq \mu_{m,n}(\lambda, \rho) \geqq \mu_{m,n+1}(\lambda, \rho)$.

(e) *If* $\lambda \geqq \rho$, *then* $\mu_{m-1,n}(\lambda, \rho) \leqq \mu_{m,n}(\lambda, \rho) \leqq \mu_{m,n+1}(\lambda, \rho)$.

As a consequence of (d) and (e), the iterated weak limits

$$\mu_r(\lambda, \rho) = \lim_{m\to-\infty} \lim_{n\to+\infty} \mu_{m,n}(\lambda, \rho), \quad \text{and}$$
$$\mu_l(\lambda, \rho) = \lim_{n\to+\infty} \lim_{m\to-\infty} \mu_{m,n}(\lambda, \rho)$$

both exist and satisfy $\mu_r(\lambda, \rho) \leqq \mu_l(\lambda, \rho)$ if $\lambda \leqq \rho$ and $\mu_l(\lambda, \rho) \leqq \mu_r(\lambda, \rho)$ if $\lambda \geqq \rho$. The main objective of this section is to evaluate these two measures explicitly for the values of λ and ρ covered in Theorem 1.1, and in fact to prove that for such λ and ρ, $\mu_r(\lambda, \rho) = \mu_l(\lambda, \rho)$.

PROPOSITION 2.2. *For any* $\lambda, \rho \in [0, 1]$, $\mu_r(\lambda, \rho)$ *and* $\mu_l(\lambda, \rho)$ *are exchangeable measures.*

PROOF. Observe that $\lim_{n\to\infty} \mu_{m,n}(\lambda, \rho)$ and $\lim_{n\to\infty} \mu_{m+1,n}(\lambda, \rho)$ are translates of one another, so that $\mu_r(\lambda, \rho)$ is translation invariant. Since $\lim_{m\to-\infty; n\to+\infty} \Omega_{m,n}^{\lambda,\rho} f = \Omega f$ for all functions which depend on only finitely many coordinates, $\mu_r(\lambda, \rho)$ is invariant for $S(t)$. Therefore $\mu_r(\lambda, \rho)$ is exchangeable by Theorem 1.1 of [6]. The proof for $\mu_l(\lambda, \rho)$ is similar.

Since $\mu_{m,n}(\lambda, \rho)$ is invariant for $S_{m,n}(t)$, $\int \Omega_{m,n}^{\lambda,\rho} f \, d\mu_{m,n}(\lambda, \rho) = 0$ for $f(\eta) = \eta(x)$, $m \leqq x \leqq n$. Writing this out yields the fact that

$$\begin{aligned} c_{m,n}(\lambda, \rho) = \lambda & \textstyle\sum_{x<m; u<y\leqq n} p(x, y)\mu_{m,n}(\lambda, \rho)\{\eta(y) = 0\} \\ & + \textstyle\sum_{m\leqq x\leqq u; u<y\leqq n} p(x, y)\mu_{m,n}(\lambda, \rho)\{\eta(x) = 1, \eta(y) = 0\} \\ & + (1 - \rho) \textstyle\sum_{m\leqq x\leqq u; y>n} p(x, y)\mu_{m,n}(\lambda, \rho)\{\eta(x) = 1\} \\ & - \rho \textstyle\sum_{x>n; m\leqq y\leqq u} p(x, y)\mu_{m,n}(\lambda, \rho)\{\eta(y) = 0\} \\ & - \textstyle\sum_{u<x\leqq n; m\leqq y\leqq u} p(x, y)\mu_{m,n}(\lambda, \rho)\{\eta(x) = 1, \eta(y) = 0\} \\ & - (1 - \lambda) \textstyle\sum_{u<x\leqq n; y<m} p(x, y)\mu_{m,n}(\lambda, \rho)\{\eta(x) = 1\} \end{aligned} \tag{2.3}$$

is independent of u for $m - 1 \leqq u \leqq n$. Since $c_{m,n}(\lambda, \rho)$ depends on m and n only through the sum $n - m$, and since $p(0, x)$ has a finite first moment, the joint limit $c(\lambda, \rho) = \lim_{m\to-\infty; n\to+\infty} c_{m,n}(\lambda, \rho)$ exists and

$$\begin{aligned} c(\lambda, \rho) &= \textstyle\sum_{x\leqq u<y} p(x, y)\mu_r(\lambda, \rho)\{\eta(x) = 1, \eta(y) = 0\} \\ &\quad - \textstyle\sum_{y\leqq u<x} p(x, y)\mu_r(\lambda, \rho)\{\eta(x) = 1, \eta(y) = 0\} \\ &= \gamma\mu_r(\lambda, \rho)\{\eta(x) = 1, \eta(y) = 0\} \end{aligned} \tag{2.4}$$

for $x \neq y$, which is independent of x and y by Proposition 2.2. Similarly,

$$c(\lambda, \rho) = \gamma\mu_l(\lambda, \rho)\{\eta(x) = 1, \eta(y) = 0\}. \tag{2.5}$$

PROPOSITION 2.6.

$$c(\lambda, \rho) \leqq \gamma \min[\lambda(1 - \lambda), \rho(1 - \rho)] \quad \text{if} \quad \lambda \leqq \rho, \quad \text{and} \tag{2.7}$$

$$c(\lambda, \rho) \geqq \gamma \max[\lambda(1 - \lambda), \rho(1 - \rho)] \quad \text{if} \quad \lambda \geqq \rho. \tag{2.8}$$

PROOF. Writing (2.3) for $u = m - 1$ and $u = n$ respectively yields

$$\begin{aligned} c_{m,n}(\lambda, \rho) = \lambda & \textstyle\sum_{x<m\leqq y\leqq n} p(x, y)\mu_{m,n}(\lambda, \rho)\{\eta(y) = 0\} \\ & - (1 - \lambda) \textstyle\sum_{y<m\leqq x\leqq n} p(x, y)\mu_{m,n}(\lambda, \rho)\{\eta(x) = 1\}, \quad \text{and} \end{aligned}$$

$$\begin{aligned} c_{m,n}(\lambda, \rho) = (1 - \rho) & \textstyle\sum_{m\leqq x\leqq n<y} p(x, y)\mu_{m,n}(\lambda, \rho)\{\eta(x) = 1\} \\ & - \rho \textstyle\sum_{m\leqq y\leqq n<x} p(x, y)\mu_{m,n}(\lambda, \rho)\{\eta(y) = 0\}. \end{aligned}$$

Using (a), (b), and (c) of Proposition 2.1 then gives

$$c_{m,n}(\lambda, \rho) \leqq \lambda(1-\lambda) \textstyle\sum_{x<m\leqq y\leqq n} [p(x, y) - p(y, x)] \quad \text{if } \lambda \leqq \rho$$
$$c_{m,n}(\lambda, \rho) \geqq \lambda(1-\lambda) \textstyle\sum_{x<m\leqq y\leqq n} [p(x, y) - p(y, x)] \quad \text{if } \lambda \geqq \rho$$
$$c_{m,n}(\lambda, \rho) \leqq \rho(1-\rho) \textstyle\sum_{m\leqq x\leqq n<y} [p(x, y) - p(y, x)] \quad \text{if } \lambda \leqq \rho$$
$$c_{m,n}(\lambda, \rho) \geqq \rho(1-\rho) \textstyle\sum_{m\leqq x\leqq n<y} [p(x, y) - p(y, x)] \quad \text{if } \lambda \geqq \rho\,,$$

and the result follows by taking limits.

THEOREM 2.9. (a) *If* $\lambda \geqq \frac{1}{2}$ *and* $\rho \leqq \frac{1}{2}$, *then* $\mu_l(\lambda, \rho) = \mu_r(\lambda, \rho) = \nu_{\frac{1}{2}}$.
(b) *If* $\rho \geqq \frac{1}{2}$ *and* $\lambda + \rho > 1$, *then* $\mu_l(\lambda, \rho) = \mu_r(\lambda, \rho) = \nu_\rho$.
(c) *If* $\lambda \leqq \frac{1}{2}$ *and* $\lambda + \rho < 1$, *then* $\mu_l(\lambda, \rho) = \mu_r(\lambda, \rho) = \nu_\lambda$.

PROOF. We will prove this for $\mu_r(\lambda, \rho)$ only, since the proof for $\mu_l(\lambda, \rho)$ is identical. By Proposition 2.2 and de Finetti's theorem, $\mu_r(\lambda, \rho) = \int_0^1 \nu_\tau \beta(d\tau)$ for some probability measure $\beta(d\tau)$ on [0, 1], which of course depends on λ and ρ. By (2.4), $c(\lambda, \rho) = \gamma \int_0^1 \tau(1-\tau)\beta(d\tau) \leqq \gamma/4$. By (2.8), on the other hand, $c(1, \frac{1}{2}) \geqq \gamma/4$ and $c(\frac{1}{2}, 0) \geqq \gamma/4$, so that if $\lambda = 1$ and $\rho = \frac{1}{2}$ or $\lambda = \frac{1}{2}$ and $\rho = 0$, $\beta\{\frac{1}{2}\} = 1$ and $\mu_r(1, \frac{1}{2}) = \mu_r(\frac{1}{2}, 0) = \nu_{\frac{1}{2}}$. Now suppose $\lambda \geqq \frac{1}{2}$ and $\rho \leqq \frac{1}{2}$, and take limits in (b) and (c) of Proposition 2.1 to obtain

$$\nu_{\frac{1}{2}} = \mu_r(\tfrac{1}{2}, 0) \leqq \mu_r(\lambda, \rho) \leqq \mu_r(1, \tfrac{1}{2}) = \nu_{\frac{1}{2}}\,,$$

which gives part (a) of the theorem. If $\lambda \geqq \rho \geqq \frac{1}{2}$, then $\beta(d\tau)$ concentrates on $[\rho, \lambda]$ by (a), (b), and (c) of Proposition 2.1, and $\int_\rho^\lambda \tau(1-\tau)\beta(d\tau) \geqq \rho(1-\rho)$ by (2.8). Therefore $\beta\{\rho\} = 1$, and hence $\mu_r(\lambda, \rho) = \nu_\rho$. A similar argument shows that $\mu_r(\lambda, \rho) = \nu_\lambda$ if $\frac{1}{2} \geqq \lambda \geqq \rho$. Finally, if $\lambda \leqq \rho$ and $\lambda + \rho > 1$, $\beta(d\tau)$ concentrates on $[\lambda, \rho]$ and $\int_\lambda^\rho \tau(1-\tau)\beta(d\tau) \leqq \rho(1-\rho)$, so $\beta\{\rho\} = 1$ and $\mu_r(\lambda, \rho) = \nu_\rho$, with a similar argument showing that $\mu_r(\lambda, \rho) = \nu_\lambda$ if $\lambda \leqq \rho$ and $\lambda + \rho < 1$.

Note that in case $0 \leqq \lambda < \frac{1}{2}$ and $\lambda + \rho = 1$, these techniques give only that $\mu_r(\lambda, \rho)$ and $\mu_l(\lambda, \rho)$ are convex combinations of ν_λ and ν_ρ. In fact, it can be shown that in this case, $\mu_r(\lambda, \rho) = \nu_\lambda$ and $\mu_l(\lambda, \rho) = \nu_\rho$, so that the comparisons used in the next section would not yield results.

3. Proofs of the main results. The following provides the key link between Theorems 2.9 and 1.1.

THEOREM 3.1. *Suppose* μ *is the product measure on* X *with*

$$\mu\{\eta : \eta(x) = 1\} = \lambda \quad \textit{if} \quad x \leqq 0 \qquad\qquad (3.2)$$
$$= \rho \quad \textit{if} \quad x > 0\,.$$

If ν *is any weak limit point of* $\mu S(t)$ *as* $t \to \infty$, *then*

$$\mu_l(\lambda, \rho) \leqq \nu \leqq \mu_r(\lambda, \rho) \quad \textit{if} \quad \lambda \geqq \rho\,, \qquad \textit{and}$$
$$\mu_r(\lambda, \rho) \leqq \nu \leqq \mu_l(\lambda, \rho) \quad \textit{if} \quad \lambda \leqq \rho\,.$$

PROOF. We will carry out the proof that $\nu \leqq \mu_r(\lambda, \rho)$ if $\lambda \geqq \rho$, since the proofs are similar in the other cases. Fix $k \leqq 0$, and note that

$$\mu S(t) = \lim_{m\to-\infty} \lim_{n\to+\infty} \mu S_{m,n}(t) \leqq \lim_{n\to+\infty} \mu S_{k,n}(t)\,, \qquad (3.3)$$

where the equality is a consequence of the Trotter–Kurtz convergence theorem for semigroups [2], and the inequality follows from (e) of Proposition 2.1. Since μ and ν_ρ on $\{0, 1\}^{\{k,k+1,\cdots\}}$ are two product measures whose marginal probabilities differ at only finitely many points, a simple coupling argument applied to the process with semigroup $\lim_{n\to\infty} S_{k,n}(t)$ yields the conclusion that

$$\lim_{t\to\infty} \lim_{n\to\infty} [\mu S_{k,n}(t) - \nu_\rho S_{k,n}(t)] = 0 . \tag{3.4}$$

Next, since $\nu_\rho S_{k,n}(t)$ is increasing in both n and t by Theorems 2.4 and 2.13 of [5],

$$\lim_{t\to\infty} \lim_{n\to\infty} \nu_\rho S_{k,n}(t) = \lim_{n\to\infty} \lim_{t\to\infty} \nu_\rho S_{k,n}(t) . \tag{3.5}$$

Putting (3.3), (3.4) and (3.5) together yields

$$\nu \leqq \lim_{n\to\infty} \lim_{t\to\infty} \nu_\rho S_{k,n}(t) = \lim_{n\to\infty} \mu_{k,n}(\lambda, \rho) .$$

Since this is true for all $k \leqq 0$, it follows that $\nu \leqq \mu_r(\lambda, \rho)$.

Proof of Theorem 1.1. Theorems 2.9 and 3.1 together give the required result in case μ is the product measure with marginals given by (3.2). Since the limit is translation invariant in each case, the result holds for translates of this μ as well. Given a product measure μ for which

$$\mu\{\eta(x) = 1\} = \lambda \quad \text{for} \quad x < m \quad \text{and} \tag{3.6}$$
$$\mu\{\eta(x) = 1\} = \rho \quad \text{for} \quad x > n \quad \text{and}$$

$$\mu\{\eta(x) = 1\} \text{ lies between } \lambda \text{ and } \rho \text{ for } m \leqq x \leqq n , \tag{3.7}$$

there are product measures μ_1 and μ_2 which are translates of the one with marginals (3.2) and which satisfy $\mu_1 \leqq \mu \leqq \mu_2$. Therefore the required result holds for such μ by Theorem 2.3 of [5]. Let $N = 2^{n-m+1}$ and let $\eta_1, \cdots, \eta_N$ be the points in $X_{m,n}$. For $1 \leqq i \leqq N$, let μ_i be the product measure with marginals given by (3.6) and $\mu_i\{\eta(x) = 1\} = \eta_i(x)$ for $m \leqq x \leqq n$, and let μ be the following convex combination on the μ_i's:

$$\mu = \sum_{i=1}^N |\prod_{x=m}^n [\delta(x)]^{\eta_i(x)}[1 - \delta(x)]^{1-\eta_i(x)}| \mu_i .$$

This is a product measure whose marginals satisfy (3.6) and (3.7), provided that $\delta(x)$ lies between λ and ρ for all $x \in D_{m,n}$. Therefore the required result holds for each μ_i if $\lambda \neq \rho$, and thus by continuity and Theorem 2.3 of [5], even if $\lambda = \rho$. Hence we conclude that the required result holds for all product measures satisfying (3.6) alone. To extend the result to product measures satisfying (1.2), we then use again Theorem 2.3 of [5] and the continuity of the limits in (a), (b), and (c) of Theorem 1.1 in λ and ρ for λ and ρ other than $\lambda < \frac{1}{2}$, $\lambda + \rho = 1$.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA
LOS ANGELES, CALIFORNIA 90024