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CONVERGENCE RATES FOR THE ISOTROPE DISCREPANCY[1]

By Winfried Stute

University of Bochum

For each sequence of independent and identically distributed $\mathbb{R}^k$-valued random variables, $k \geqq 3$, with distribution μ defined on some probability space $(\Omega, \mathscr{F}, \mathbb{P})$, let

$$D_n(\omega, \mu) \equiv \sup_C |\mu_n^\omega(C) - \mu(C)|, \qquad n \in \mathbb{N}, \omega \in \Omega,$$

be the so-called isotrope discrepancy (at stage n), where μ_n^ω denotes the nth empirical distribution pertaining to ω and where the supremum is taken over the class of all convex measurable sets $C \subset \mathbb{R}^k$. It is proved that almost everywhere and in the mean $D_n(\cdot)$ converges to zero as $n^{-2/(k+1)}$ (up to a logarithmic factor), provided μ is absolutely continuous with a bounded density function of compact support.

1. Introduction and main results. The main purpose of the present paper is to derive various results on the asymptotic behavior of the isotrope discrepancy. To begin with, let $\omega = \{\mathbf{x}_i\}_{i \in \mathbb{N}}$ be an infinite sequence of points in $[0, 1]^k$, $k \in \mathbb{N}$. Then ω is said to be "uniformly distributed" iff for each k-dimensional interval Q

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^n \chi_Q(\mathbf{x}_i) = \int_{[0,1]^k} \chi_Q(\mathbf{x})\, d\lambda(\mathbf{x})$$

where χ_A, $A \subset \mathbb{R}^k$, denotes the indicator function of A and integration is taken w.r.t. k-dimensional Lebesgue-measure λ. Using the well-known convergence theorem of Pólya–Cantelli it is easy to establish that the above convergence is uniform over the class of all intervals Q, i.e., if one defines

$$D_n'(\omega) \equiv \sup_Q \left| \frac{1}{n} \sum_{i=1}^n \chi_Q(\mathbf{x}_i) - \int_{[0,1]^k} \chi_Q(\mathbf{x})\, d\lambda(\mathbf{x}) \right|$$

then $D_n'(\omega) \to 0$ as $n \to \infty$. Now, if one puts

$$D_n(\omega) \equiv \sup_C \left| \frac{1}{n} \sum_{i=1}^n \chi_C(\mathbf{x}_i) - \int_{[0,1]^k} \chi_C(\mathbf{x})\, d\lambda(\mathbf{x}) \right|$$

where the supremum is taken over all $C \in \mathscr{C}_k$, the class of all convex measurable subsets of $\mathbb{R}^k$ (including the empty set), it was shown by E. Hlawka (1961) that even in this case $D_n(\omega) \to 0$ as $n \to \infty$. In accordance with S. K. Zaremba (1970)

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the number $D_n(\omega)[D_n'(\omega)]$ will be henceforth called the isotrope [extreme] discrepancy of ω (at stage n) and can obviously be regarded as describing the imperfection of the equidistribution of the first n points of ω over $[0, 1]^k$.

Next, let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of independent identically distributed (i.i.d.) $\mathbb{R}^k$-valued random variables defined on some p-(= probability) space $(\Omega, \mathscr{F}, \mathbb{P})$ with distribution μ. Then, if $\mu_n^\omega \equiv (1/n)(\varepsilon_{f_1(\omega)} + \cdots + \varepsilon_{f_n(\omega)})$, $n \in \mathbb{N}$, $\omega \in \Omega$, denotes the empirical distribution pertaining to $f_1(\omega), \cdots, f_n(\omega)$, one is led to define the (generalized) extreme and isotrope discrepance of ω as

$$D_n'(\omega) = D_n'(\omega, \mu) \equiv \sup_Q |\mu_n^\omega(Q) - \mu(Q)| \qquad \text{and}$$
$$D_n(\omega) = D_n(\omega, \mu) \equiv \sup_C |\mu_n^\omega(C) - \mu(C)|\,, \quad \text{respectively.}$$

As to the limit behavior of $D_n'(\omega)$ it was shown by Richter (1974) and Wichura (1973) that under no restrictions on μ

(1) $$\limsup_{n\to\infty} n^{\frac{1}{2}}(2\log\log n)^{-\frac{1}{2}} D_n'(\omega) \leqq \tfrac{1}{2} \qquad \text{for } \mathbb{P}\text{-almost all} \quad \omega \in \Omega$$

which is best possible according to the ordinary L.I.L. For a comprehensive account on these and similar results, see Niederreiter (1973).

Now it is natural to ask whether one could replace $D_n'(\omega)$ in (1) by the isotrope discrepancy $D_n(\omega)$. For $k = 2$ the answer is positive if $\mu = \mathscr{U}[0, 1]^2$, the uniform distribution over $[0, 1]^2$, as it was shown by W. Philipp (1973). On the other hand (cf. Ranga Rao (1962), page 680),

$$\mathbb{P}(\{\omega \in \Omega : \lim_{n\to\infty} D_n(\omega) = 0\}) = 0\ ,$$

if $\mu = \mathscr{U}S^1$, the uniform distribution over $S^1 \equiv \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1\}$, i.e., concerning (1) the answer is far from being positive in this case. Thus for dimension $k \geqq 2$ the rate of almost sure convergence of D_n may heavily depend on μ. On the other hand, as was shown in Stute (1976), Theorem 1.4, for every distribution μ fulfilling the following condition

(++) $$\mu \ll \textstyle\bigotimes_{i=1}^k \mu_i \qquad \text{for some} \quad \mu_i \in ca_+(\mathbb{R}, \mathscr{B}_1)\ ,^2 \qquad i = 1, \cdots, k\ ,$$

one obtains the Glivenko–Cantelli result

$$\mathbb{P}(\{\omega \in \Omega : \lim_{n\to\infty} D_n(\omega) = 0\}) = 1\ .$$

Now, if one imposes some additional conditions on μ, it is possible to derive the following result on the rate of almost sure convergence of D_n.

THEOREM 1.1. *Let* $\{f_n\}_{n\in\mathbb{N}}$ *be a sequence of* i.i.d. $\mathbb{R}^k$*-valued random variables,* $k \geqq 3$, *defined on some p-space* $(\Omega, \mathscr{F}, \mathbb{P})$ *with distribution* $\mu = \mathbb{P}f_1^{-1}$. *Suppose that there exist nonatomic* $\mu_i \in ca_+(\mathbb{R}, \mathscr{B}_1)$, $i = 1, \cdots, k$, *such that*

(2) $$\mu \ll \textstyle\bigotimes_{i=1}^k \mu_i \equiv \nu$$

and

(3) $$\left\|\frac{d\mu}{d\nu}\right\|_{\infty,\nu} \equiv R < \infty\ .$$

[2] Here and in what follows $ca_+(\mathbb{R}^k, \mathscr{B}_k)$ denotes the space of all finite nonnegative Borel measures on $\mathscr{B}_k$.

Then for $\mathbb{P}$*-almost all* $\omega \in \Omega$

$$D_n(\omega) = O\left(\left(\frac{\log n}{n}\right)^{1/k}\right). \tag{4}$$

Using the same techniques as in Stute (1974), Theorem 1.10, it is easy to see that (3) cannot be dispensed with in general.

On the other hand, considering only sequences $\{f_n\}_{n \in \mathbb{N}}$ with absolutely continuous distribution μ, (4) can be remarkably improved in the following way.

THEOREM 1.2. *Under the general assumptions of Theorem* 1.1, *but with* $k \geqq 4$, *suppose that* μ *has a Lebesgue-density function* f *such that*

$$\|f\|_{\infty,\lambda} \equiv R < \infty \tag{3'}$$

and

$$f \quad \text{has compact support.} \tag{5}$$

Then for $\mathbb{P}$*-almost all* $\omega \in \Omega$

$$D_n(\omega) = O\left(\left(\frac{\log n}{n}\right)^{2/(k+1)}\right). \tag{6}$$

In particular (6) extends a recent result of M. Zuker (1974), Theorem V.5.b, which states that for each $\varepsilon > 0$ $n^{2/(k+3+\varepsilon)}D_n(\omega) \to 0$ for $\mathbb{P}$-almost all $\omega \in \Omega$. Theorem 1.10 in Stute (1974) provides an example that (5) is essential in order to ensure (6).

The following theorem shows that (6) cannot be improved very much in the case $\mu = \mathscr{U}[0, 1]^k$, the uniform distribution over $[0, 1]^k$.

THEOREM 1.3. *Let* $\{f_n\}_{n \in \mathbb{N}}$ *be a sequence of* i.i.d. $\mathbb{R}^k$*-valued random variables,* $k \geqq 2$, *with common distribution* $\mu = \mathscr{U}[0, 1]^k$. *Then for* $\mathbb{P}$*-almost all* $\omega \in \Omega$

$$n^{2/(k+1)}(\log n)^{(k-1)/(k+1)}D_n(\omega) \quad \text{is bounded away from zero as} \quad n \to \infty .$$

Combining this result with Theorem 1.2 it follows in the case $\mu = \mathscr{U}[0, 1]^k$, $k \geqq 4$, that for $\mathbb{P}$-almost all $\omega \in \Omega$, $D_n(\omega)$ tends to zero at a rate of $n^{-2/(k+1)}$ (up to a logarithmic factor), while the uniform L.I.L. (1) with D_n' replaced by D_n yields the precise rate of convergence in the case $k = 1, 2$. The question whether the same holds for $k = 3$ still awaits an answer, whereas the answer is negative in the case $k \geqq 4$ as it might be easily derived from Theorem 1.3. On the other hand, it would be interesting to know how to construct explicitly arbitrarily long sequences ω of points in $[0, 1]^k$ such that $D_n(\omega)$ tends to zero as $n^{-2/(k+1)}$. Finally it is still an open problem whether there exists a p-distribution μ fulfilling (2) and (3) such that $\mathbb{P}$-almost everywhere D_n tends to zero at the prescribed order of $n^{-1/k}$. Concerning the speed of the so-called mean Glivenko-Cantelli convergence the same methods of proof will yield the following result.

THEOREM 1.4. *For each sequence* $\{f_n\}_{n \in \mathbb{N}}$ *of* i.i.d. $\mathbb{R}^k$*-valued random variables with distribution* μ *defined on some p-space* $(\Omega, \mathscr{F}, \mathbb{P})$

$$\mathbb{E}_{\mathbb{P}}(\alpha_n D_n(\cdot)) \quad \text{is bounded as} \quad n \to \infty ,$$

where $\alpha_n = (n/\log n)^{1/k}$, *if* $k \geqq 3$ *and if* μ *fulfills* (2) *and* (3) *of Theorem* 1.1, *and* $\alpha_n = (n/\log n)^{2/(k+1)}$, *if* $k \geqq 4$ *and if* μ *fulfills* (3′) *and* (5) *of Theorem* 1.2.

Furthermore, if $\mu = \mathscr{U}[0, 1]^k$, $k \geqq 2$, one obtains that

$$n^{2/(k+1)}(\log n)^{(k-1)/(k+1)}\mathbb{E}_{\mathbb{P}}(D_n(\cdot)) \quad \text{is bounded away from zero as} \quad n \to \infty .$$

In particular Theorem 1.4 extends Stute (1974), Theorem 1.9, where it is shown with different methods, that under the hypotheses of Theorem 1.1

$$\mathbb{E}_{\mathbb{P}}(D_n(\cdot)) = O(n^{-1/(k+2)}) .$$

REMARK. As to the case $k = 2$ ($k = 3$ resp.) it should be noted that under the assumptions of Theorem 1.1 (Theorem 1.2 resp.) the same methods of proof apply in order to show that $\mathbb{P}$-almost everywhere and in the mean

$$n^{\frac{1}{2}}(\log n)^{-\frac{1}{2}}D_n(\cdot) \quad \text{is bounded as} \quad n \to \infty .$$

2. Proof of Theorem 1.1. The proof of (4) is mainly based on techniques developed in Stute (1976) to derive a Glivenko–Cantelli theorem for the class of all convex measurable sets in $\mathbb{R}^k$. In addition we will apply the following strong version of Kolmogorov's upper exponential bound, which may be proved in essentially the same way as in Loève (1963), page 255.

LEMMA 2.1. *Let* $\{g_n\}_{n \in \mathbb{N}}$ *be a sequence of independent real-valued random variables defined on some p-space* $(\Omega, \mathscr{F}, \mathbb{P})$ *satisfying, for each* $n \in \mathbb{N}$, *the following conditions*:

$$\sup_{\omega \in \Omega} |g_n(\omega)| \leqq 1 , \qquad \mathbb{E}_{\mathbb{P}}(g_n) = 0 , \qquad \sigma_n^2 \equiv \mathbb{E}_{\mathbb{P}}(g_n^2) > 0 .$$

Let $\tau_n^2 \equiv \sum_{s=1}^n \sigma_s^2$. *Then for every* $\varepsilon > 0$ *and all* $t \in [0, \tau_n]$

$$\mathbb{P}(\textstyle\bigcup_{j=1}^n \{\omega \in \Omega : \sum_{i=1}^j g_i(\omega) \geqq \tau_n \varepsilon\}) \leqq \exp[-t\varepsilon + (t^2/2)(1 + (2\tau_n)^{-1}t)] .$$

In order to apply the techniques of Stute (1976), for each measurable space $(X, \mathscr{A})$ let $(\Pi(\mathscr{A}), \prec)$ be the class of all finite partitions of X into $\mathscr{A}$-sets, directed by refinement. Then for each indexed subset $\{\pi^s : s \in I\}$ of $\Pi(\mathscr{A})$ and every $C \in \mathscr{A}$ let

$$C_+^s \equiv W(C, \pi^s) \equiv \bigcup \{B \in \pi^s : B \cap C \neq \emptyset\}$$
$$C_-^s \equiv V(C, \pi^s) \equiv \bigcup \{B \in \pi^s : B \subset C\} .$$

As an immediate consequence $\pi^s \prec \pi^t$ implies

$$C_-^s \subset C_-^t \subset C \subset C_+^t \subset C_+^s .$$

In the sequel we will consider partitions $\pi \in \Pi(\mathscr{B}_k)$ of the following type: let $-\infty < t_0 < t_1 < \cdots < t_i < \infty$ be $i + 1$ ($i \geqq 0$) distinct points on the real line. Put

$$I_0 \equiv (-\infty, t_0] ,$$
$$I_j \equiv (t_{j-1}, t_j] , \qquad j = 1, \cdots, i$$
$$I_{i+1} \equiv (t_i, \infty) ,$$

and $\pi \equiv \{I_{j_1} \times \cdots \times I_{j_k} : 0 \leqq j_r \leqq i + 1 \text{ for all } r = 1, \cdots, k\}$.

Partitions π of the above type will be called partitions of type $\mathscr{F}_k(i)$. In order to make the proof of (4) more transparent the main arguments will be shown separately in Lemmas 2.5–2.8. In this context Lemmas 2.2–2.4 will be very useful.

LEMMA 2.2. *Let $\pi \in \Pi(\mathscr{B}_k)$ be of type $\mathscr{F}_k(i)$, $k \geqq 2$. Then for each $C \in \mathscr{C}_k$ which is contained in some $\mathbb{R}^{k-1} \times I_j$, $j \in \{0, 1, \cdots, i+1\}$, there exists $C' \in \mathscr{C}_{k-1}$ such that*

$$V(C, \pi) = V(C' \times I_j, \pi) . \tag{7}$$

PROOF. Let $p: \mathbb{R}^k \to \mathbb{R}^{k-1}$ be defined for all $\mathbf{x} = (x_1, \cdots, x_k) \in \mathbb{R}^k$ by

$$p(\mathbf{x}) \equiv (x_1, \cdots, x_{k-1}) .$$

With $C'' \equiv co(V(C, \pi))$ (where $co(H)$ means the convex hull of $H \subset \mathbb{R}^k$) one obtains

$$C'' = co(p(V(C, \pi)) \times I_j) = co(p(V(C, \pi))) \times I_j .$$

Put $C' \equiv co(p(V(C, \pi)))$. Then, in order to prove (7), it suffices to show that

$$V(C, \pi) = V(C'', \pi) .$$

First, by definition of C'', each $B \in \pi$ with $B \subset C$ is contained in C'', i.e.,

$$V(C, \pi) \subset V(C'', \pi) .$$

To prove the converse inclusion let $B \in \pi$ with $B \subset C''$ be given. Since by definition $V(C, \pi)$ is contained in C it follows from the convexity of C that $C'' \subset C$ whence $B \subset C$. This shows $V(C'', \pi) \subset V(C, \pi)$ and completes the proof of Lemma 2.2. □

To state the next lemma for each $\pi \in \Pi(\mathscr{B}_k)$ let

$$\mathscr{V}(\pi) \equiv \{V(C, \pi) : C \in \mathscr{C}_k\} \qquad \text{and} \qquad \mathscr{W}(\pi) \equiv \{W(C, \pi) : C \in \mathscr{C}_k\} .$$

LEMMA 2.3. *Suppose that $\pi \in \Pi(\mathscr{B}_k)$, $k \geqq 1$, is of type $\mathscr{F}_k(i)$, $i \geqq 0$. Then*

$$|\mathscr{V}(\pi)| \leqq (i+2)^{2(i+2)^{k-1}} \qquad \textit{and} \qquad |\mathscr{W}(\pi)| \leqq (i+2)^{2(i+2)^{k-1}} .$$

PROOF. Since the assertion is obvious for $k = 1$, we may assume w.l.o.g. that $k \geqq 2$. As before let $p: \mathbb{R}^k \to \mathbb{R}^{k-1}$ be defined for all $\mathbf{x} = (x_1, \cdots, x_k) \in \mathbb{R}^k$ by $p(\mathbf{x}) = (x_1, \cdots, x_{k-1})$. By assumption there exist pairwise disjoint intervals I_j, $j = 0, 1, \cdots, i+1$, such that

$$\pi \equiv \{I_{j_1} \times \cdots \times I_{j_k} : 0 \leqq j_r \leqq i+1 \ \text{ for all } \ r = 1, \cdots, k\} .$$

Let

$$\bar{\pi} \equiv \{I_{j_1} \times \cdots \times I_{j_{k-1}} : 0 \leqq j_r \leqq i+1 \ \text{ for all } \ r = 1, \cdots, k-1\} .$$

Then for each $j \in \{0, 1, \cdots, i+1\}$ and every $C \in \mathscr{C}_k$ with $C \subset \mathbb{R}^{k-1} \times I_j$ one obtains

$$p(W(C, \pi)) = W(p(C), \bar{\pi}) . \tag{8}$$

Indeed, for each $B \in \bar{\pi}$ one has that

$$B \cap p(C) \neq \emptyset \quad \text{if and only if} \quad (B \times I_j) \cap C \neq \emptyset .$$

In the case of $V(C, \pi)$ apply Lemma 2.2 to obtain $C' \in \mathscr{C}_{k-1}$ such that

$$V(C, \pi) = V(C' \times I_j, \pi) .$$

Since for each $B \in \bar{\pi}$

$$B \subset p(C' \times I_j) = C' \quad \text{if and only if } B \times I_j \subset C' \times I_j ,$$

one obtains

$$\begin{aligned} p(V(C, \pi)) &= p(V(C' \times I_j, \pi)) \\ &= V(C', \bar{\pi}) . \end{aligned} \tag{9}$$

Thus, if one defines T_W and T_V for each $C \in \mathscr{C}_k$ by

$$T_W(W(C, \pi)) \equiv (p(W(C \cap (\mathbb{R}^{k-1} \times I_j), \pi)))_{j=0,\ldots,i+1}$$

and

$$T_V(V(C, \pi)) \equiv (p(V(C \cap (\mathbb{R}^{k-1} \times I_j), \pi)))_{j=0,\ldots,i+1} ,$$

then, by (8) and (9), one obtains natural injections

$$T_W : \mathscr{W}(\pi) \to \textstyle\prod_{j=0}^{i+1} \mathscr{W}(\bar{\pi})$$

and

$$T_V : \mathscr{V}(\pi) \to \textstyle\prod_{j=0}^{i+1} \mathscr{V}(\bar{\pi}) .$$

Since by definition $\bar{\pi}$ is of type $\mathscr{F}_{k-1}(i)$ the assertion of Lemma 2.3 follows now by induction on k. □

To state the next lemma let μ_1 be an arbitrary probability measure on $\mathscr{B}_1$. Suppose that $\mu_1(\{x\}) = 0$ for each $x \in \mathbb{R}$. Then, for each $n \in \mathbb{N}$, there exist $2^n - 1$ distinct points

$$-\infty < t_{n,0} < t_{n,1} < \cdots < t_{n,2^n-2} < +\infty$$

such that

$$\mu_1(-\infty, t_{n,0}] = 2^{-n} ,$$

$$\mu_1(t_{n,j}, t_{n,j+1}] = 2^{-n} , \qquad j = 0, \cdots, 2^n - 3 ,$$

and

$$\mu_1(t_{n,2^n-2}, \infty) = 2^{-n} .$$

Let $\pi^n \in \Pi(\mathscr{B}_k)$ be the corresponding partition of type $\mathscr{F}_k(2^n - 2)$. Then it is easy to see that the points $t_{n,j}$ may be chosen so that

$$\pi^n \prec \pi^{n+1} \qquad \text{for all} \quad n \in \mathbb{N} .$$

Put $\nu' \equiv \bigotimes_{i=1}^{n} \mu_1$ and

$$\pi_C{}^n \equiv \{B \in \pi^n : B \cap C \neq \emptyset \neq B - C\} , \qquad C \in \mathscr{C}_k .$$

LEMMA 2.4. *In the above notation there exists finite $c_k' > 0$ (depending only on k) such that*

$$\sup_{C \in \mathscr{C}_k} \nu'(\textstyle\bigcup \pi_C{}^n) \leqq c_k' 2^{-n} . \tag{10}$$

PROOF. Put $K \equiv (t_{n,0}, t_{n,2^n-2}]^k$. Since $\nu'(I) = 2^{-nk}$, $I \in \pi^n$, and $\nu'(K^c) \leqq 2k2^{-n}$ it obviously suffices to show the existence of some real number c_k', which depends only on k, such that

$$\sup_{C \in \mathscr{C}_k, C \subset K} |\pi_C^n| \leqq c_k' 2^{n(k-1)} .$$

For this, put

$$I_j \equiv (t_{n,j}, t_{n,j+1}] , \quad j = 0, \cdots, 2^n - 3 \quad \text{and}$$
$$J \equiv \{0, 1, \cdots, 2^n - 3\}^k .$$

For each $\mathbf{j} = (j_1, \cdots, j_k) \in J$ and every $\mathbf{h} = (h_1, \cdots, h_k) \in \{0, 1\}^k$ let

$$N(\mathbf{j}, \mathbf{h}) = \{p \in \mathbb{N} : (j_1 + p(-1)^{h_1}, \cdots, j_k + p(-1)^{h_k}) \in J\} .$$

Put

$$J_1 \equiv \{(j_1, \cdots, j_k) \in J : j_r \notin \{0, 2^n - 3\} \text{ for all } r = 1, \cdots, k\} .$$

Since $|J - J_1| \leqq 2k(2^n - 2)^{k-1}$, it remains to show that for some constant c_k'' and every convex measurable subset C of K

$$|\tilde{\pi}_C^n| \leqq c_k''(2^n - 2)^{k-1} ,$$
$$\text{where} \quad \tilde{\pi}_C^n = \{I = I_{j_1} \times \cdots \times I_{j_k} \in \pi_C^n : (j_1, \cdots, j_k) \in J_1\} .$$

Indeed, for each $I = I_{j_1} \times \cdots \times I_{j_k} \in \tilde{\pi}_C^n$ and all $\mathbf{h} = (h_1, \cdots, h_k) \in \{0, 1\}^k$, $p \in N(\mathbf{j}, \mathbf{h})$, let

$I_{p,\mathbf{h}} = I_{j_1+p(-1)^{h_1}} \times \cdots \times I_{j_k+p(-1)^{h_k}}$; suppose furthermore that for every $\mathbf{h} \in \{0, 1\}^k$ there exists $p_\mathbf{h} \in N(\mathbf{j}, \mathbf{h})$ with $I_{p_\mathbf{h},\mathbf{h}} \cap C \neq \emptyset$. Choose any $x_\mathbf{h} \in I_{p_\mathbf{h},\mathbf{h}} \cap C$ and put $S = \{x_\mathbf{h} : \mathbf{h} \in \{0, 1\}^k\}$. Since $S \subset C$ it follows from the convexity of C that $co(S) \subset C$, whence, by $I \subset co(S)$, one obtains $I \subset C$, a contradiction to $I \in \tilde{\pi}_C^n$. Thus the above argument shows that one can find $\mathbf{h}_I = (h_1, \cdots, h_k) \in \{0, 1\}^k$ such that $I_{p,\mathbf{h}_I} \subset C^c$ for all $p \in N(\mathbf{j}, \mathbf{h}_I)$. Let p_I be the uniquely determined element of $N(\mathbf{j}, \mathbf{h}_I)$ with

$$\mathbf{j}_I \equiv (j_1 + p_I(-1)^{h_1}, \cdots, j_k + p_I(-1)^{h_k}) \in J - J_1$$

and define $F : \tilde{\pi}_n^C \to J - J_1$ by $F(I) = \mathbf{j}_I$, $I \in \tilde{\pi}_n^C$. Then it follows from the definition of F that for each

$$\mathbf{h} \in \{0, 1\}^k \quad \text{and every} \quad \mathbf{j} \in J - J_1 \quad |\{I \in \tilde{\pi}_C^n : F(I) = \mathbf{j}, \mathbf{h}_I = \mathbf{h}\}| \leqq 1 .$$

This implies that $|\tilde{\pi}_C^n| \leqq 2^k 2k(2^n - 2)^{k-1}$ and completes the proof of Lemma 2.4. □

To simplify the arguments in the following lemmas we may and do assume w.l.o.g. that the marginal components of ν in (2) are all equal to μ_1. If not replace ν by $\bar{\nu} \equiv \bigotimes_{i=1}^k (\mu_1 + \cdots + \mu_k)$, which is possible since $||d\nu/d\bar{\nu}||_{\infty,\bar{\nu}} \leqq 1$, whence $||d\mu/d\bar{\nu}||_{\infty,\bar{\nu}} \equiv R' < \infty$. Dividing through by $\mu_1(\mathbb{R}) > 0$ if necessary we may assume w.l.o.g. $\mu_1(\mathbb{R}) = 1$. In what follows k will be an arbitrary fixed integer $\geqq 2$.

Before stating the next lemmas we need some additional notations. For every $A \in \mathscr{B}_k$ put $\sigma^2(A) \equiv \mu(A)(1-\mu(A))$. For each $n \geqq 3$ let

$$\tau(n) \equiv \langle \log\log\log n \cdot \log_2 e/(k-1)\rangle \quad \text{and}$$
$$T(n) \equiv \langle (\log n - \log\log n)\log_2 e/k\rangle$$

where $\log_2 x$, $x > 0$, denotes the logarithm of x w.r.t. base 2 and $\langle x\rangle$, $x \in \mathbb{R}$, means the greatest integer $m \leqq x$.

Finally let $\pi^n \in \Pi(\mathscr{B}_k)$, $n \in \mathbb{N}$, denote the finite partition of type $\mathscr{F}_k(2^n - 2)$ obtained from μ_1 as before, and let n_0 be such that $\inf_{n \geqq n_0} \tau(n) \geqq 1$.

LEMMA 2.5. *For each $n \geqq n_0$ let A_n be defined by*

$$A_n \equiv \bigcup_{A \in \mathscr{Y}(\pi^{\tau(n)})} \{\omega \in \Omega : |\mu_n^\omega(A) - \mu(A)| \geqq 4n^{-1/k}(\log n)^{1/k}\}\,.$$

Then under the hypotheses of Theorem 1.1

$$\mathbb{P}(\limsup_{n\to\infty} A_n) = 0\,. \tag{11}$$

PROOF. For each $n \geqq n_0$ and every $A \in \mathscr{Y}(\pi^{\tau(n)})$ let

$$\begin{aligned} A_n(A) &\equiv \{\omega \in \Omega : |\mu_n^\omega(A) - \mu(A)| \geqq 4n^{-1/k}(\log n)^{1/k}\} \\ &= \{\omega \in \Omega : |\textstyle\sum_{1\leqq i\leqq n)} \chi_A(f_i(\omega)) - \mu(A))| \geqq 4n^{1-1/k}(\log n)^{1/k}\}\,. \end{aligned}$$

Apply Lemma 2.1 with $t = \sigma(A)(\log n)^{\frac{1}{2}}$ to obtain for each $A \in \mathscr{Y}(\pi^{\tau(n)})$ with $\sigma(A) > 0$

$$\begin{aligned} \mathbb{P}(A_n(A)) &\leqq 2\exp[-4n^{\frac{1}{2}-1/k}(\log n)^{\frac{1}{2}+1/k} + (\log n)(1+\tfrac{1}{2})/2] \\ &\leqq 2n\exp[-4n^{\frac{1}{2}-1/k}(\log n)^{\frac{1}{2}+1/k}]\,. \end{aligned} \tag{12}$$

Since the left-hand side of the first inequality is equal to zero if $\sigma(A) = 0$ it follows that $\mathbb{P}(A_n) \leqq |\mathscr{Y}(\pi^{\tau(n)})| \cdot 2n\exp[-4n^{\frac{1}{2}-1/k}(\log n)^{\frac{1}{2}+1/k}]$.

Apply Lemma 2.3 and the definition of $\tau(n)$ to obtain

$$|\mathscr{Y}(\pi^{\tau(n)})| \leqq 2^{2\tau(n)2^{(k-1)\tau(n)}} \leqq 2^{2\tau(n)\log\log n} \leqq (\log n)^{2\tau(n)}$$

which together with (12) implies that

$$\mathbb{P}(A_n) \leqq 2n(\log n)^{2\tau(n)}\exp[-4n^{\frac{1}{2}-1/k}(\log n)^{\frac{1}{2}+1/k}] = O(n^{-2})\,.$$

The assertion now follows from the first Borel–Cantelli lemma. □

Besides the lemma just proved, the following statement will be crucial for the proof of Theorem 1.1. First put $\delta = k(\frac{1}{2} - 1/k)/(k-1) \geqq 0$. Then

$$(k-1)(1-\delta) = k/2 \tag{13}$$

and

$$(k-1)\delta/k = \tfrac{1}{2} - 1/k\,. \tag{14}$$

For each $n \geqq n_0$ let $D(C, s)$ and $E_n(C, r, s)$ be defined for all $r > 0$ by

$$\begin{aligned} &D(C, s) \equiv C_-^{s} - C_-^{s-1}\,, \\ &E_n(C, r, s) \equiv \{\omega \in \Omega : |(\mu_n^\omega - \mu)D(C, s)| \geqq rn^{-\frac{1}{2}}s^{\frac{1}{2}}2^{(k-1)s\delta}\}\,, \\ &C \in \mathscr{C}_k;\ s = \tau(n)+1, \cdots, T(n)\,. \end{aligned}$$

Put $E_n(r) \equiv \bigcup_{s=\tau(n)+1}^{T(n)} \bigcup_{C \in \mathscr{C}_k} E_n(C, r, s)$. Then one obtains

LEMMA 2.6. *Under the hypotheses of Theorem* 1.1 *for all sufficiently large* $r > 0$

$$\mathbb{P}(\limsup_{n\to\infty} E_n(r)) = 0 . \tag{15}$$

PROOF. For each $n \geqq n_0$ let $m \in \mathbb{N}$ be such that $2^m < n \leqq 2^{m+1}$. Put

$$F_{m+1}(C, r, s) \equiv \bigcup_{1\leqq j\leqq 2^{m+1}} \{\omega \in \Omega : |\sum_{i=1}^{j} (\chi_{D(C,s)}(f_i(\omega)) - \mu(D(C, s)))| \geqq r2^{(m+1)/2}s^{\frac{1}{2}}2^{(k-1)s\delta-\frac{1}{2}}\} ,$$

$C \in \mathscr{C}_k$, $s = \tau(2^m) + 1, \cdots, T(2^{m+1})$. Then $2^m < n \leqq 2^{m+1}$ implies that, for all $s = \tau(n) + 1, \cdots, T(n)$, $E_n(C, r, s) \subset F_{m+1}(C, r, s)$, whence

$$E_n(r) \subset F_{m+1}(r) \tag{16}$$

where $F_{m+1}(r) \equiv \bigcup_{s=\tau(2^m)+1}^{T(2^{m+1})} \bigcup_{C\in\mathscr{C}_k} F_{m+1}(C, r, s)$.

Thus to prove (15) it suffices to show according to the first Borel–Cantelli lemma that

$$\sum_{m\geqq m_0} \mathbb{P}(F_{m+1}(r)) < \infty , \quad \text{where} \quad m_0 \quad \text{is chosen such that} \quad 2^{m_0} \geqq n_0 . \tag{17}$$

To this extent, let $m \geqq m_0$, $C \in \mathscr{C}_k$ and $\tau(2^m) < s \leqq T(2^{m+1})$ be given. We will apply Lemma 2.1 with

$$t = \sigma(D(C, s))s^{\frac{1}{2}}2^{ks/2} = \sigma(D(C, s))s^{\frac{1}{2}}2^{(k-1)(1-\delta)s}$$

in order to obtain an upper bound for $\mathbb{P}(F_{m+1}(C, r, s))$, which is possible since

$$t \leqq \sigma(D(C, s))T(2^{m+1})^{\frac{1}{2}}2^{kT(2^{m+1})/2} \leqq \sigma(D(C, s))2^{(m+1)/2} .$$

So by Lemma 2.1

$$\mathbb{P}(F_{m+1}(C, r, s)) \leqq 2\exp[-r2^{-\frac{1}{2}}s2^{(k-1)s} + 3\mu(D(C, s))s2^{ks}/4] . \tag{18}$$

In order to find an upper bound for

$$\Delta_s \equiv \sup_{C\in\mathscr{C}_k} \mu(D(C, s))$$

we remark that by (2), (3), and Lemma 2.4 for some finite c_k'

$$\Delta_s \leqq \sup_{C\in\mathscr{C}_k} \mu(C_+^{s-1} - C_-^{s-1}) \leqq Rc_k'2^{-(s-1)} .$$

Put $c = 2Rc_k'$ and let $r_0 > 0$ be defined by the equality $-5 = -r_0 2^{-\frac{1}{2}} + 3c/4$. Then, by (18), for all $r \geqq r_0$ $\mathbb{P}(F_{m+1}(C, r, s)) \leqq 2\exp[-5s2^{(k-1)s}]$.

On the other hand, it follows from Lemma 2.3 that for each s there exist at most $2^{4s2^{(k-1)s}}$ sets of the form $D(C, s)$, $C \in \mathscr{C}_k$. Thus

$$\begin{aligned} \mathbb{P}(F_{m+1}(r)) &\leqq 2 \sum_{s=\tau(2^m)+1}^{T(2^{m+1})} \exp[-s2^{(k-1)s}] \\ &\leqq 2 \sum_{s=\tau(2^m)+1}^{T(2^{m+1})} \exp[-s2^{(k-1)\tau(2^m)}] \\ &\leqq 2 \sum_{s=\tau(2^m)+1}^{T(2^{m+1})} \exp[-s2^{1-k}\log\log 2^m] \\ &= O(\exp[-\tau(2^m)2^{1-k}\log\log 2^m]) = O(m^{-2}) . \end{aligned} \tag{19}$$

This proves (17) and completes the proof of Lemma 2.6. □

The proof of the next lemma is, except for details, the same as that given for Lemma 2.6 and will be omitted.

LEMMA 2.7. *For each $n \geqq n_0$ and $r > 0$ let $G_n(r)$ be defined by*

$$G_n(r) \equiv \bigcup_{C \in \mathscr{C}_k} \{\omega \in \Omega : |\mu_n^\omega(C_+^{T(n)} - C_-^{T(n)}) - \mu(C_+^{T(n)} - C_-^{T(n)})| \geqq rn^{-\frac{1}{2}}T(n)^{\frac{1}{2}}2^{(k-1)T(n)\delta}\}.$$

Then under the hypotheses of Theorem 1.1 for all sufficiently large $r > 0$

$$\mathbb{P}(\limsup_{n\to\infty} G_n(r)) = 0. \tag{20}$$

We are now in the position to give the proof of Theorem 1.1. Put

$$A_0 = \limsup_{n\to\infty} A_n,$$
$$E_0 \equiv \limsup_{n\to\infty} E_n(r),$$
$$G_0 \equiv \limsup_{n\to\infty} G_n(r),$$

where $r > 0$ is chosen large enough in order to ensure (15) and (20). Let $H_0 = A_0^c \cap E_0^c \cap G_0^c$. Then by (11), (15), and (20) $\mathbb{P}(H_0) = 1$. Thus, in order to show (4), it suffices to prove the following

LEMMA 2.8. *Suppose that $k \geqq 3$; then in the above notation for each $\omega \in H_0$*

$$D_n(\omega) = O((\log n/n)^{1/k}). \tag{21}$$

PROOF. For each $\omega \in H_0$ and every $n \geqq n_0$ let

$$c_1(n) = c_1(n, \omega) \equiv \sup_{C \in \mathscr{C}_k} |\mu_n^\omega(C_-^{\tau(n)}) - \mu(C_-^{\tau(n)})|,$$
$$c_2(n) = c_2(n, \omega) \equiv \sum_{s=\tau(n)+1}^{T(n)} \sup_{C \in \mathscr{C}_k} |\mu_n^\omega(D(C, s)) - \mu(D(C, s))|,$$
$$c_3(n) = c_3(n, \omega) \equiv \sup_{C \in \mathscr{C}_k} |\mu_n^\omega(C_+^{T(n)} - C_-^{T(n)}) - \mu(C_+^{T(n)} - C_-^{T(n)})|$$

and

$$c_4(n) = c_4(n, \omega) \equiv \sup_{C \in \mathscr{C}_k} |\mu_n^\omega(C - C_-^{T(n)}) - \mu(C - C_-^{T(n)})|.$$

Then

$$D_n(\omega) \leqq c_1(n) + c_2(n) + c_4(n). \tag{22}$$

Since by (3) and (10)

$$\sup_{C \in \mathscr{C}_k} \mu(C_+^{T(n)} - C_-^{T(n)}) \leqq Rc_k' 2^{-T(n)},$$

one obtains

$$\sup_{C \in \mathscr{C}_k} \mu_n^\omega(C - C_-^{T(n)}) \leqq \sup_{C \in \mathscr{C}_k} \mu_n^\omega(C_+^{T(n)} - C_-^{T(n)}) \leqq c_3(n) + Rc_k' 2^{-T(n)},$$
$$\text{whence} \quad c_4(n) \leqq c_3(n) + 2Rc_k' 2^{-T(n)}.$$

Next let $n_1 \geqq n_0$ be chosen so large that for each $n \geqq n_1$

$$c_1(n) < 4(\log n/n)^{1/k},$$
$$c_2(n) < \sum_{s=\tau(n)+1}^{T(n)} rn^{-\frac{1}{2}}s^{\frac{1}{2}}2^{(k-1)s\delta} \leqq rn^{-\frac{1}{2}}T(n)^{\frac{1}{2}} \sum_{s=1}^{T(n)} 2^{(k-1)s\delta},$$

and

$$c_3(n) < rn^{-\frac{1}{2}}T(n)^{\frac{1}{2}}2^{(k-1)T(n)\delta} \leqq r(\log n/n)^{1/k}.$$

Since $\delta > 0$ iff $k \geqq 3$ it follows from (22) that

$$\begin{aligned} D_n(\omega) &\leqq (4 + r + 4Rc_k')(\log n/n)^{1/k} + (rn^{-\frac{1}{2}}T(n)^{\frac{1}{2}} \textstyle\sum_{s=1}^{T(n)} 2^{(k-1)s\delta}) \\ &= O((\log n/n)^{1/k} + n^{-\frac{1}{2}}T(n)^{\frac{1}{2}}2^{(k-1)\delta T(n)}) = O((\log n/n)^{1/k}). \end{aligned} \tag{23}$$

This proves (21) and completes the proof of Theorem 1.1. □

3. Proof of Theorem 1.2. The main arguments are nearly the same as those applied in proving Theorem 1.1. To begin with, let k be an arbitrary fixed integer $\geqq 2$.

Concerning the validity of (4) the proof of Theorem 1.1 heavily depended, for each $n \in \mathbb{N}$, on the existence of finite $\mathscr{C}_k$-approximating systems $\mathscr{V}(\pi^n)$ and $\mathscr{W}(\pi^n)$ (cf. Lemma 2.3 and Lemma 2.4). Now, as it may be clear from the arguments in the preceding chapter, it is possible to improve (4) if one could replace $\mathscr{V}(\pi^n)$ and $\mathscr{W}(\pi^n)$ by $\mathscr{C}_k$-approximating systems with smaller cardinality. The following two lemmas will go in this direction. First let $K = [a, b]^k$, $a, b \in Z$, be an arbitrary large cube with $\mu(K^c) = 0$. For each $C \in \mathscr{C}_k \cap K = \{C \in \mathscr{C}_k : C \subset K\}$ and every $\varepsilon > 0$ let

$$C^\varepsilon \equiv \{\mathbf{x} \in \mathbb{R}^k : K_\varepsilon(\mathbf{x}) \cap C \neq \emptyset\} \qquad \text{and} \qquad C_\varepsilon \equiv \{\mathbf{x} \in \mathbb{R}^k : K_\varepsilon(\mathbf{x}) \subset C\}$$

denote the outer and inner (convex) ε-parallel sets, respectively, where $K_\varepsilon(\mathbf{x}) \equiv \{\mathbf{y} \in \mathbb{R}^k : \|\mathbf{y} - \mathbf{x}\| \leqq \varepsilon\}$ means the solid sphere with center $\mathbf{x}$ and radius ε w.r.t. the Euclidean norm.

The next lemma plays the same role for (6) as Lemma 2.3 for (4).

LEMMA 3.1. *For each $n \in \mathbb{N}$ there exist finite subfamilies $\mathscr{V}^n$ and $\mathscr{W}^n$ of $\mathscr{B}_k \cap K$ such that for each $C \in \mathscr{C}_k \cap K$ there exist $V^n(C) \in \mathscr{V}^n$ and $W^n(C) \in \mathscr{W}^n$ with*

$$V^n(C) \subset V^{n+1}(C) \subset C \subset W^{n+1}(C) \subset W^n(C)\,, \tag{24}$$

$$W^n(C) - V^n(C) \subset C^{2^{-n}} - C_{2^{-n}}\,. \tag{25}$$

Furthermore for some finite $L > 0$

$$\max(|\mathscr{V}^n|, |\mathscr{W}^n|) \leqq 2^{Ln2^{n(k-1)/2}}\,.$$

As will be clear soon, $\mathscr{V}^n$ and $\mathscr{W}^n$ play the same role for Theorem 1.2 as $\mathscr{V}(\pi^n)$ and $\mathscr{W}(\pi^n)$ for Theorem 1.1.

PROOF. For the proof of the above statement we will follow the proof of Dudley (1974), Theorem 4.1, in order to obtain for each $0 < \varepsilon < \pi/4$ a finite subfamily $\mathscr{W}_\varepsilon$ of $\mathscr{B}_k \cap K$ such that for each $C \in \mathscr{C}_k \cap K$ there exists $W(C, \varepsilon) \in \mathscr{W}_\varepsilon$ with

$$C \subset W(C, \varepsilon) \subset C^\varepsilon\,. \tag{26}$$

Furthermore for some finite $L_0 > 0$,

$$|\mathscr{W}_\varepsilon| \leqq \exp(-L_0 \varepsilon^{(1-k)/2} \ln \varepsilon)\,. \tag{27}$$

Let

$$W^n(C) \equiv \textstyle\bigcap_{r=1}^n W(C, 2^{-r})\,, \qquad V^n(C) \equiv \bigcup_{r=1}^n W(C_{2^{-r}}, 2^{-r})\,, \qquad n \in \mathbb{N}\,.$$

Then, if one puts

$$\mathscr{V}^n \equiv \{V^n(C) : C \in \mathscr{C}_k \cap K\}\,, \qquad \mathscr{W}^n \equiv \{W^n(C) : C \in \mathscr{C}_k \cap K\}\,,$$

(24) and (25) immediately follow from (26) and the inclusion $(C_\varepsilon)^\varepsilon \subset C$, $\varepsilon > 0$.

On the other hand, since

$$\sum_{r=1}^{n} r2^{r(k-1)/2} \leqq n \sum_{r=1}^{n} 2^{r(k-1)/2} \leqq 4n2^{n(k-1)/2},$$

it follows from (27) that for some $L > 0$,

$$\max(|\mathscr{V}^n|, |\mathscr{W}^n|) \leqq 2^{Ln2^{n(k-1)/2}}. \qquad \square$$

For each $C \in \mathscr{C}_k \cap K$ and $n \in \mathbb{N}$ put $C_+^n \equiv W^n(C)$ and $C_-^n \equiv V^n(C)$. Let ν denote the uniform distribution over K.

LEMMA 3.2. *There exists $c_k'' > 0$ (depending only on k) such that*

$$\sup_{C \in \mathscr{C}_k \cap K} \nu(C_+^n - C_-^n) \leqq c_k'' 2^{-n}.$$

PROOF. For each $n \geqq 4$ divide $[a, b]$ into pairwise disjoint intervals of equal length $\varepsilon(n) \equiv 2^{-n}$. As in the preceding chapter let $\pi^n \in \Pi(\mathscr{B}_k)$ denote the corresponding partition of type $\mathscr{F}_k(2^n - 2)$. We are going to show that for each $C \in \mathscr{C}_k \cap K$

$$C^{\varepsilon(n)} - C_{\varepsilon(n)} \subset \pi^{n-3}_{C_{\varepsilon(n)}} \cup \pi^{n-3}_{C\varepsilon(n)}. \tag{28}$$

Indeed, for each $x \in C^{\varepsilon(n)} - C_{\varepsilon(n)}$ let A_x denote the uniquely determined element of π^{n-3} with $x \in A_x$. In particular

$$A_x \cap C^{\varepsilon(n)} - C_{\varepsilon(n)} \neq \emptyset.$$

Suppose now that $A_x \subset C^{\varepsilon(n)} - C_{\varepsilon(n)}$, which is possible only for bounded A_x. Let y be the midpoint of A_x. Since $y \notin C_{\varepsilon(n)}$ and $K_{\varepsilon(n)}(y) \subset A_x$ there exists $z \in K_{\varepsilon(n)}(y) - C$ for which, by convexity of C, one can find $u \in K_{2\varepsilon(n)}(z) - C^{\varepsilon(n)}$. But $K_{2\varepsilon(n)}(z) \subset A_x$, so that $u \in A_x - C^{\varepsilon(n)}$, a contradiction to our assumption. Thus $(A_x - C^{\varepsilon(n)}) \cup (A_x \cap C_{\varepsilon(n)}) \neq \emptyset$, whence (28) holds true.

The assertion follows now from (28) and Lemma 2.4 with $c_k'' = 16c_k'$. $\square$

The proof of the next statement is, except for details, the same as that given for Lemma 2.5. For each $n \geqq 3$ let $\tau(n)$ and $T(n)$ redefined as

$$\tau(n) \equiv \langle (2 \log\log\log n \cdot \log_2 e)/(k-1) \rangle \quad \text{and}$$
$$T(n) \equiv \langle 2(\log n - \log\log n) \log_2 e/(k+1) \rangle.$$

Put

$$A_n = \bigcup_{A \in \mathscr{V}^{\tau(n)}} \{\omega \in \Omega : |\mu_n^\omega(A) - \mu(A)| \geqq 4(\log n/n)^{2/(k+1)}\}, \quad n \geqq n_0;$$

then

LEMMA 3.3. *Under the hypotheses of Theorem* 1.2

$$\mathbb{P}(\limsup_{n\to\infty} A_n) = 0. \tag{29}$$

The next lemma plays the same role for Theorem 1.2 as Lemma 2.6 does for Theorem 1.1.

To begin with, for each $k \geqq 3$ let

$$\delta \equiv (k-3)/2(k-1) \geqq 0.$$

Then $(k-1)(1-\delta)/2 = (k+1)/4$ and $(k-1)\delta/(k+1) - \frac{1}{2} = -2/(k+1)$. For each $n \geqq n_0$ let $E_n(C, r, s)$ be defined for all $r > 0$ by

$$E_n(C, r, s) \equiv \{\omega \in \Omega : |\mu_n^{\omega}(D(C, s)) - \mu(D(C, s))| \geqq rn^{-\frac{1}{2}}s^{\frac{1}{2}}2^{(k-1)s\delta/2}\}$$
$$C \in \mathscr{C}_k \cap K ; \quad s = \tau(n) + 1, \cdots, T(n) .$$

Put

$$E_n(r) \equiv \bigcup_{s=\tau(n)+1}^{T(n)} \bigcup_{C \in \mathscr{C}_k \cap K} E_n(C, r, s) ;$$

then one obtains

LEMMA 3.4. *Under the hypotheses of Theorem* 1.2 *for all sufficiently large* $r > 0$

$$\mathbb{P}(\limsup_{n\to\infty} E_n(r)) = 0 . \tag{30}$$

PROOF. As in the proof of Lemma 2.6 it will be sufficient to show that for all large $r > 0$

$$\sum_{m \geqq m_0} \mathbb{P}(F_{m+1}(r)) < \infty ,$$

where

$$F_{m+1}(r) \equiv \bigcup_{s=\tau(2^m)+1}^{T(2^{m+1})} \bigcup_{C \in \mathscr{C}_k \cap K} F_{m+1}(C, r, s)$$

and

$$F_{m+1}(C, r, s) \equiv \bigcup_{1 \leqq j \leqq 2^{m+1}} \{\omega \in \Omega : |\sum_{i=1}^{j} (\chi_{D(C,s)}(f_i(\omega)) - \mu(D(C, s)))|$$
$$\geqq r2^{(m+1)/2}s^{\frac{1}{2}}2^{((k-1)s\delta-1)/2}\}$$
$$C \in \mathscr{C}_k \cap K ; \quad s = \tau(2^m) + 1, \cdots, T(2^{m+1}) .$$

To show $\sum_{m \geqq m_0} \mathbb{P}(F_{m+1}(r)) < \infty$, apply Lemma 2.1 for each $C \in \mathscr{C}_k \cap K$ and every s such that $\tau(2^m) < s \leqq T(2^{m+1})$ with

$$t \equiv \sigma(D(C, s))s^{\frac{1}{2}}2^{(k+1)s/4} = \sigma(D(C, s))s^{\frac{1}{2}}2^{(k-1)(1-\delta)s/2}$$

to obtain an upper bound for $\mathbb{P}(F_{m+1}(C, r, s))$. Since by definition of $T(2^{m+1})$ $t \leqq \sigma(D(C, s))2^{(m+1)/2}$, it follows from Lemma 2.1 that

$$\mathbb{P}(F_{m+1}(C, r, s)) \leqq 2 \exp[-r2^{-\frac{1}{2}}s2^{(k-1)s/2} + 3\mu(D(C, s))s2^{(k+1)s/2}/4] .$$

Since by (5) and Lemma 3.2

$$\Delta_s \equiv \sup_{C \in \mathscr{C}_k \cap K} \mu(D(C, s)) \leqq Rc_k''2^{-(s-1)} ,$$

the above considerations show that we may apply the techniques of Lemma 2.6 in order to show $\mathbb{P}(F_{m+1}(r)) = O(m^{-2})$, provided $r \geqq r_0$, where $r_0 > 0$ is defined by the equality $-(L+1) = -r_0 2^{-\frac{1}{2}} + 6Rc_k''/4$ and $L > 0$ is the same constant as occurring in Lemma 3.1. □

The following lemma corresponds to Theorem 1.2 as Lemma 2.7 to Theorem 1.1.

LEMMA 3.5. *For each* $n \geqq n_0$ *let* $G_n(r)$, $r > 0$, *be defined by*

$$G_n(r) \equiv \bigcup_{C \in \mathscr{C}_k \cap K} \{\omega \in \Omega : |\mu_n^{\omega}(C_+^{T(n)} - C_-^{T(n)}) - \mu(C_+^{T(n)} - C_-^{T(n)})|$$
$$\geqq rn^{-\frac{1}{2}}T(n)^{\frac{1}{2}}2^{(k-1)T(n)\delta/2}\} .$$

Then under the hypotheses of Theorem 1.2 *for all sufficiently large r*

(31) $$\mathbb{P}(\limsup_{n\to\infty} G_n(r)) = 0 .$$

Now, by (29), (30), and (31), the proof of Theorem 1.2 may be finished by using the same arguments as in Lemma 2.8. □

4. Proof of Theorem 1.3. In order to obtain sharp results on the order of magnitude of the logarithmic factor in Theorem 1.3 the following obvious lemma will be useful.

LEMMA 4.1. *Let* $f_1, \cdots, f_n$, $n \in \mathbb{N}$, *be* i.i.d. *random variables on some p-space* $(\Omega, \mathscr{F}, \mathbb{P})$ *with values in some measurable space* $(X, \mathscr{A})$. *Suppose that* $S_1, \cdots, S_m$, $m \leqq n$, *are pairwise disjoint sets in* $\mathscr{A}$ *with* $\mathbb{P}(f_1 \in S_j) = x$ *for all* $j = 1, \cdots, m$. *Let*

$$S \equiv \bigcap_{j=1}^m \bigcup_{i=1}^n \{f_i \in S_j\} .$$

Then

(32) $$\mathbb{P}(S) = \sum_{j=0}^m (-1)^j \binom{m}{j} (1 - jx)^n .$$

Besides the lemma just cited, the following simple remark is essential for the proof of Theorem 1.3.

LEMMA 4.2 (cf. Zaremba (1970), Lemma 3.1). *Let* $\omega = \{\mathbf{x}_i\}_{i\in\mathbb{N}}$ *be an infinite sequence of points in* $\mathbb{R}^k$. *Suppose that for given* $q \in \mathbb{N}$ *and* $n \in \mathbb{N}$ *there exists a measurable convex set* C_0 *such that* ∂C_0 (= *boundary of* C_0) *contains q points of* $\{\mathbf{x}_1, \cdots, \mathbf{x}_n\}$. *Then*

(33) $$D_n(\omega) \geqq q/2n$$

for all $\mu \in ca_+(\mathbb{R}^k, \mathscr{B}_k)$ *fulfilling* $\sup_{C\in\mathscr{C}_k} \mu(\partial C) = 0$.

We are now in the position to give the

PROOF OF THEOREM 1.3. Let $K' \equiv K_{\frac{1}{2}}(\mathbf{1/2})$ be the sphere with center $\mathbf{1/2} = (\frac{1}{2}, \cdots, \frac{1}{2}) \in \mathbb{R}^k$ and radius $\frac{1}{2}$. Then for each $0 < \theta < \pi/2$ there exist $m = m(\theta)$ spherical caps in K' with pairwise disjoint interior sets $S_1(\theta), \cdots, S_{m(\theta)}(\theta)$ of equal Lebesgue measure $\mu(\theta)$, where

(34) $$m(\theta) \sim \theta^{1-k}$$

(35) $$\mu(\theta) \sim c\theta^{k+1} \quad \text{as} \quad \theta \to 0 , \quad \text{where} \quad c > 0 .$$

(Cf. Zaremba (1970), pages 133–134.) By (35) there exists $\theta_0 > 0$ and $c' > 0$ such that

(36) $$\mu(\theta) \geqq c'\theta^{k+1} \quad \text{provided} \quad \theta < \theta_0 .$$

Choose $c'' > 0$ so large that

(37) $$(k - 1)/(k + 1) - c'c'' < -1 .$$

Then, for each $n \in \mathbb{N}$, let θ_n be defined by

$$\theta_n \equiv (c''n^{-1} \log n)^{1/(k+1)} .$$

Put

$$S_n \equiv \bigcap_{j=1}^{m(\theta_n)} \bigcup_{i=1}^{n} \{f_i \in S_j(\theta_n)\} .$$

Then it follows from (32) that

$$\mathbb{P}(S_n) = \sum_{j=0}^{m(\theta_n)} (-1)^j \binom{m(\theta_n)}{j} (1 - j\mu(\theta_n))^n \geqq 2 - (1 + e^{-n\mu(\theta_n)})^{m(\theta_n)} , \quad \text{i.e.,}$$

$$\text{(38)} \qquad \mathbb{P}(S_n^c) \leqq (1 + \exp(-n\mu(\theta_n)))^{m(\theta_n)} - 1 \leqq \exp(m(\theta_n)\exp(-n\mu(\theta_n))) - 1 .$$

Using (34) and (36) one obtains $n_0 \in \mathbb{N}$ and $c_0 > 0$ such that

$$\begin{aligned} m(\theta_n)e^{-n\mu(\theta_n)} &\leqq c_0 \theta_n^{1-k} \exp(-n\theta_n^{k+1} c') \\ &\leqq n^{(k-1)/(k+1) - c'c''} , \quad \text{provided} \quad n \geqq n_0 . \end{aligned}$$

Apply (37) to obtain

$$\text{(39)} \qquad \sum_{n \in \mathbb{N}} m(\theta_n) \exp(-n\mu(\theta_n)) < \infty .$$

In particular $\lim_{n\to\infty} m(\theta_n) \exp(-n\mu(\theta_n)) = 0$ whence

$$m(\theta_n) \exp(-n\mu(\theta_n)) \sim \exp(m(\theta_n) \exp(-n\mu(\theta_n))) - 1 .$$

So, by (38) and (39) $\sum_{n\geqq 1} \mathbb{P}(S_n^c) < \infty$, whence by the first Borel–Cantelli lemma

$$\text{(40)} \qquad \mathbb{P}(\limsup_{n\to\infty} S_n^c) = 0 \qquad \text{and} \qquad \mathbb{P}(\liminf_{n\to\infty} S_n) = 1 .$$

On the other hand, we may infer from the definition of S_n that for each $\omega \in S_n$, $n \in \mathbb{N}$, and every $j \in \{1, \cdots, m(\theta_n)\}$ there exists at least one $i_j \in \{1, \cdots, n\}$ with $f_{i_j}(\omega) \in S_j(\theta_n)$. Let

$$C_n(\omega) \equiv co(\{f_{i_j}(\omega) : j = 1, \cdots, m(\theta_n)\}) .$$

Then it follows from geometrical considerations that $f_{i_j}(\omega) \in \partial C_n(\omega)$ for all $j = 1, \cdots, m(\theta_n)$. Using (33) and (34) one obtains that for all sufficiently large $n \in \mathbb{N}$

$$\begin{aligned} D_n(\omega) &\geqq c''' n^{-1} (n/\log n)^{(k-1)/(k+1)} \\ &= c''' n^{-2/(k+1)} (\log n)^{(1-k)/(k+1)} \end{aligned}$$

where $c''' > 0$ is some constant depending only on k. The assertion of Theorem 1.3 now is an easy consequence of (40). □

5. Proof of Theorem 1.4. First we remark that concerning the proof of Lemma 2.6 and Lemma 3.4 the events $F_m(r)$ have been introduced only to cover the case $k = 2$ and $k = 3$, respectively. For example, in the case of Theorem 1.1, by redefining $\tau(n)$ as

$$\tau(n) \equiv \langle \log\log n \cdot \log_2 e/(k-1) \rangle$$
$$(\tau(n) \equiv \langle 2 \log\log n \cdot \log_2 e/(k-1) \rangle \quad \text{in the case of Theorem 1.2}) ,$$

it would be possible to show directly by using the same arguments (e.g., cf. (19)) that for each $k \geqq 3$ ($k \geqq 4$ in the case of Theorem 1.2) and all large $r > 0$

$$\text{(41)} \qquad \mathbb{P}(A_n) , \qquad \mathbb{P}(E_n(r)) \qquad \text{and} \qquad \mathbb{P}(G_n(r)) \quad \text{are} \quad O(n^{-2}) .$$

To prove the first statement in Theorem 1.4 we remark (cf. the proof of Lemma 2.8) that for each fixed $r > 0$ fulfilling (41) there exists $r_0 > 0$ such that

$$\{\omega \in \Omega : \alpha_n D_n(\omega) \geqq r_0\} \subset A_n \cup E_n(r) \cup G_n(r)$$

which implies that

$$\begin{aligned} \mathbb{E}_{\mathbb{P}}(\alpha_n D_n(\cdot)) &= \textstyle\int_{\{\alpha_n D_n < r_0\}} \alpha_n D_n(\omega)\, d\mathbb{P}(\omega) + \int_{\{\alpha_n D_n \geqq r_0\}} \alpha_n D_n(\omega)\, d\mathbb{P}(\omega) \\ &\leqq r_0 + \alpha_n \mathbb{P}(\{\alpha_n D_n \geqq r_0\}) \\ &\leqq r_0 + \alpha_n(\mathbb{P}(A_n) + \mathbb{P}(E_n(r)) + \mathbb{P}(G_n(r))) \\ &= r_0 + \alpha_n O(n^{-2}) \to r_0 \quad \text{as} \quad n \to \infty . \end{aligned}$$

Using the same arguments as in the proof of Theorem 1.3, the last assertion immediately follows from Chebyshev's inequality and $\lim_{n\to\infty} \mathbb{P}(S_n) = 1$. □

Finally, to prove the last statement of the first chapter, we remark that (23) may equally well be applied to the case $k = 2$ (where $\delta = 0$), i.e., under the assumptions of Theorem 1.1 for $\mathbb{P}$-almost all $\omega \in \Omega$

$$D_n(\omega) = O((\log n/n)^{\frac{1}{2}}) + O(T(n)^{\frac{3}{2}}/n^{\frac{1}{2}}) = O((\log n)^{\frac{3}{2}}/n^{\frac{1}{2}}) .$$

As to the mean Glivenko–Cantelli convergence, the proof will be, except for some minor modifications, the same as that given for Theorem 1.4 and may be omitted here.

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MATHEMATISCHES INSTITUT
RUHR-UNIVERSITÄT BOCHUM
UNIVERSITÄTSSTRASSE 150, GEB NA
463 BOCHUM, W. GERMANY