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ON THE INDIVIDUAL ERGODIC THEOREM FOR SUBSEQUENCES[1]

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We define a concept of *saturation* for a sequence of integers $\{k_j\}$. In the main theorem we prove that if $\{k_j\}$ saturates and T is any weakly mixing measure-preserving transformation on an arbitrary probability space, then there exists a dense set $\mathscr{D}_T \subset L^2$ such that for $f \in \mathscr{D}_T$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} f(T^{k_j}x) = E(f) \quad \text{a.e.}$$

This has the following application to probability theory:

Let $Y_1, Y_2, \cdots$ be independent and identically distributed positive (or negative) integer-valued random variables with $E(Y_1) < \infty$. Let

$$k_j(\omega) = \sum_{l=1}^{j} Y_l(\omega) \qquad j = 1, 2, \cdots.$$

Then there exists a set C of probability one such that for $\omega \in C$ and for any weakly mixing measure preserving transformation T on an arbitrary probability space

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} f(T^{k_j(\omega)}x) = E(f) \quad \text{a.e.}$$

for all $f \in L^1$.

1. Introduction. Let $\{k_j\}$ be a sequence of integers. Suppose that for every weakly mixing transformation T on an arbitrary probability space

$$\lim_{N\to\infty} \left\| \frac{1}{N} \sum_{j=1}^{N} f(T^{k_j}x) - E(f) \right\| = 0 \tag{1.1}$$

for all $f \in L^2$. The analogue to the individual ergodic theorem would then be that (1.1) implies

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} f(T^{k_j}x) = E(f) \quad \text{a.e.} \qquad \text{for all} \quad f \in L^1. \tag{1.2}$$

This, however, is not true, as Krengel in [4] shows that there exists a sequence $\{k_j\}$ for which (1.1) holds for all mixing transformations but for which (1.2) does not hold even when we restrict ourselves to bounded functions. In Theorem (4.1) we give a condition on the sequence $\{k_j\}$ which implies:

(1.3) For every weakly mixing measure preserving transformation T on an arbitrary probability space there exists a dense subspace $\mathscr{D}_T \subset L^2$ such that for $f \in \mathscr{D}_T$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} f(T^{k_j}x) = E(f) \quad \text{a.e.}$$

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Let Z, Z^+, Z^- denote the integers, positive integers and negative integers, respectively. Let $Y_1, Y_2, \cdots$ be a sequence of independent identically distributed integervalued random variables with $E|Y_1| < \infty$ and let

$$k_j(\omega) = \sum_{l=1}^{j} Y_l(\omega)\,, \qquad j = 1, 2, \cdots .$$

In (6.3) we show that the sequence $\{k_j(\omega)\}$ almost surely satisfies (1.3). In (6.4) we show that if the state space of the Y_j's is Z^+ or Z^- then the sequence $\{k_j(\omega)\}$ almost surely satisfies (1.2).

To avoid technicalities we restrict ourselves to probability spaces. However, with minor modifications all the proofs hold also for infinite measure spaces.

2. Notation and preliminaries. Let T denote a weakly mixing invertible, measurable, measure-preserving transformation on an arbitrary probability space, U the unitary operator on L^2 associated with T, and $\{F_\lambda\}_{\lambda \in [0,2\pi]}$ the resolution of the identity for U.

Halmos proves the following lemma in [3]:

(2.1) LEMMA. *T is weakly mixing if the only eigenvalue of the induced unitary operator U is the number 1 and this eigenvalue is simple.*

The following corollary is an easy consequence of (2.1):

(2.2) COROLLARY. *If T is weakly mixing, then $(F_\lambda f, f)$ is continuous except possibly at 2π and $F(\{2\pi\})f = E(f)$.*

By a *sequence* $\{k_j\}$ we will mean a function from Z^+ to Z (i.e., not necessarily one to one). By a *subsequence* of Z or Z^+ we will mean a strictly increasing sequence in Z or Z^+.

Let J be a *closed* interval contained in the open interval $(0, 2\pi)$. We say that the sequence $\{k_j\}$ satisfies a *uniform order condition on J*, if there exists a subsequence $\{N_l\}$ of Z^+ such that

$$\sum_{l=1}^{\infty} \sup_{\alpha \in J} \left| \frac{1}{N_l} \sum_{j=1}^{N_l} e^{ik_j\alpha} \right|^2 < \infty$$

and

$$\lim_{l\to\infty} \frac{N_l}{N_{l+1}} = 1 .$$

For a sequence $\{k_j\}$ let $\mathscr{J}$ be the class of J on which $\{k_j\}$ satisfies a uniform order condition. Let $\mathscr{K} = [0, 2\pi] \setminus (\bigcup_{J \in \mathscr{J}} (\text{interior } J))$. We say that $\mathscr{J}$ *saturates* if $\mathscr{K}$ is at most countable. Note that $\mathscr{K}$ always contains the two-point set $\{0, 2\pi\}$. For a sequence $\{k_j\}$ define

$$k_N^* = \max_{1 \leq j \leq N} |k_j|\,, \qquad M(n) = \max_{-n \leq j \leq n} \sum_{l=1}^{\infty} \chi_{\{j\}}(k_l)\,.$$

Note that $M(n)$ counts the greatest number of times that an integer between $-n$ and n has been repeated by the sequence $\{k_j\}$.

3. Some auxiliary results.

(3.1) LEMMA. *If J is a closed interval contained in the interior of $[0, 2\pi]$ then $L^\infty \cap F(J)L^2$ is dense in $F(J)L^2$.*

PROOF. Let $f \in F(J)L^2$ and $\varepsilon > 0$ choose $h \in L^\infty$ such that

$$\|f - h\|^2 < \frac{\varepsilon}{2}. \tag{1}$$

By (2.2) it is clear that $(F_\gamma f, f)$ is continuous on J; therefore we can find a closed interval $\tilde{J}$ in the interior of J such that

$$\|F(\tilde{J})h - F(J)h\|^2 < \frac{\varepsilon}{2}. \tag{2}$$

Let $\phi(\lambda)$ be a twice differentiable function on $[0, 2\pi]$ such that $0 \leqq \phi(\lambda) \leqq 1$, $\phi(\lambda) = 0$ for $\lambda \in J^c$, and $\phi(\lambda) = 1$ for $\lambda \in \tilde{J}$. It is clear that $\sum_{j=-\infty}^{\infty} |\hat{\phi}(j)| < \infty$.

Let

$$g(x) = \frac{1}{2\pi} \int_0^{2\pi} \phi(\lambda)\, dF_\lambda h = \frac{1}{2\pi} \int_0^{2\pi} \sum_{-\infty}^{\infty} \hat{\phi}(j) e^{ij\lambda}\, dF_\lambda h$$

$$= \sum_{-\infty}^{\infty} \hat{\phi}(j) \frac{1}{2\pi} \int_0^{2\pi} e^{ij\lambda}\, dF_\lambda h = \sum_{-\infty}^{\infty} \hat{\phi}(j) U^j h .$$

From the properties of ϕ follows $g \in L^\infty \cap F(J)L^2$. By (2) and the properties of ϕ,

$$\|g(x) - h(x)\|^2 = \frac{1}{2\pi} \int_0^{2\pi} |\phi(\lambda) - \chi_J(\lambda)|^2\, d(F_\lambda h, h) \tag{3}$$

$$\leqq \frac{1}{2\pi} \int_0^{2\pi} \chi_{J\setminus\tilde{J}}(\lambda)\, d(F_\lambda h, h) \leqq \frac{\varepsilon}{2}.$$

Putting together the estimates in (1) and (3) finishes the proof.

(3.2) LEMMA. *If $\{k_j\}$ satisfies a uniform order condition on J, then for $h \in L^\infty \cap F(J)L^2$*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} h(T^{k_j}x) = E(h) \quad \text{a.e.}$$

PROOF. First, $E(h) = 0$ since J is a closed interval contained in the interior of $[0, 2\pi]$. Let $\{N_l\}$ be the subsequence of Z^+ in the definition of uniform order. For $\varepsilon > 0$ let

$$A_{l,\varepsilon} = \left\{x \,\middle|\, \left|\frac{1}{N_l} \sum_{j=1}^{N_l} h(T^{k_j}x)\right| < \varepsilon\right\} \qquad \text{for} \quad l = 1, 2, \cdots .$$

Then

$$P(A_{l,\varepsilon}^c) \leqq \frac{1}{\varepsilon^2} \int_X \left|\frac{1}{N_l} \sum_{j=1}^{l} h(T^{k_j}x)\right|^2 P(dx)$$

$$= \frac{1}{\varepsilon^2} \frac{1}{2\pi} \int_0^{2\pi} \left|\frac{1}{N_l} \sum_{j=1}^{N_l} e^{ik_j\lambda}\right|^2 d(F_\lambda h, h)$$

$$\leqq \frac{\|h\|^2}{2\pi\varepsilon^2} \sup_{\lambda \in J} \left|\frac{1}{N_l} \sum_{j=1}^{N_l} e^{ik_j\lambda}\right|^2 .$$

The last inequality is a consequence of $h \in F(J)L^2$. Since $\{k_j\}$ satisfies a uniform order condition on J,

$$\sum_{l=1}^{\infty} P(A_{l,\varepsilon}^c) \leqq \frac{\|h\|^2}{2\pi\varepsilon^2} \sum_{l=1}^{\infty} \sup_{\lambda \in J} \left| \frac{1}{N_l} \sum_{j=1}^{N_l} e^{ik_j\lambda} \right|^2 < \infty .$$

Let A_ε be the set of x contained in all except possibly finitely many of the sets $A_{l,\varepsilon}$.

By the Borel–Cantelli lemma $P(A_\varepsilon) = 1$. In Lemma (3.3) we prove that if $x \in A_\varepsilon$ then

$$\limsup_{N\to\infty} \left| \frac{1}{N} \sum_{j=1}^{N} h(T^{k_j}x) \right| < \varepsilon .$$

Now let $A = \bigcap_{k=1}^{\infty} A_{1/k}$. Then $P(A) = 1$ and for $x \in A$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} h(T^{k_j}x) = 0 .$$

(3.3) LEMMA. *Let A_ε be the set in the proof of* (3.2). *Then for $x \in A_\varepsilon$*

$$\limsup_N \left| \frac{1}{N} \sum_{j=1}^{N} h(T^{k_j}x) \right| < \varepsilon .$$

PROOF. Let $\{N_l\}$ and $\{A_{l,\varepsilon}\}$ be as defined in the proof of (3.2). If $x \in A_\varepsilon$, there exists a positive integer $n(x)$ such that $x \in A_{l,\varepsilon}$ for $l \geqq n(x)$. Let $N > N_{n(x)}$; then there exists $l \geqq n(x)$ such that $N_{l+1} \geqq N > N_l$. Therefore

$$\begin{aligned}
\left| \frac{1}{N} \sum_{j=1}^{N} h(T^{k_j}x) \right| &\leqq \left| \frac{1}{N} \sum_{j=1}^{N_l} h(T^{k_j}x) \right| + \left| \frac{1}{N} \sum_{j=N_l+1}^{N} h(T^{k_j}x) \right| \\
&\leqq \frac{N_l}{N} \left| \frac{1}{N_l} \sum_{j=1}^{N_l} h(T^{k_j}x) \right| + \|h\|_\infty \left(\frac{N_{l+1} - N_l}{N_l} \right) \\
&\leqq \left| \frac{1}{N_l} \sum_{j=1}^{N_l} h(T^{k_j}x) \right| + \|h\|_\infty \left(\frac{N_{l+1}}{N_l} - 1 \right) \\
&< \varepsilon + \|h\|_\infty \left(\frac{N_{l+1}}{N_l} - 1 \right) .
\end{aligned}$$

The last inequality follows since $x \in A_{l,\varepsilon}$ for $l \geqq n(x)$. From the definition of uniform order $(N_{l+1}/N_l - 1) \to 0$ as $l \to \infty$, which finishes the proof.

(3.4) LEMMA. *Suppose $\{k_j\}$ satisfies an order condition on $J_1, \cdots, J_n$. Then there exist closed nonoverlapping intervals $I_1, \cdots, I_m$ such that each I_j is contained in some J_l, $\bigcup_{l=1}^{n} J_l = \bigcup_{l=1}^{m} I_l$, and $\{k_j\}$ satisfies an order condition on each I_l, $l = 1, 2, \cdots, m$.*

PROOF. The fact that we can find the partition I_l with the desired set theoretic properties is obvious; that $\{k_j\}$ satisfies an order condition on $I_1, I_2, \cdots, I_m$ follows from the fact that each I_j is contained in some J_l.

4. The main theorem.

(4.1) THEOREM. *Let $\{k_j\}$ be a sequence of integers and suppose $\mathscr{J}$ saturates. Then $\{k_j\}$ satisfies* (1.3).

PROOF. The fact that $\mathscr{D}_T$ is a subspace is obvious. Let $f \in L^2$ be such that $E(f) = 0$. Let $\mathscr{K} = \{\delta_j\}_{j=1}^{\omega}$, where ω is finite or infinite, be the set in the definition of saturation. Also $(F_\lambda f, f)$ is continuous, since $E(f) = 0$.

Fix $\varepsilon > 0$. From the continuity of $(F_\lambda f, f)$ it follows that there exists a neighborhood $\mathscr{O}_j$ for each δ_j such that

(1) $$(F(\mathscr{O}_j)f, f) < \frac{\varepsilon}{2^{j+1}} \qquad j = 1, 2, \cdots, \omega .$$

Let $\mathscr{O} = \bigcup_{j=1}^{\omega} \mathscr{O}_j$. Then it is clear that

$$\|F(\mathscr{O}^c)f - f\| < \frac{\varepsilon}{2} \cdot$$

Since $\mathscr{O}^c$ is closed and $\mathscr{O}^c \subset \bigcup_{J \in \mathscr{J}}$ (interior J) there exist intervals $J_1, \cdots, J_n \in \mathscr{J}$ such that $\bigcup_{l=1}^{n} J_l \supset \mathscr{O}^c$. Apply Lemma (3.4) to find nonoverlapping intervals $I_1, \cdots, I_m \in \mathscr{J}$ such that

(2) $$\bigcup_{l=1}^{m} I_l = \bigcup_{l=1}^{n} J_l \supset \mathscr{O}^c .$$

Since $(F_\lambda f, f)$ is continuous, $F(\bigcup_{l=1}^{m} I_l)f = \sum_{l=1}^{m} F(I_l)f$.

Now apply (3.1) and (3.2) to each I_l to find $h_l \in L^\infty \cap F(I_l)L^2$ such that

(3) $$\|h_l - F(I_l)f\|^2 < \frac{\varepsilon}{2m} \qquad \text{for} \quad l = 1, 2, \cdots, m$$

and $\lim_{N\to\infty} (1/N) \sum_{j=1}^{N} h_l(T^{k_j}x) = 0$ a.e. for $l = 1, 2, \cdots, m$. Let $h = \sum_{l=1}^{m} h_l(x)$. Then

$$\begin{aligned}\|h - f\|^2 &= \|h - \textstyle\sum_{l=1}^{m} F(I_l)f\|^2 + \|F((\bigcup_{l=1}^{m} I_l)^c)f\|^2 \\ &\leqq \textstyle\sum_{l=1}^{m} \|h_l - F(I_l)f\|^2 + \|F(\mathscr{O})f\|^2 < \varepsilon .\end{aligned}$$

The second-to-last inequality follows from (2) and the last inequality from (1) and (3). The pointwise convergence of h is guaranteed by the choice of the h_l's. This proves the theorem for f with $E(f) = 0$; for the general $f \in L^2$ find h as above for $f - E(f)$ and let $\tilde{h} = h + E(f)$.

The following theorem is the special case of a theorem in [2] (page 3).

(4.2) THEOREM. *If there exists a constant $K > 0$ such that for $f \in L^1$*

$$P\left\{x \,\middle|\, \sup_N \left|\frac{1}{N} \sum_{j=1}^{N} f(T^{k_j}x)\right| > \lambda\right\} \leqq \frac{K\|f\|_1}{\lambda} ,$$

then the set of f's in L^1 for which $\lim_{N\to\infty} (1/N) \sum_{j=1}^{N} f(T^{k_j}x)$ *exists* a.e. *is closed in L^1.*

Now we are in the position to prove the following theorem.

(4.3) THEOREM. *Suppose $\{k_j\}$ is a sequence for which $\mathscr{J}$ saturates. If in addition*

$$\sup_N \frac{k_N{}^* \cdot M(k_N{}^*)}{N} = r < \infty ,$$

then for all $f \in L^1$, $\lim_{N\to\infty} (1/N) \sum_{j=1}^{N} f(T^{k_j}x) = E(f)$ a.e.

PROOF. We have

$$P\left\{x \,\middle|\, \sup_N \left|\frac{1}{N}\sum_{j=1}^N f(T^{k_j}x)\right| > \lambda\right\}$$

$$\leqq P\left\{x \,\middle|\, \sup_N \frac{k_N{}^* \cdot M(k_N{}^*)}{N}\cdot\frac{1}{k_N{}^*}\sum_{j=-k_N*}^{k_N*}|f(T^jx)| > \lambda\right\}$$

$$\leqq P\left\{x \,\middle|\, \sup_N \frac{1}{N}\sum_{j=-N}^{N}|f(T^jx)| > \frac{\lambda}{r}\right\} < \frac{\|f\|_1\cdot r}{\lambda}.$$

The last inequality follows from the maximal ergodic lemma.

5. A sufficient condition for saturation. Let $\{k_j\}$ be a sequence, and let $\{k_1, k_2, \cdots, k_N\}$ denote the set which contains $k_1, \cdots, k_N$ and lists each integer according to its multiplicity. The cardinality will be denoted by $|\cdot|$.

(5.1) THEOREM. *Suppose for a sequence $\{k_j\}$ there exists a finite set $K \subset Z^+$ such that for each N there exists $j_N \in K$ satisfying*

$$\left(1 - \frac{|\{k_1 + j_N, \cdots, k_N + j_N\} \cap \{k_1, \cdots, k_N\}|}{N}\right) = O\left(\frac{1}{N^r}\right)$$

for some $0 < r < 1$. Then $\mathcal{J}$ saturates.

PROOF. Let $S_N = \{k_1, \cdots, k_N\}$. Then

$$\left|(1 - e^{ij_N\alpha})\frac{1}{N}\sum_{j=1}^N e^{ik_j\alpha}\right| = \left|\frac{1}{N}\left(\sum_{j=1}^N e^{ik_j\alpha} - \sum_{j=1}^N e^{i(k_j+j_N)\alpha}\right)\right|$$

$$\leqq \frac{|S_N \Delta S_N + j_N|}{N} = O\left(\frac{1}{N^r}\right).$$

Let J be any closed interval which does not contain a root of unity of order $\prod_{j\in K} j$. Then $\min_{j\in K}\inf_{\alpha\in J}|1 - e^{ij\alpha}| = \gamma > 0$. Therefore

$$\sup_{\alpha\in J}\left|\frac{1}{N}\sum_{j=1}^N e^{ik_j\alpha}\right| \leqq \frac{1}{\gamma}O\left(\frac{1}{N^r}\right) = O\left(\frac{1}{N^r}\right).$$

Then the subsequence $N_l = l^{[1/r]2}$, $l = 1, 2, \cdots$, has the desired properties, where $[\cdot]$ denotes the integral part function. Now $\mathcal{J}$ saturates because

$$[0, 2\pi]\setminus\bigcup_{J\in\mathcal{J}}(\text{interior } J)$$
$$= \left\{0, \frac{2\pi}{M}, \frac{4\pi}{M}, \cdots, \frac{(n-1)2\pi}{M}, 2\pi\right\} \qquad \text{where} \quad M = \prod_{j\in K} j.$$

Note that a sequence $\{k_j\}$ can satisfy a uniform order condition even though it is "thin" in the integers, i.e.,

$$\lim_{N\to\infty}\frac{|[-N, N] \cap \{k_1, k_2, \cdots\}|}{2N} = 0.$$

Here is an example:

$1, 2^1, 2^1 + 1, 2^2, 2^2 + 1, 2^2 + 2, \cdots, 2^n, 2^n + 1, 2^n + 2, \cdots, 2^n + n, 2^{n+1}, \cdots.$

Let k_j be the jth integer in this sequence. It is not hard to show that

$$\left|\frac{|\{k_1, \cdots, k_N\} \cap \{k_1+1, \cdots, k_N+1\}|}{N} - 1\right| = O\left(\frac{1}{N^{\frac{1}{2}}}\right);$$

then by (5.1) $\mathscr{J}$ saturates.

6. Applications to probability theory. Let $Y_1, Y_2, \cdots,$ be a sequence of independent identically distributed integer-valued random variables. Let $k_j(\omega) = \sum_{i=1}^{j} Y_i(\omega)$, for $j = 1, 2, \cdots$. In [1], Blum and Cogburn prove

(6.1) THEOREM. *If $E(Y_1) < \infty$ and $E(e^{i\alpha Y_1}) \neq 1$ for $\beta \leqq \alpha \leqq \gamma$ then* a.s.

$$\lim_{N\to\infty} \sup_{\beta\leqq\alpha\leqq\gamma} |\textstyle\sum_{j=1}^{N} e^{ik_j(\omega)\alpha}| = 0 .$$

The authors prove this theorem by actually proving the following sharper version:

(6.2) THEOREM. *If $E(Y_1) < \infty$ and $E(e^{i\alpha Y_1}) \neq 1$ for $\beta \leqq \alpha \leqq \gamma$ then there exists a set C with $P(C) = 1$ such that for $\omega \in C$*

$$\sup_{\beta\leqq\alpha\leqq\gamma} \left|\frac{1}{N} \sum_{j=1}^{N} e^{ik_j(\omega)\alpha}\right| = O(N^{-\frac{1}{3}})$$

This theorem enables us to prove:

(6.3) THEOREM. *If $E(Y_1) < \infty$ then the $\mathscr{J}(\omega)$ associated with $\{k_j(\omega)\}$ saturates* a.s.

PROOF. Assume first that Y_1 is nonlattice, i.e., that $E(e^{i\alpha Y_1}) \neq 1$ for $0 < \alpha < 2\pi$. Let $J_n = [1/n, 2\pi - (1/n)]$. Then by (6.2) there exists C_n with $P(C_n) = 1$ such that for $\omega \in C_n$, $\{k_j(\omega)\}$ satisfies a uniform order condition on J_n. Let $C = \bigcap_{n=3}^{\infty} C_n$. Then $P(C) = 1$ and for $\omega \in C$, $\{k_j(\omega)\}$ satisfies a uniform order condition on J_n for $n \geqq 3$. Now $\mathscr{J}(\omega)$ saturates for $\omega \in C$ because $[0, 2\pi]\backslash\bigcup_{J\in\mathscr{J}(\omega)}$ (interior J) $= \{0, 2\pi\}$. If Y_1 is lattice, with lattice distance $n > 1$, then $E(e^{i\alpha Y_1}) \neq 1$ for $(k-1)2\pi/n < \alpha < k2\pi/n$, $k = 1, 2, \cdots, n$. Now argue for each interval $[(k-1)2\pi/n, k2\pi/n]$ as we did for $[0, 2\pi]$ in the nonlattice case.

With the additional restriction that the state space of the Y_j's be Z^+ or Z^- we can prove:

(6.4) THEOREM. *Suppose Y_1 has state space Z^+ or Z^- and $E(Y_1) < \infty$. Then there exists a set C with $P(C) = 1$ such that for $\omega \in C$ and every T*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} f(T^{k_j(\omega)}x) = E(f) \quad \text{a.e.} \qquad \textit{for all} \quad f \in L^1 .$$

PROOF. We will prove the theorem for state space Z^+, since the same proof with obvious modifications works for Z^-.

Since the Y_j's have state space Z^+ it follows that for the sequence $\{k_j(\omega)\}$, $k_N^*(\omega) = k_N(\omega)$ and $M(n, \omega)$ is 0 or 1 for all $n \in Z$. Also by the law of large numbers $\lim_{N\to\infty} k_N(\omega)/N = E(Y_1)$ a.s. Now the theorem follows from (4.3) and (6.3). □

7. Epilogue. It should be noted that we can extend the condition of uniform order to sequences $\{\mu_N\}$ of probability measures on Z. Namely, $\{\mu_N\}$ satisfies a uniform order condition on the closed interval $J \subset (0, 2\pi)$ if there exists a subsequence $\{N_l\}$ such that

$$\sum_{l=1}^{\infty} \sup_{\alpha \in J} |\hat{\mu}_{N_l}(\alpha)|^2 < \infty \qquad \text{and} \qquad \lim_{l\to\infty} \max_{N_l \leq N < N_{l+1}} |\mu_N - \mu_{N_l}|(Z) = 0 .$$

Then Lemmas (3.2)—(3.4) have their obvious analogues for $\{\mu_N\}$ and the theorem corresponding to (4.1) states:

(7.1) THEOREM. *If $\{\mu_N\}$ saturates, then for every T there exists a dense subspace $\mathscr{D}_T \subset L^2$ such that for $f \in \mathscr{D}_T$*

$$\lim_{N\to\infty} \int_Z f(T^j x)\mu_N(dj) = E(f) \quad \text{a.e.}$$

Except for obvious modifications the proofs remain the same.

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