The Annals of Probability
1977, Vol. 5, No. 6, 875-887

MARTINGALE INVARIANCE PRINCIPLES

BY PETER HALL[1]

Mathematical Institute, Oxford

Let $\{(S_{nj}, \mathscr{F}_{nj}), 1 \leqq j \leqq k_n\}$ be a square-integrable martingale for each $n = 1, 2, 3, \cdots$. Define $X_{nj} = S_{nj} - S_{n,j-1}$ $(S_{n0} = 0)$, $U_{nj}^2 = \sum_{i=1}^{j} X_{ni}^2$, $U_n^2 = U_{nk_n}^2$, and for each $z \in [0, 1]$ let $\xi_n(z) = U_n^{-1} \sum_{j=1}^{k_n} X_{nj} I(U_n^{-2} U_{nj}^2 \leqq z)$ and $\eta_n(z) = \sum_{j=1}^{k_n} X_{nj} I(U_n^{-2} U_{nj}^2 \leqq z)$; ξ_n and η_n are random elements of $D[0, 1]$. Sufficient conditions are given for ξ_n to converge in distribution to Brownian motion and for η_n to converge to a mixture of Brownian motion distributions. We give several applications and examples.

1. Introduction and summary. Let $\{(S_{nj}, \mathscr{F}_{nj}), 1 \leqq j \leqq k_n\}$ be a zero-mean, square-integrable (forward) martingale for each $n = 1, 2, \cdots$, and define $S_n = S_{nk_n}$, $X_{nj} = S_{nj} - S_{n,j-1}$ $(S_{n0} = 0)$,

$$U_{nj}^2 = \sum_{i=1}^{j} X_{ni}^2 , \qquad U_n^2 = U_{nk_n}^2 , \qquad V_{nj}^2 = \sum_{i=1}^{j} E(X_{ni}^2 \mid \mathscr{F}_{n,i-1}) \quad \text{and} \quad V_n^2 = V_{nk_n}^2 .$$

$\{(S_{nj}, \mathscr{F}_{nj}), 1 \leqq j \leqq k_n, n \geqq 1\}$ is called a triangular martingale array.

In recent years a large body of central limit theory has been developed for triangular martingale arrays. The objective has usually been to study the convergence

$$S_n \to_{\mathscr{D}} N(0, 1) \tag{1.1}$$

and it has been found necessary to impose a normalisation condition on some estimate of the variance of S_n. Brown (1971) and Scott (1973) impose the condition

$$V_n^2 \to_p 1 \tag{1.2}$$

while McLeish (1974) uses

$$U_n^2 \to_p 1 . \tag{1.3}$$

McLeish has shown that if

$$\lim_{\lambda \to \infty} \limsup_{n \to \infty} P(V_n > \lambda) = 0 \tag{1.4}$$

and if the conditional Lindeberg condition

$$\text{for all} \quad \varepsilon > 0 , \qquad \sum_j E[X_{nj}^2 I(|X_{nj}| > \varepsilon) \mid \mathscr{F}_{n,j-1}] \to_p 0 \tag{1.5}$$

is satisfied, then

$$\max_{j \leq k_n} (U_{nj}^2 - V_{nj}^2) \to_p 0 . \tag{1.6}$$

That is, the variance estimates U_{nj}^2 and V_{nj}^2 are asymptotically equivalent.

Received December 29, 1975; revised February 16, 1977.

[1] Now at Melbourne University.

AMS 1970 *subject classifications.* Primary 60F05; Secondary 60G45.

Key words and phrases. Martingales, invariance principles, central limit theorem, mixtures.

Section 2 contains our invariance principles. Let us discuss their more familiar one-dimensional analogues. In the theorem we use McLeish's techniques to show that (1.3) can be relaxed if we consider the convergence

$$S_n/U_n \to_{\mathscr{D}} N(0, 1) \tag{1.7}$$

rather than (1.1). There appears to be little published work in this area. Blackwell and Freedman (1973) and Freedman (1975) have shown that if S_n is considered as occurring at time V_n^2 rather than at time n then the process S_n resembles Brownian motion. Hence it is natural to study the convergence of S_n/V_n, and in view of (1.6) this is usually equivalent to studying the convergence of S_n/U_n. Chatterji (1974) and Eagleson (1975) have proved results on convergence to mixtures of normal laws. They give conditions under which

$$S_n \to_{\mathscr{D}} \Phi_T , \tag{1.8}$$

the distribution with characteristic function $E\exp(-\frac{1}{2}t^2T)$, where T is a random variable. Their results are largely contained in our corollary. In a different approach to the problem, McLeish and Drogin (1972) have shown that normalisation conditions like (1.2) and (1.3) may be avoided if instead of (1.1), we consider the convergence

$$S_{\nu_n} \to_{\mathscr{D}} N(0, 1) ,$$

where ν_n is a random variable.

Instead of condition (1.3), we ask that it be possible to approximate to U_n^2 by a variable u_n^2 which does not depend very much on the nth row of the array. Our theorem is stated for "near martingales" rather than martingales, and specialised to martingales in the corollary.

Section 3 is devoted to examples, while Section 4 contains some proofs of results in Section 2.

Finally, let us give some definitions and notation. If α_n, β_n and γ_n $(1 \leqq n \leqq \infty)$ are random elements of $D[0, 1]$, we will understand the condition

$$(\alpha_n, \beta_n, \gamma_n) \to_{\mathscr{D}} (\alpha_\infty, \beta_\infty, \gamma_\infty)$$

to mean that for all α_∞-, β_∞- and γ_∞-continuity sets A, B and C, respectively,

$$P(\alpha_n \in A, \beta_n \in B, \gamma_n \in C) \to P(\alpha_\infty \in A, \beta_\infty \in B, \gamma_\infty \in C)$$

(see Billingsley (1968), Theorem 3.1). $I(A)$ denotes the indicator function of the set A, and $\tilde{A}$ denotes the complement of A. Almost sure (a.s.) convergence, convergence in probability and convergence in distribution (weak convergence) are denoted by $\to_{\text{a.s.}}$, $\to_p$ and $\to_{\mathscr{D}}$, respectively. The variables X_n are "$o(1)$ in probability" if $X_n \to_p 0$ as $n \to \infty$. The variables S_{nj} are said to be adapted to the σ-fields $\mathscr{G}_{nj}$ if each S_{nj} is $\mathscr{G}_{nj}$-measurable. If $\mathscr{F}$ and $\mathscr{G}$ are σ-fields, let $\mathscr{F} \vee \mathscr{G}$ denote the smallest σ-field generated by $\mathscr{F} \cup \mathscr{G}$.

2. Invariance principles. Let $T \geqq 0$ be a variable with distribution F, let W

be standard Brownian motion on [0, 1] and let T' be a copy of T, independent of W. Define $W_F = (T')^{\frac{1}{2}}W$, a random element of $C[0, 1]$.

THEOREM. *Suppose the square-integrable variables* $S_{n1}, S_{n2}, \cdots, S_{nk_n}$ *are adapted to the σ-fields* $\mathscr{G}_{nj}$, *where* $\mathscr{G}_{n1} \subseteqq \mathscr{G}_{n2} \subseteqq \cdots \subseteqq \mathscr{G}_{nk_n}$ $(n \geqq 1)$. *Define* $X_{nj} = S_{nj} - S_{n,j-1}$, $U_{nj}^2 = \sum_{i=1}^{j} X_{ni}^2$ *and* $U_n^2 = U_{nk_n}^2$. *For each* $z \in [0, 1]$ *let*

$$\eta_n(z) = \sum_j X_{nj} I(U_n^{-2} U_{nj}^2 \leqq z)$$

and $\xi_n(z) = U_n^{-1}\eta_n(z)$. ξ_n *and* η_n *are random elements of* $D[0, 1]$. *Suppose that*

$$\lim_{n\to\infty} E(\max_{j\leqq k_n} X_{nj}^2) = 0 \tag{2.1}$$

and that there exist variables u_n^2 *adapted to the σ-fields* $\mathscr{G}_{n1}$ *such that*

$$U_n^2 - u_n^2 \to_p 0 . \tag{2.2}$$

If

$$\lim_{\delta\to 0} \liminf_{n\to\infty} P(U_n > \delta) = 1 \tag{2.3}$$

and

$$\sum_j |E(X_{nj} \mid \mathscr{G}_{n,j-1})| \to_p 0 \tag{2.4}$$

then

$$\xi_n \to_{\mathscr{D}} W . \tag{2.5}$$

(*Condition* (1.7) *is a corollary.*) *If*

$$U_n^{-2} \max_{j\leqq k_n} X_{nj}^2 \to_p 0 , \tag{2.6}$$

$$U_n^2 \to_{\mathscr{D}} T \tag{2.7}$$

and

$$U_n^{-1} \sum_j |E(X_{nj} \mid \mathscr{G}_{n,j-1})| \to_p 0 , \tag{2.8}$$

then

$$(\eta_n, U_n^2) \to_{\mathscr{D}} (W_F, T') . \tag{2.9}$$

(*Conditon* (1.8) *is a corollary.*)

COROLLARY. *Let* $\{(S_{nj}, \mathscr{F}_{nj}), 1 \leqq j \leqq k_n, n \geqq 1\}$ *be a zero-mean, square-integrable triangular martingale array, and define* X_{nj}, U_n^2, ξ_n *and* η_n *as in the theorem. Suppose* (2.1) *holds,*

$$U_n^2 \to_p T , \tag{2.10}$$

and either

$$T \text{ is measurable in } \bigcap_{n=1}^{\infty} \mathscr{F}_{n1} \tag{2.11}$$

or

$$k_n \uparrow \infty \quad \text{and} \quad \mathscr{F}_{nj} \subseteqq \mathscr{F}_{n+1,j} \text{ for all } j \leqq k_n . \tag{2.12}$$

Then (2.9) *holds, and if* $T > 0$ a.s., (2.5) *is true.*

If (2.3), (2.4) and (2.6)—(2.8) hold simultaneously, then of course (2.5) and (2.9) can be combined, so that $(\xi_n, \eta_n, U_n^2) \to_{\mathscr{D}} (W, W_F, T')$. The processes ξ_n and η_n can often be approximated by simpler processes. For example, suppose the triangular array is derived from a martingale $\{(S_n, \mathscr{F}_n), n \geq 1\}$, in the sense that $S_{nj} = c_n^{-1}S_j$ for constants c_n. Let $X_n = S_n - S_{n-1}$, $\eta_n^*(z) = c_n^{-1} \sum_1^n X_j I(c_n^{-2}c_j^2 \leq z)$ and $\xi_n^* = U_n^{-1}c_n\eta_n^*$. If $c_n^{-2} \sum_1^n X_j^2 \to_{\text{a.s.}} T > 0$ a.s. then $||\xi_n - \xi_n^*||$ and $||\eta_n - \eta_n^*|| \to_p 0$, where $||\ ||$ denotes the uniform norm on $D[0, 1]$.

3. Some examples. This section is devoted to examples of the invariance principles in Section 2.

EXAMPLE 3.1. Let X_n, $n \geq 1$ be martingale differences, $S_n = \sum_1^n X_j$ and $U_n^2 = \sum_1^n X_j^2$. Suppose $c_n^{-2}E(\max_{j\leq k_n} X_j^2) \to 0$ and $c_n^{-2}U_{k_n}^2 \to_p T > 0$ a.s. Define $\eta_n(z) = c_n^{-1} \sum_j X_j I(U_{k_n}^{-2} U_j^2 \leq z)$ and $\xi_n = U_{k_n}^{-1} c_n \eta_n$. Then $(\xi_n, \eta_n, c_n^{-2}U_{k_n}^2) \to_{\mathscr{D}} (W, W_F, T')$. This example may be applied to prove new central limit theorems for stationary sequences, as in Scott (1973).

EXAMPLE 3.2. Suppose $\{(S_{nj}, \mathscr{F}_{nj}), 1 \leq j \leq k_n, n \geq 1\}$ is a triangular martingale array. This example shows that (2.1) and (2.10) are not sufficient for either (1.7) or (1.8).

Let $W(t)$, $t \geq 0$ be standard Brownian motion and define

$$\begin{aligned} t_{nj} &= 1/n && \text{if } 1 \leq j \leq n\,, \\ &= I(W(1) > 0)/n && \text{if } n+1 \leq j \leq 2n\,, \end{aligned}$$

$T_{nj} = \sum_{i=1}^{j} t_{ni}$, $S_{nj} = W(T_{nj})$ and $\mathscr{F}_{nj} =$ (σ-field generated by $S_{n1}, \cdots, S_{nj}$), $1 \leq j \leq 2n = k_n$. For each n, $\{(S_{nj}, \mathscr{F}_{nj}), 1 \leq j \leq 2n\}$ is a martingale, (2.1) and (2.10) hold, and $S_n = S_{n,2n} = W(1)I(W(1) \leq 0) + W(2)I(W(1) > 0)$. The distribution of S_n is not a mixture of normals and S_n/U_n does not converge weakly to $N(0, 1)$.

EXAMPLE 3.3. This example shows the need for a condition which restricts the amount of dependence between U_n^2 and the differences $X_{n1}, X_{n2}, \cdots, X_{nk_n}$.

Let $Y_1, Y_2, \cdots$ be mutually independent symmetric ± 1 variables and let $m(n)$ be an integer between 1 and n. Define $I_n = I(Y_{n-m(n)+1} + \cdots + Y_n > 0)$,

$$\begin{aligned} X_{nj} &= Y_j n^{-\frac{1}{2}} && \text{if } 1 \leq j \leq n\,, \\ &= Y_j I_n n^{-\frac{1}{2}} && \text{if } n+1 \leq j \leq 2n\,, \end{aligned}$$

$S_{nj} = \sum_{i=1}^{j} X_{ni}$ and $\mathscr{F}_{nj} =$ (σ-field generated by $S_{n1}, S_{n2}, \cdots, S_{nj}$), $1 \leq j \leq 2n = k_n$. For each n, $\{(S_{nj}, \mathscr{F}_{nj}), 1 \leq j \leq 2n\}$ is a martingale and $U_n^2 = U_{n,2n}^2 = 1 + I_n$. If $m(n) \equiv 1$ or $m(n) \to \infty$ then $U_n^2 \to_{\mathscr{D}} 1 + I$ where I is a symmetric 0—1 variable, but in no case does U_n^2 converge in probability. Condition (1.7) holds if and only if $m(n)/n \to 0$, and when this is the case, (1.7) may be obtained from the theorem by setting $\mathscr{G}_{nj} = \mathscr{F}_{nj} \vee$ (σ-field generated by U_n^2) and $u_n^2 = U_n^2$.

EXAMPLE 3.4. We begin by constructing a linear martingale $\{(S_n, \mathscr{F}_n), n \geq 1\}$.

Then we define a triangular array which satisfies the conditions of the theorem but for which there do not exist constants c_n and a nonidentically zero variable T such that $U_n^2/c_n^2 \to_{\mathscr{D}} T$.

The martingale $\{(S_n, \mathscr{F}_n), n \geqq 1\}$ is built up in blocks of differences $X_n = S_n - S_{n-1}$, with $\mathscr{F}_n =$ (σ-field generated by $X_1, X_2, \cdots, X_n$). Let $Y_1, Y_2, \cdots$ be mutually independent symmetric ± 1 variables. The first block is of size 2: $X_1 = Y_1$, $X_2 = Y_2 I(Y_1 > 0)$. Suppose we have defined N blocks and the nth was of size $s(n)$. The $(N+1)$th will be of size

$$s(N+1) = 2[\textstyle\sum_1^N s(n)]^2 = 2t(N)^2\,, \quad \text{say,}$$

and is defined as follows. Let $I_{N+1} = I(Y_{t(N)+t(N)^2} > 0)$ and

$$\begin{aligned} Y_n &= X_n && \text{if} \quad t(N) < n \leqq t(N) + t(N)^2\,,\\ &= X_n I_{N+1} && \text{if} \quad t(N) + t(N)^2 < n \leqq t(N) + 2t(N)^2\,. \end{aligned}$$

If $t(N) < n \leqq t(N+1)$ let $X_{nj} = n^{-\frac{1}{2}} X_j$,

$$\begin{aligned} u_n^2 &= n^{-1}[t(N-1)]^2(1 + I_N) + n^{-1} \textstyle\sum_{j=t(N)+1}^n X_j^2\,,\\ p(N) &= t(N-1) + t(N-1)^2\,, \end{aligned}$$

$q(N) = t(N) + t(N)^2$ and $\mathscr{G}_{nj} = \mathscr{F}_j \vee$ (σ-field generated by $Y_{p(N)}$ and $Y_{q(N)}$). The theorem implies the invariance principle (2.5), but in this case there do not exist constants c_n and a nonidentically zero variable T such that $U_n^2/c_n^2 \to_{\mathscr{D}} T$.

EXAMPLE 3.5. Finally, an example of an infinite martingale $\{(S_n, \mathscr{F}_n), n \geqq 1\}$ for which $S_n/U_n \not\to_{\mathscr{D}} N(0, 1)$. It is constructed in the same way as the infinite martingale in Example 3.4, except that here $I_{N+1} = I(\sum_{j=t(N)+1}^{t(N)+t(N)^2} Y_j > 0)$. In this case, S_n/U_n does not converge in distribution at all.

4. Proofs.

PROOF OF THEOREM. We will prove only that (2.1), (2.2), and (2.6)—(2.8) are sufficient for (2.9). First let us prove:

LEMMA. *Suppose $\{(S_{nj}, \mathscr{F}_{nj}), 1 \leqq j \leqq k_n\}$ is a martingale and $\mathscr{G}_n$ is a sub-σ-field of $\mathscr{F}_{n1}$ such that $E(X_{n1} \mid \mathscr{G}_n) = 0$. If u_n^2 is $\mathscr{G}_n$-measurable then* (2.1), (2.2), (2.6) *and* (2.7) *are sufficient for* (2.9).

Note that (2.9) is equivalent to the pair of conditions:

(4.1) for all sequences $0 = z_0 < z_1 < \cdots < z_p \leqq 1$ and real numbers $t_1, t_2, \cdots, t_p$,

$$\begin{aligned} \lim_{n\to\infty} |E \exp[i \textstyle\sum_k t_k(\eta_n(z_k) - \eta_n(z_{k-1})) + isU_n^2]\\ - E\exp[-\tfrac{1}{2}T \textstyle\sum_k t_k^2(z_k - z_{k-1}) + isT]| = 0\,, \end{aligned}$$

and

(4.2) the sequence of random elements $\{\eta_n, n \geqq 1\}$ is tight.

PROOF OF (4.1) IN LEMMA. Suppose first that for some $\lambda > 0$, $P(T > \lambda) = 0$.

u_n can be chosen such that $P(u_n^2 > 2\lambda) = 0$ and (2.2) is satisfied. Fix $\delta > 0$, define

$$u_n(\delta) = \delta I(u_n \leqq \delta) + u_n I(u_n > \delta)$$

(then $u_n = u_n(0)$), and for each real z let

$$\eta_n'(\delta, z) = \sum_j X_{nj} I(u_n(\delta)^{-2} U_{n,j-1}^2 \leqq z) .$$

For fixed δ, $\eta_n'(\delta, \cdot)$ restricted to [0, 1] is a random element of $D[0, 1]$. Since $u_n(\delta)$ is $\mathscr{G}_n$-measurable and $E(X_{n1} \mid \mathscr{G}_n) = 0$ then sequences like

$$P_{nm} = \sum_{j=1}^m X_{nj} I(u_n(\delta)^{-2} U_{n,j-1}^2 \leqq z)$$

and

$$Q_{nm} = \sum_{j=1}^m X_{nj} I(z_1 < u_n(\delta)^{-2} U_{n,j-1}^2 \leqq z_2)$$

are zero-mean martingales with respect to the σ-fields $\mathscr{F}_{nm}$ $(1 \leqq m \leqq k_n)$.

LEMMA A.

(4.3) *For all* $\varepsilon > 0$ *and* $z \in [0, 1]$,

$$\lim_{\delta \to 0} \limsup_{n \to \infty} P(|\eta_n(z) - \eta_n'(\delta, z)| > \varepsilon) = 0 .$$

PROOF OF LEMMA A.

$$|U_n^{-2} U_{nj}^2 - u_n(\delta)^{-2} U_{n,j-1}^2| \leqq U_n^{-2} X_{nj}^2 + |u_n(\delta)^{-2} U_n^2 - 1|$$

and so on the set

$$F_n = \{U_n^{-2} \max_{j \leqq k_n} X_{nj}^2 \leqq \Delta/2;\ |u_n(\delta)^{-2} U_n^2 - 1| \leqq \Delta/2\}$$

we have the following inequalities for all j and z:

$$I(u_n(\delta)^{-2} U_{n,j-1}^2 \leqq z - \Delta) \leqq I(U_n^{-2} U_{nj}^2 \leqq z) \leqq I(u_n(\delta)^{-2} U_{n,j-1}^2 \leqq z + \Delta) .$$

Hence on F_n,

$$\begin{aligned} |\eta_n(z) - \eta_n'(\delta, z)| &= |\textstyle\sum_j X_{nj}\{I(U_n^{-2} U_{nj}^2 \leqq z) - I(u_n(\delta)^{-2} U_{n,j-1}^2 \leqq z)\}| \\ &\leqq \max_{0 \leqq m_1 < m_2 \leqq k_n} |\textstyle\sum_{j=m_1+1}^{m_2} X_{nj} I(z - \Delta < u_n(\delta)^{-2} U_{n,j-1}^2 \leqq z + \Delta)| \\ &\leqq 2 \max_{m \leqq k_n} |\textstyle\sum_{j=1}^m X_{nj} I(z - \Delta < u_n(\delta)^{-2} U_{n,j-1}^2 \leqq z + \Delta)| . \end{aligned}$$

On the set $G_n = \{|U_n^2 - u_n^2| \leqq \Delta;\ u_n \leqq \delta\}$,

$$\begin{aligned} |\eta_n(z)|, |\eta_n'(\delta, z)| &\leqq \max_{m \leqq k_n} |\textstyle\sum_{j=1}^m X_{nj}| \\ &= \max_{m \leqq k_n} |\textstyle\sum_{j=1}^m X_{nj} I(U_{n,j-1}^2 \leqq \Delta + \delta^2)| . \end{aligned}$$

Therefore

$$\begin{aligned} &P(|\eta_n(z) - \eta_n'(\delta, z)| > \varepsilon) \\ &\quad \leqq P(\max_{m \leqq k_n} |\textstyle\sum_{j=1}^m X_{nj} I(z - \Delta < u_n(\delta)^{-2} U_{n,j-1}^2 \leqq z + \Delta)| > \varepsilon/2) \\ &\qquad + P(\max_{m \leqq k_n} |\textstyle\sum_{j=1}^m X_{nj} I(U_{n,j-1}^2 \leqq \Delta + \delta^2)| > \varepsilon/2) \\ &\qquad + P(\tilde{F}_n \cap \tilde{G}_n) \\ &\quad \leqq P(U_n^{-2} \max_{j \leqq k_n} X_{nj}^2 > \Delta/2) + P(|U_n^2 - u_n^2| > \Delta) \\ &\qquad + P(|u_n(\delta)^{-2} U_n^2 - 1| > \Delta/2;\ u_n > \delta) . \end{aligned} \tag{4.4}$$

(2.2) and (2.6) imply that the first two terms on the right-hand side are $o(1)$. The last term does not exceed

$$P(|U_n^2 - u_n^2| > \Delta\delta^2/2) \to 0 \quad \text{as} \quad n \to \infty .$$

Hence

$$P(\tilde{F}_n \cap \tilde{G}_n) = o(1) . \tag{4.5}$$

Apply Kolmogorov's inequality to the martingale $\{(Q_{nm}, \mathscr{F}_{nm}), 1 \leqq m \leqq k_n\}$, to prove that

$$\begin{aligned} &P(\max_{m \leqq k_n} |\sum_{j=1}^m X_{nj} I(z - \Delta < u_n(\delta)^{-2} U_{n,j-1}^2 \leqq z + \Delta)| > \varepsilon/2) \\ &\quad \leqq 4\varepsilon^{-2} E[\sum_{j=1}^{k_n} X_{nj}^2 I(z - \Delta < u_n(\delta)^{-2} U_{n,j-1}^2 \leqq z + \Delta)] \\ &\quad \leqq 8\varepsilon^{-2}\lambda E[u_n(\delta)^{-2} \sum_j X_{nj}^2 I(z - \Delta < u_n(\delta)^{-2} U_{n,j-1}^2 \leqq z + \Delta)] \\ &\quad \leqq 8\varepsilon^{-2}\lambda E[2\Delta + \delta^{-2} \max_{j \leqq k_n} X_{nj}^2] \\ &\quad = 16\varepsilon^{-2}\lambda\Delta + o(1) . \end{aligned} \tag{4.6}$$

Similarly, applying Kolmogorov's inequality to the martingale $\{(P_{nm}, \mathscr{F}_{nm}), 1 \leqq m \leqq k_n\}$:

$$P(\max_{m \leqq k_n} |\sum_{j=1}^m X_{nj} I(U_{n,j-1}^2 \leqq \Delta + \delta^2)| > \varepsilon/2) \leqq 4\varepsilon^{-2}(\Delta + \delta^2) + o(1) . \tag{4.7}$$

Combining (4.4)—(4.7), we obtain the inequality

$$P(|\eta_n(z) - \eta_n'(\delta, z)| > \varepsilon) \leqq 16\varepsilon^{-2}\lambda\Delta + 4\varepsilon^{-2}(\Delta + \delta^2) + o(1) .$$

Let $n \to \infty$ and then $\Delta \to 0$ and $\delta \to 0$ to establish (4.3). □

In view of (2.2) and (4.3), (4.1) will follow if we show that

$$\begin{aligned} \lim_{\delta \to 0} \limsup_{n \to \infty} |E \exp[i \textstyle\sum_k t_k(\eta_n'(\delta, z_k) - \eta'(\delta, z_{k-1})) + isu_n^2] \\ - E \exp[-\tfrac{1}{2}T \textstyle\sum_k t_k^2(z_k - z_{k-1}) + isT]| = 0 . \end{aligned} \tag{4.8}$$

Define

$$A_{nj} = A_{nj}(\delta) = X_{nj} \sum_k t_k I(z_{k-1} < u_n(\delta)^{-2} U_{n,j-1}^2 \leqq z_k) \qquad 1 \leqq j \leqq k_n$$

and

$$B_n^2 = B_n^2(\delta) = \sum_j A_{nj}^2 = \sum_j X_{nj}^2 \sum_k t_k^2 I(z_{k-1} < u_n(\delta)^{-2} U_{n,j-1}^2 \leqq z_k) .$$

Note that

$$\sum_j A_{nj} = \sum_k t_k(\eta_n'(\delta, z_k) - \eta_n'(\delta, z_{k-1})) = C_n , \quad \text{say.}$$

The A_{nj} are martingale differences. We can write $e^{ix} = (1 + x) \exp(-\frac{1}{2}x^2 + r(x))$ where $|r(x)| \leqq |x|^3$ for $|x| < 1$. Define

$$\begin{aligned} I_n &= I_n(\delta) = \exp(iC_n + isu_n^2) , \\ T_n &= T_n(\delta) = \textstyle\prod_j (1 + iA_{nj}) , \\ W_n &= W_n(\delta) = \exp(-\tfrac{1}{2}B_n^2 + \textstyle\sum_j r(A_{nj}) + isu_n^2) , \\ w_n &= \exp(-\tfrac{1}{2}u_n^2\sigma^2 + isu_n^2) \end{aligned}$$

where

$$\sigma^2 = \sum_k t_k^2(z_k - z_{k-1}) ,$$

and

$$x = E \exp(-\tfrac{1}{2}T\sigma^2 + isT) .$$

Let $t = \max_k |t_k|$. The term within modulus signs in (4.8) equals

$$E(T_n W_n) - x = ET_n(W_n - w_n) + E(T_n - 1)w_n + E(w_n) - x .$$

$$|E(T_n - 1)w_n| \leqq E|E(T_n \mid \mathscr{G}_n) - 1| = 0 .$$

Since $U_n^2 \to_{\mathscr{D}} T$, $U_n^2 - u_n^2 \to_p 0$ and the functions $f(u) = \exp(-\frac{1}{2}\sigma^2 u)$ and $g(u) = \exp(isu)$ are uniformly bounded and continuous on $[0, \infty[$, then $E(w_n) \to x$ as $n \to \infty$. Therefore (4.8) will follow from the condition

$$\lim_{\delta\to 0} \limsup_{n\to\infty} E|T_n(\delta)(W_n(\delta) - w_n)| = 0 . \tag{4.9}$$

First let us prove

LEMMA B.

(4.10) *For all* $\varepsilon > 0$, $\lim_{\delta\to 0} \limsup_{n\to\infty} P(|W_n(\delta) - w_n| > \varepsilon) = 0$.

PROOF OF LEMMA B. If $\max_{j\leqq k_n} |A_{nj}| < 1$ then

$$|\textstyle\sum_j r(A_{nj})| \leqq \sum_j |A_{nj}|^3 \leqq B_n^2 \max_{j\leqq k_n} |A_{nj}|$$
$$\leqq t^3 U_n^2 \max_{j\leqq k_n} |X_{nj}| \to_p 0 \quad \text{as} \quad n \to \infty .$$

Since $\max_{j\leqq k_n} |A_{nj}| \to_p 0$, then

$$\textstyle\sum_j r(A_{nj}(\delta)) \to_p 0 \quad \text{as} \quad n \to \infty . \tag{4.11}$$

On the set F_n,

$$\begin{aligned} &\textstyle\sum_k t_k^2 \sum_j X_{nj}^2 I(z_{k-1} + \Delta < U_n^{-2}U_{nj}^2 \leqq z_k - \Delta) \\ &\leqq B_n^2 \\ &= \textstyle\sum_k t_k^2 \sum_j X_{nj}^2 I(z_{k-1} < u_n(\delta)^{-2}U_{n,j-1}^2 \leqq z_k) \\ &\leqq \textstyle\sum_k t_k^2 \sum_j X_{nj}^2 I(z_{k-1} - \Delta < U_n^{-2}U_{nj}^2 \leqq z_k + \Delta) . \end{aligned} \tag{4.12}$$

In view of (2.6), for each $z \in [0, 1]$,

$$D_n^2(z) = U_n^{-2} \textstyle\sum_j X_{nj}^2 I(U_n^{-2}U_{nj}^2 \leqq z) \to_p z .$$

If $z > 1$ then $D_n^2(z) = 1$ and if $z < 0$, $D_n^2(z) = 0$. If Δ is so small that each $z_{k-1} + \Delta \leqq z_k - \Delta$, the inequalities (4.12) can be rewritten as

$$\textstyle\sum_k t_k^2[D_n^2(z_k - \Delta) - D_n^2(z_{k-1} + \Delta)]$$
$$\leqq U_n^{-2}B_n^2 \leqq \textstyle\sum_k t_k^2[D_n^2(z_k + \Delta) - D_n^2(z_{k-1} - \Delta)] .$$

Hence on the set $F_n \cap \{\max_{0\leqq k\leqq p} |D_n^2(z_k \pm \Delta) - (z_k \pm \Delta)| \leqq \Delta\}$,

$$\sigma^2 - 4pt^2\Delta \leqq U_n^{-2}B_n^2 \leqq \sigma^2 + 4pt^2\Delta ,$$

and so on the set $F_n \cap \{\max_{0\leqq k\leqq p} |D_n^2(z_k \pm \Delta) - (z_k \pm \Delta)| \leqq \Delta\} \cap \{U_n^2 \leqq 2\lambda\}$,

$$U_n^2\sigma^2 - 8pt^2\Delta\lambda \leqq B_n^2 \leqq U_n^2\sigma^2 + 8pt^2\Delta\lambda .$$

Let $\varepsilon > 0$ and choose Δ so small that $8pt^2\Delta\lambda < \varepsilon$. Then

$$P(|B_n^2 - U_n^2\sigma^2| > \varepsilon; F_n) \leqq P(\max_{0\leqq k\leqq p} |D_n^2(z_k \pm \Delta) - (z_k \pm \Delta)| > \Delta)$$
$$+ P(U_n^2 > 2\lambda) .$$

Let $n \to \infty$. Since $U_n^2 \to_{\mathscr{D}} T \leqq \lambda$ a.s. then $P(U_n^2 > 2\lambda) \to 0$, and so for all sufficiently small Δ and all δ,

$$\lim_{n\to\infty} P(|B_n^2(\delta) - U_n^2\sigma^2| > \varepsilon; F_n(\delta, \Delta)) = 0 . \tag{4.13}$$

On the set G_n, $U_n^2 \leqq \Delta + \delta^2$ and so

$$|B_n^2 - U_n^2\sigma^2| \leqq 2t^2U_n^2 \leqq 2t^2(\Delta + \delta^2) .$$

Choose Δ and δ so small that $2t^2(\Delta + \delta^2) < \varepsilon$. Then for all n,

$$P(|B_n^2 - U_n^2\sigma^2| > \varepsilon; G_n(\delta, \Delta)) = 0 . \tag{4.14}$$

Combining (4.5), (4.13) and (4.14), we see that for all sufficiently small Δ and δ,

$$P(|B_n^2 - U_n^2\sigma^2| > \varepsilon) = o(1) .$$

Together with (2.2) and (4.11), this is sufficient to establish (4.10). □

Finally let us prove (4.9). Let $\varepsilon > 0$.

$$E|T_n(W_n - w_n)| \leqq (E|T_n|^2)^{\frac{1}{2}}\varepsilon + \int_{\{|W_n - w_n| > \varepsilon\}} |T_n(W_n - w_n)| \, dP .$$

The last term does not exceed

$$\int_{\{|W_n - w_n| > \varepsilon\}} (|I_n| + |T_n w_n|) \, dP \leqq P(|W_n - w_n| > \varepsilon) + (E|T_n^2|)^{\frac{1}{2}} P(|W_n - w_n| > \varepsilon)^{\frac{1}{2}} .$$

Hence for all $\varepsilon > 0$,

$$E|T_n(W_n - w_n)| \leqq (E|T_n^2|)^{\frac{1}{2}}(\varepsilon + P(|W_n - w_n| > \varepsilon)^{\frac{1}{2}}) + P(|W_n - w_n| > \varepsilon) . \tag{4.15}$$

Let $J_n = \max\{j \leqq k_n \mid u_n(\delta)^{-2}U_{n,j-1}^2 \leqq 1\}$. Since $A_{nj}^2 \leqq t^2X_{nj}^2$ then

$$\begin{aligned} E|T_n|^2 = E \textstyle\prod_j (1 + A_{nj}^2) &\leqq E \exp(t^2U_{n,J_n-1}^2)(1 + t^2X_{n,J_n}^2) \\ &\leqq E \exp(t^2u_n(\delta)^2)(1 + t^2 \max_{j\leqq k_n} X_{nj}^2) \\ &\leqq \exp(2\lambda t^2)(1 + t^2E(\max_{j\leqq k_n} X_{nj}^2)) \\ &\to \exp(2\lambda t^2) \qquad \text{as} \quad n \to \infty . \end{aligned}$$

(4.9) now follows from (4.10) and (4.15).

This completes the proof of (4.1) in the lemma when for some $\lambda > 0$, $P(T > \lambda) = 0$. Now let us consider the case of an arbitrary T. Let $\lambda > 0$ be a continuity point of T and define $\bar{X}_{nj} = X_{nj} I(U_{n,j-1}^2 \leqq \lambda)$, $\bar{S}_{nj} = \sum_{i=1}^j \bar{X}_{ni}$, $\bar{U}_{nj}^2 = \sum_{i=1}^j \bar{X}_{ni}^2$, $\bar{U}_n^2 = \bar{U}_{nk_n}^2$, $\bar{u}_n^2 = u_n^2 I(u_n^2 \leqq \lambda) + \lambda I(u_n^2 > \lambda)$ and $\bar{T} = TI(T \leqq \lambda) + \lambda I(T > \lambda)$. Since $\bar{X}_{nj}^2 \leqq X_{nj}^2$,

$$\lim_{n\to\infty} E(\max_{j\leqq k_n} \bar{X}_n^2) = 0 . \tag{2.1$'$}$$

$|\bar{U}_n^2 - \bar{u}_n^2| \leqq |U_n^2 - u_n^2| + \max_{j\leqq k_n} X_{nj}^2$, and so

$$\bar{U}_n^2 - \bar{u}_n^2 \to_p 0 . \tag{2.2$'$}$$

$\bar{U}_n^{-2} \max_{j\leqq k_n} \bar{X}_{nj}^2 \leqq \max(U_n^{-2} \max_{j\leqq k_n} X_{nj}^2, \lambda^{-1} \max_{j\leqq k_n} X_{nj}^2) \to_p 0$, and so

$$\bar{U}_n^{-2} \max_{j\leqq k_n} \bar{X}_{nj}^2 \to_p 0 . \tag{2.6$'$}$$

Since λ is a continuity point of T,

$$\tilde{U}_n^2 = U_n^2 I(U_n^2 \leqq \lambda) + \lambda I(U_n^2 > \lambda) \to_{\mathscr{D}} \bar{T}.$$

$\tilde{U}_n^2 \leqq \bar{U}_n^2 \leqq \tilde{U}_n^2 + \max_{j \leqq k_n} X_{nj}^2$ and hence

$$\bar{U}_n^2 \to_{\mathscr{D}} \bar{T}. \tag{2.7$'$}$$

Define $\bar{\eta}_n$ for the martingale $\{(\bar{S}_{nj}, \mathscr{F}_{nj}), 1 \leqq j \leqq k_n\}$ in the same way that we defined η_n for $\{(S_{nj}, \mathscr{F}_{nj}), 1 \leqq j \leqq k_n\}$. In view of (2.1)$'$, (2.2)$'$, (2.6)$'$ and (2.7)$'$, the proof given above establishes that

$$\lim_{n\to\infty} |E \exp[i \textstyle\sum_k t_k(\bar{\eta}_n(z_k) - \bar{\eta}_n(z_{k-1})) + is\bar{U}_n^{-2}] \cdot - E\exp[-\frac{1}{2}T^2 + is\bar{T}]| = 0. \tag{4.16}$$

$\bar{\eta}_n = \eta_n$ and $\bar{U}_n^2 = U_n^2$ on the set $\{U_n^2 \leqq \lambda\}$, and so (4.1) follows from (4.16) and the fact that $\limsup_{n\to\infty} P(U_n^2 > \lambda) \to 0$ as $\lambda \to \infty$. □

PROOF OF (4.2) IN LEMMA. Suppose first that for some $\lambda > 0$, $P(T > \lambda) = 0$, and choose the variables u_n such that $P(u_n^2 > 2\lambda) = 0$. We will follow Brown's (1971) proof of tightness and show that

$$\text{for all } \varepsilon > 0, \quad \lim_{h\to 0} \limsup_{n\to\infty} \textstyle\sum_{kh<1} P(\max_{kh<z\leqq(k+1)h} |\eta_n(z) - \eta_n(kh)| > \varepsilon) = 0 \tag{4.17}$$

(see Parthasarathy (1967), page 222). On the set F_n,

$$\max_{j\leqq k_n} |U_n^{-2}U_{nj}^2 - u_n(\delta)^{-2}U_{n,j-1}^2| \leqq \Delta$$

and so on F_n,

$$\sup_{kh<z\leqq(k+1)h} |\eta_n(z) - \eta_n(kh)| \leqq 2\max_{m\leqq k_n} |\textstyle\sum_{j=1}^m X_{nj} I(kh - \Delta < u_n(\delta)^{-2}U_{n,j-1}^2 \leqq (k+1)h + \Delta)|.$$

On the set G_n,

$$\sup_{kh<z\leqq(k+1)k} |\eta_n(z) - \eta_n(kh)| \leqq 2\max_{m\leqq k_n} |\textstyle\sum_{j=1}^m X_{nj} I(U_{n,j-1}^2 \leqq \Delta + \delta^2)|.$$

Therefore

$$\begin{aligned} &P(\sup_{kh<z\leqq(k+1)h} |\eta_n(z) - \eta_n(kh)| > \varepsilon) \\ &\quad = p_{nk} \quad \text{(say)} \\ &\quad \leqq P(\max_{m\leqq k_n} |\textstyle\sum_{j=1}^m X_{nj} I(kh - \Delta < u_n(\delta)^{-2}U_{n,j-1}^2 \leqq (k+1)h + \Delta)| > \varepsilon/2) \\ &\qquad + P(\max_{m\leqq k_n} |\textstyle\sum_{j=1}^m X_{nj} I(U_{n,j-1}^2 \leqq \Delta + \delta^2)| > \varepsilon/2) \\ &\qquad + P(\tilde{F}_n \cap \tilde{G}_n). \end{aligned} \tag{4.18}$$

Apply Brown's Lemma 4 to the martingale

$$M_{nm} = \textstyle\sum_{j=1}^m X_{nj} I(kh - \Delta < u_n(\delta)^{-2}U_{n,j-1}^2 \leqq (k+1)h) \qquad 1 \leqq m \leqq k_n$$

with $M_n = M_{nk_n}$ to prove that the first term in (4.18) does not exceed

$$(\varepsilon/4)^{-1} \textstyle\int_{\{|M_n|>\varepsilon/4\}} |M_n|\, dP \leqq (\varepsilon/4)^{-1} \int_{\{|M_n|>C\}} |M_n|\, dP + (\varepsilon/4)^{-1} \int_{\tilde{L}_n} |M_n|\, dP + (\varepsilon/4)^{-1} \int_{L_n \cap \{C \geqq |M_n| > \varepsilon/4\}} |M_n|\, dP \tag{4.19}$$

where C is a large positive constant, $\varepsilon_1 > 0$ and L_n is the set

$$L_n = \{|\eta_n(kh - \Delta) - \eta_n'(\delta, kh - \Delta)| < \varepsilon_1;\ |\eta_n((k+1)h + \Delta) - \eta_n'(\delta, (k+1)h + \Delta)| < \varepsilon_1\}\,.$$

A Chebyshev-type inequality bounds the first term on the right-hand side of (4.19) by

$$\begin{aligned}(4/\varepsilon C)E|M_n|^2 &= (4/\varepsilon C)E[\textstyle\sum_j X_{nj}^2 I(kh - \Delta < u_n(\delta)^{-2}U_{n,j-1}^2 \leqq (k+1)h + \Delta)]\\ &\leqq (8\lambda/\varepsilon C)E[u_n(\delta)^{-2} \textstyle\sum_j X_{nj}^2 I(kh - \Delta < u_n(\delta)^{-2}U_{n,j-1}^2 \leqq (k+1)h + \Delta)]\\ &\leqq (8\lambda/\varepsilon C)E[2 + \delta^{-2}\max_{j \leqq k_n} X_{nj}^2] \qquad \text{if} \quad (k+1)h + \Delta \leqq 2\\ &= 16\lambda/\varepsilon C + o(1)\,.\end{aligned}$$

The second term on the right-hand side of (4.19) does not exceed

$$(4/\varepsilon)(P(\tilde{L}_n)E|M_n|^2)^{\frac12} \leqq (4/\varepsilon)(P(\tilde{L}_n)4\lambda)^{\frac12} + o(1)\,.$$

$$\begin{aligned}|M_n| &= |\eta_n'(\delta, (k+1)h + \Delta) - \eta_n'(\delta, kh - \Delta)|\\ &\leqq |\eta_n((k+1)h + \Delta) - \eta_n(kh - \Delta)| + 2\varepsilon_1 \qquad \text{on} \quad L_n\\ &= |M_n'| + 2\varepsilon_1\,, \quad \text{say,}\end{aligned}$$

and so the third term on the right-hand side of (4.19) does not exceed

$$(4/\varepsilon)(2\varepsilon_1 + \textstyle\int_{\{C+2\varepsilon_1 \geqq |M_n'| > \varepsilon/4 - 2\varepsilon_1\}} |M_n'|\, dP)\,.$$

The finite-dimensional distributions of η_n converge to those of W_F, and hence

$$\lim_{n\to\infty} \textstyle\int_{\{C+2\varepsilon_1 \geqq |M_n'| > \varepsilon/4 - 2\varepsilon_1\}} |M_n'| = E[\int_{\varepsilon/4-2\varepsilon_1}^{C+2\varepsilon_1} x\, dP(|N(0, T(h + 2\Delta))| \leqq x)$$

where T is held constant in the innermost integral on the right. (A slight adjustment has to be made for the first and last terms, when $k = 0$ or $(k+1)h \geqq 1$.) Kolmogorov's inequality bounds the second term in (4.18) by $(\varepsilon/2)^{-2}(\Delta + \delta^2) + o(1)$, and by (4.5) the last term is $o(1)$. Combining all the results from (4.18) down, we see that

$$\begin{aligned}\limsup_{n\to\infty} p_{nk} \leqq\ & 16\lambda/\varepsilon C + (4/\varepsilon)(\limsup_{n\to\infty} P(\tilde{L}_n)4\lambda)^{\frac12}\\ &+ (4/\varepsilon)E[\textstyle\int_{\varepsilon/4-2\varepsilon_1}^{C+2\varepsilon_1} x\, dP\,(|N(0, T(h + 2\Delta))| \leqq x)]\\ &+ (4/\varepsilon^2)(\Delta + \delta^2)\,.\end{aligned}$$

Let $\delta \to 0$ and then $\varepsilon_1 \to 0$, $\Delta \to 0$ and $C \to \infty$, to prove that

$$\begin{aligned}\limsup_{n\to\infty} p_{nk} &\leqq (4/\varepsilon)E[\textstyle\int_{\varepsilon/4}^{\infty} x\, dP\,(|N(0, Th)| \leqq x)]\\ &= (4/\varepsilon)E[(2/\pi)^{\frac12}(Th)^{\frac12} \exp(-\frac12(\varepsilon/4)^2 T^{-1}h^{-1})]\,.\end{aligned}$$

Returning to (4.17),

$$\limsup_{n\to\infty} \textstyle\sum_{kh<1} p_{nk} \leqq (4/\varepsilon)(2/\pi)^{\frac12}E[(T/h)^{\frac12} \exp(-\frac12(\varepsilon/4)^2T^{-1}h^{-1})] + \delta_h$$

where δ_h is a correction for the first and last terms in the series. The integrand of the expectation on the right converges a.s. to 0 as $h \to 0$ and is dominated by $(4/\varepsilon)T \leqq (4/\varepsilon)\lambda$. Hence the expectation itself converges to 0, and since $\delta_h \to 0$, this establishes (4.17).

This proves (4.2) in the case where T is essentially bounded. A proof in the more general case follows via a truncation argument like that used in the proof of (4.1). □

Now let us prove the theorem. Let $Y_{nj} = X_{nj} - E(X_{nj} \mid \mathscr{G}_{n,j-1})$, $T_{nj} = \sum_{i=1}^{j} Y_{nj}$, $V_{nj}^2 = \sum_{i=1}^{j} Y_{ni}^2$ and $V_n^2 = V_{nk_n}^2$. Let β_n be the random element of $D[0, 1]$ defined by

$$\beta_n(z) = \sum_j Y_{nj} I(V_n^{-2} V_{nj}^2 \leqq z) .$$

Condition (2.1) implies that

$$E(\max_{j \leq k_n} Y_{nj}^2) \rightarrow_p 0$$

and so the martingales $\{(T_{nj}, \mathscr{G}_{nj}), 1 \leqq j \leqq k_n\}$, $n \geqq 1$, satisfy the analogue of (2.1).

$$\begin{aligned} U_n^{-2} \max_{j \leq k_n} |U_{nj}^2 - V_{nj}^2| &\leqq 2U_n^{-2}(\textstyle\sum_j X_{nj}^2)^{\frac{1}{2}}(\sum_j |E(X_{nj} \mid \mathscr{G}_{n,j-1})|^2)^{\frac{1}{2}} \\ &\quad + U_n^{-2} \textstyle\sum_j |E(X_{nj} \mid \mathscr{G}_{n,j-1})|^2 \\ &= 2\delta_n + \delta_n^2 \end{aligned}$$

where $\delta_n = U_n^{-1}(\sum_j |E(X_{nk} \mid \mathscr{G}_{n,j-1})|^2)^{\frac{1}{2}}$. By (2.8), $\delta_n \rightarrow_p 0$ and so

$$U_n^{-2} \max_{j \leq k_n} |U_{nj}^2 - V_{nj}^2| \rightarrow_p 0 . \tag{4.20}$$

In particular, $U_n^2 - V_n^2 \rightarrow_p 0$ and the analogues of (2.2) and (2.7) hold for the martingales $\{(T_{nj}, \mathscr{G}_{nj}), 1 \leqq j \leqq k_n\}$. The analogue of (2.11) holds, and the lemma gives the analogues of (4.1) and (4.2):

$$\begin{aligned} \lim_{n\to\infty} |E \exp[i \textstyle\sum_k t_k(\beta_n(z_k) - \beta_n(z_{k-1})) + isV_n^2] \\ - E \exp[-\frac{1}{2}T \textstyle\sum_k t_k^2(z_k - z_{k-1}) + isT]| = 0 , \end{aligned} \tag{4.1$'$}$$

and

(4.2)′ the sequence of random elements $\{\beta_n\}_1^\infty$ is tight.

The theorem will follow from the result

$$\sup_{z \in [0,1]} |\eta_n(z) - \beta_n(z)| \rightarrow_p 0 . \tag{4.21}$$

On the set $\{\sup_{j \leq k_n} |U_n^{-2}U_{nj}^2 - V_n^{-2}V_{nj}^2| \leqq \delta\}$ we have

$$\begin{aligned} \sup_{z \in [0,1]} |\eta_n(z) - \beta_n(z)| &\leqq \sup_{z,w \in [0,1]; |z-w| \leq \delta} |\beta_n(w) - \beta_n(z)| \\ &\quad + U_n^{-1} \textstyle\sum_j |E(X_{nj} \mid \mathscr{G}_{n,j-1})| \end{aligned}$$

and so for any $\varepsilon > 0$,

$$\begin{aligned} &P(\sup_{z \in [0,1]} |\eta_n(z) - \beta_n(z)| > \varepsilon) \\ &\quad \leqq P(\sup_{|z-w| \leq \delta} |\beta_n(w) - \beta_n(z)| > \varepsilon/2) \\ &\quad\quad + P(U_n^{-1} \textstyle\sum_j |E(X_{nj} \mid \mathscr{G}_{n,j-1})| > \varepsilon/2) \\ &\quad\quad + P(\sup_{j \leq k_n} |U_n^{-2}U_{nj}^2 - V_n^{-2}V_{nj}^2| > \delta) . \end{aligned} \tag{4.22}$$

In view of (2.8) and (4.20), the second and third terms are $o(1)$ as $n \rightarrow \infty$, and because of the tightness of $\{\beta_n\}_1^\infty$,

$$\lim_{\delta\to 0} \limsup_{n\to\infty} P(\sup_{|z-w|\le\delta} |\beta_n(w) - \beta_n(z)| > \varepsilon) = 0 .$$

Hence (4.21) follows from (4.22), and the theorem is proved. □

PROOF OF COROLLARY. We will only prove that (2.1), (2.10), and (2.11) or (2.12) are sufficient for (2.13). If (2.11) holds, apply the Theorem with $\mathscr{G}_{nj} = \mathscr{F}_{nj}$ and $u_n^2 = T$. (2.8) follows from (2.6), and so (2.9) holds. If (2.12) is true, choose integers $l_n \leqq k_n$ such that $l_n \uparrow \infty$ and $l_n U_n^{-1} \max_{j\le k_n} |X_{nj}| \to_p 0$. T is measurable in the σ-field generated by $\bigcup_n \mathscr{F}_{nk_n} = \bigcup_n \mathscr{F}_{nl_n}$. Let $\mathscr{G}_n = \mathscr{F}_{nl_n}$. For each $\varepsilon > 0$ we can find an n and a $\mathscr{G}_n$-measurable variable u^2 such that $P(|T - u^2| > \varepsilon) < \varepsilon$. Hence we can choose a sequence $\{u_n^2\}$ adapted to $\{\mathscr{G}_n\}$ so that (2.2) is satisfied. Let $\mathscr{G}_{nj} = \mathscr{G}_n$ if $j \leqq l_n$; $\mathscr{F}_{nj}$ if $j > l_n$. Then

$$U_n^{-1} \sum_j |E(X_{nj} \mid \mathscr{G}_{n,j-1})| = U_n^{-1} \sum_{j=1}^{l_n} |X_{nj}| \leqq l_n U_n^{-1} \max_{j\le k_n} |X_{nj}| \to_p 0 .$$

This proves (2.8), and (2.9) now follows from the theorem.

Acknowledgments. The author would like to express his gratitude to Dr. C.C. Heyde for suggesting this problem and encouraging the research, and to Dr. G. K. Eagleson for several helpful remarks. He thanks the referee for his constructive criticism which has done much to clarify the presentation.

Note added in proof. Since the preparation of this paper, related work due to Rootzen (two papers, *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **38** 199–216) has appeared. It contains NSC's for martingale invariance principles using functionals slightly different from our own, and a 1-dimensional version of our corollary.

REFERENCES

[1] BILLINGSLEY, P. (1968). *Convergence of Probability Measures.* Wiley, New York.
[2] BLACKWELL, D. and FREEDMAN, D. A. (1973). On the amount of variance needed to escape from a strip. *Ann. Probability* **1** 772–787.
[3] BREIMAN, L. (1968). *Probability.* Addison-Wesley, Reading, Mass.
[4] BROWN, B. M. (1971). Martingale central limit theorems. *Ann. Math. Statist.* **42** 59–66.
[5] CHATTERJI, S. D. (1974). A principle of subsequences in probability theory: the central limit theorem. *Advances in Math.* **13** 31–54.
[6] DROGIN, R. (1972). An invariance principle for martingales. *Ann. Math. Statist.* **43** 602–620.
[7] EAGLESON, G. K. (1975). Martingale convergence to mixtures of infinitely divisible laws. *Ann. Probability* **3** 557–562.
[8] FREEDMAN, D. A. (1975). On tail probabilities for martingales. *Ann. Probability* **3** 100–118.
[9] MCLEISH, D. L. (1974). Dependent central limit theorems and invariance principles. *Ann. Probability* **2** 620–628.
[10] PARTHASARATHY, K. R. (1967). *Probability Measures on Metric Spaces.* Academic Press, New York.
[11] SCOTT, D. J. (1973). Central limit theorems for martingales and for processes with stationary increments using a Skorokhod representation approach. *Advances in Appl. Probability* **5** 119–137.

DEPARTMENT OF STATISTICS
UNIVERSITY OF MELBOURNE
PARKVILLE, VICTORIA 3052
AUSTRALIA