The Annals of Probability
1978, Vol. 6, No. 1, 174–181

CONVERTING DEPENDENT MODELS INTO INDEPENDENT ONES, PRESERVING ESSENTIAL FEATURES[1]

BY N. LANGBERG, F. PROSCHAN AND A. J. QUINZI

The Florida State University and Temple University

Let T denote the life length of a series system of n components having respective life lengths $T_1, \cdots, T_n$, not necessarily independent. We give necessary and sufficient conditions for the existence of a set of *independent* random variables $\{H_I\}$, I a subset of $\{1, \cdots, n\}$, such that the life length of the original series system and the occurrence of its failure pattern (set of components whose simultaneous failure coincides with that of the system) have the same joint distribution as the life length of a derived series system of components having life lengths $\{H_I\}$ and the occurrence of the corresponding failure pattern of the derived system. We also exhibit explicitly the distributions of these independent random variables $\{H_I\}$. This extends the results of Miller while using more elementary methods.

1. Introduction and summary. Because it is generally easier to analyze a model involving independent random variables than one involving dependent ones, it is useful to have ways of converting models involving dependent variables into models involving independent ones. Of course it is desirable to make such a conversion by preserving essential features of the original (dependent) model. Esary and Marshall (1974) provide such a result by showing that, given a set of (dependent) random life lengths with exponential minima, there exists a set of independent random life lengths such that the distribution of any coherent life function of the original life lengths is preserved. Langberg, Proschan and Quinzi (1977) simplify the methods of Esary and Marshall, display explicitly the distributions of the independent life lengths, and extend the Esary–Marshall result to include additional classes of life distributions. Miller (1977) replaces a dependent model by an independent one, while simultaneously preserving the distribution of the minimum and the probabilities corresponding to certain "failure patterns." Tsiatis (1975) proves a similar result in the context of competing risk theory by assuming that the joint distribution function in the dependent model has continuous partial derivatives. In this paper we extend Miller's result to include (i) necessary as well as sufficient conditions for the replacement of a dependent model by an independent one and (ii) an explicit form for each of the distributions in the dependent model, whereas Miller (1977) proves existence only. An advantage of our approach over that of Miller's is that we are able to

Received February 12, 1976; revised June 3, 1977.

AMS 1970 *subject classifications.* Primary 62N05; Secondary 62E10.

Key words and phrases. Dependent, independent, life length, coherent system, failure pattern, equivalence in life length and failure pattern, fatal shock model, multivariate exponential distribution.

[1] Research sponsored by the Air Force Office of Scientific Reseach under Grant AFOSR 74-2581B.

obtain more general and more useful results by employing techniques which are more elementary.

In Section 2 we present some preliminaries. Section 3 is devoted to illustrating the key idea behind the proof of the main result. In Section 4 we state the main result plus several implications. Section 5 consists of proofs of the results in Section 4.

2. Preliminaries. Throughout this paper we use the following notation. Let $\mathscr{I}$ denote the collection of all nonempty subsets of $\{1, \cdots, n\}$. For each collection $\{H_I, I \in \mathscr{I}\}$ of random variables, the symbol $\mathbf{H}$ denotes the vector of H_I's, where the subscripts I are ordered lexicographically. For random variables X and Y, $X =_{st} Y$ indicates that X and Y have the same distribution. A *life length* T is a nonnegative random variable such that $\lim_{t\to\infty} P(T > t) = 0$.

Let $\mathbf{T} = (T_1, \cdots, T_n)$ be the vector of component life lengths of an n-component system. We say that *failure pattern* I occurs if the simultaneous failures of the components in subset I and of no other components causes (i.e., coincides with) the failure of the system. Define

$$\begin{aligned} \xi(\mathbf{T}) &= I \qquad \text{if failure pattern } I \text{ occurs;} \\ &= \varnothing \qquad \text{otherwise.} \end{aligned}$$

Thus, for example, if $T = \min(T_i, 1 \leqq i \leqq n)$ is the random life length of an n-component series system, then $\xi(\mathbf{T}) = \{i\}$ if and only if $T_i < T_j$ for each $j \neq i$. Let $\mathbf{S}$ and $\mathbf{T}$ represent the vectors of component life lengths of two systems whose system life lengths are S and T, respectively. We say that the two systems are *equivalent in life length and patterns* ($\mathbf{S} =_{LP} \mathbf{T}$) if $P(S > t, \xi(\mathbf{S}) = I) = P(T > t, \xi(\mathbf{T}) = I)$ for each $t \geqq 0$ and each $I \in \mathscr{I}$. Thus, two systems which are equivalent in life length and patterns are such that (i) their life lengths have the same distribution and (ii) the corresponding failure patterns in the two systems have the same probability of occurrence. It is important to note that although we have chosen to employ the language of reliability theory (series system, component, etc.) the results presented here apply to any model where observations include (1) the time at which a particular event occurred and (2) which cause(s) (among a finite number of causes) resulted in the occurrence of the event.

The main result of Miller (1977) can be paraphrased as follows:

THEOREM 2.1 (MILLER). *Let T_i be the life length of component i, $i = 1, \cdots, n$, and let T be the life length of the corresponding series system. Define $F(t) = P(T > t)$. Assume that the functions*

$$F_I(t) = P(T \leqq t, \xi(\mathbf{T}) = I), \qquad I \in \mathscr{I}, \tag{2.1}$$

have no discontinuities in common and that $P[T_i = T_j] = 0$ for all $i \neq j$. Then there exists a vector $\mathbf{S} = (S_1, \cdots, S_n)$ of independent random variables such that $\mathbf{T} =_{LP} \mathbf{S}$ and at least one of the S_i is almost surely finite. The distributions of $S_1, \cdots, S_n$ are uniquely determined on $\{x : \bar{F}(x) > 0\}$.

Thus, whenever the functions (2.1) have no common discontinuities and there are no ties, then one can replace the vector $\mathbf{T} = (T_1, \cdots, T_n)$ of (dependent) life lengths by a vector $\mathbf{S} = (S_1, \cdots, S_n)$ of independent random variables such that $\mathbf{S} =_{LP} \mathbf{T}$. In Section 5 we show that the assumption of no common discontinuities is a necessary as well as a sufficient condition for the replacement of a dependent model by an independent one. Moreover, we provide explicit expressions for the appropriate distributions in the independent model.

3. Illustration of key ideas. In this section the reasoning used to arrive at the general result is illustrated by considering two special cases.

For simplicity we consider the case of a two-component system. Let T_1 and T_2 denote the component life lengths (in general, mutually dependent) in a two-component series system. Let $T = \min(T_1, T_2)$ denote the corresponding system life length. If $\bar{F}_I(\cdot)$ denotes the survival probability corresponding to the time of occurrence of failure pattern I, then

$$\begin{aligned} \bar{F}_1(t) &= P(T > t, T_1 < T_2)\,, \\ \bar{F}_2(t) &= P(T > t, T_2 < T_1)\,, \\ \bar{F}_{12}(t) &= P(T > t, T_1 = T_2)\,, \end{aligned} \tag{3.1}$$

where we write $\bar{F}_i$ for $\bar{F}_{\{i\}}$, $i = 1, 2$, and $\bar{F}_{12}$ for $\bar{F}_{\{1,2\}}$. We do *not* exclude the possibility that $P(T_1 = T_2) > 0$. The problem is to replace the vector $\mathbf{T} = (T_1, T_2)$ of life lengths by a vector $\mathbf{S} = (S_1, S_2)$ such that $\mathbf{S} =_{LP} \mathbf{T}$, where the components of the vector $\mathbf{S}$ are expressible in terms of *independent* random variables.

Let S_1 and S_2 denote the component life lengths in a two-component series system. Suppose that each component fails if it receives a shock. Independent sources of shock are present in the environment—one source for each of the three nonempty subsets of $\{1, 2\}$. A shock from source I simultaneously kills all components in subset I and no other components, $I = \{1\}, \{2\}, \{1, 2\}$. One can imagine each shock being originated by a corresponding "hammerman." Let H_I denote the time (measured from the origin) until a shock from source I occurs, $I \in \mathscr{I}$. Then $S_1 = \min(H_1, H_{12})$, $S_2 = \min(H_2, H_{12})$, and $\min(S_1, S_2) = \min(H_1, H_2, H_{12})$. The two-component model described here is the bivariate "fatal shock" model introduced by Marshall and Olkin (1967). Let $\mathbf{H} = (H_1, H_2, H_{12})$ and let

$$\begin{aligned} \xi^*(\mathbf{H}) &= I \qquad \text{if} \quad H_I < H_J \quad \text{for each} \quad J \neq I\,, \\ &= \varnothing \qquad \text{otherwise.} \end{aligned}$$

Then $P(S > t, \xi(\mathbf{S}) = I) = P(T > t, \xi(\mathbf{T}) = I)$ if and only if

$$P(H > t, \xi^*(\mathbf{H}) = I) = P(T > t, \xi(\mathbf{T}) = I) \tag{3.2}$$

for each $t \geqq 0$ and each $I = \{1\}, \{2\}, \{1, 2\}$, where $H = \min(H_1, H_2, H_{12})$. If (3.2) holds for *every* subset I of $\{1, 2\}$, we write $\mathbf{H} =_{LP} \mathbf{T}$, where now the components of $\mathbf{H}$ are the random times until shocks occur. Hence, the problem will be solved if we determine independent random variables H_1, H_2, and H_{12} such that (3.2) holds for every $t \geqq 0$ and every subset I of $\{1, 2\}$.

Let $F_i(t) = P(T \leqq t, \xi(\mathbf{T}) = \{i\})$, $i = 1, 2, 12$. To illustrate the idea behind the general result (Theorem 4.1), we consider the case where F_i has density f_i with respect to Lebesgue measure.

Let $\bar{G}_i(t) = P(H_i > t) = 1 - G_i(t)$ and let g_i be the density of G_i with respect to Lebesgue measure, $i = 1, 2, 12$. Let $\bar{F}(t) = P(T > t)$. In order that (3.2) above hold, we must have that

$$f_i = g_i \prod_{j \neq i} \bar{G}_j , \quad i = 1, 2, 12 . \tag{3.3}$$

If $H =_{st} T$, it follows from (3.3) that

$$g_i/\bar{G}_i = f_i/\bar{F} , \quad i = 1, 2, 12 . \tag{3.4}$$

We obtain a solution by inegrating both sides of (3.4):

$$\bar{G}_i(t) = \exp[-\textstyle\int_0^t (f_i(x)/\bar{F}(x))\, dx] , \quad i = 1, 2, 12 . \tag{3.5}$$

EXAMPLE 3.1. Suppose that the vector (T_1, T_2) has the Marshall–Olkin bivariate exponential (BVE) distribution with survival probability:

$$P(T_1 > t_1, T_2 > t_2) = \exp[-\lambda_1 t_1 - \lambda_2 t_2 - \lambda_{12} \max(t_1, t_2)] ,$$

for $t_i \geqq 0$, $i = 1, 2$, and $\lambda_i > 0$, $i = 1, 2, 12$. Then T_1 and T_2 are exponentially distributed and mutually dependent. Assuming that T_1 and T_2 represent the component life lengths in a two-component series system as above, we conclude from (3.5) that the original system is equivalent in life length and patterns to a system involving independent times H_1, H_2, and H_{12} until shock, where

$$\bar{G}_i(t) = P(H_i > t) = \exp(-\lambda_i t) , \quad i = 1, 2, 12 . \tag{3.6}$$

We remark that (3.6) is consistent with the well-known characterization of the BVE given by Theorem 3.2 of Marshall and Olkin (1967), namely $T_1 = \min(H_1, H_{12})$ and $T_2 = \min(H_2, H_{12})$.

4. Main result. The main result of this paper is

THEOREM 4.1. *Let $T = \min(T_i, 1 \leqq i \leqq n)$ denote the life length of an n-component series system, where T_i represents the life length of component i, $i = 1, \cdots, n$. Define $\bar{F}_I(t) = P(T > t, \xi(\mathbf{T}) = I)$, $F_I(t) = P(T \leqq t, \xi(\mathbf{T}) = I)$, $\bar{F}(t) = P(T > t)$, and $\alpha(F) = \sup\{x : \bar{F}(x) > 0\}$. Then the following statements hold*:

(i) *A necessary and sufficient condition for the existence of a set of independent random variables $(H_I, I \in \mathscr{I})$ which satisfy $\mathbf{H} =_{LP} \mathbf{T}$, where $H = \min(H_I, I \in \mathscr{I})$, is that the sets of discontinuities of the F_I be pairwise disjoint on the interval $[0, \alpha(F))$.*

(ii) *The distributions of $(H_I, I \in \mathscr{I})$ in* (i) *are uniquely determined on the interval $[0, \alpha(F))$ as follows*:

$$P(H_I > t) = \bar{G}_I(t) = \exp[-\textstyle\int_0^t (dF_I{}^C/\bar{F})] \times \prod_{a(I,j) \leqq t} \{\bar{F}(a(I,j))/[\bar{F}(a(I,j)) + f_I(a(I,j))]\} , \tag{4.1}$$

$0 \leqq t < \alpha(F)$, where $F_I{}^C$ is the continuous part of F_I, $\{a(I,j)\}_j$ is the set of discontinuities of F_I, and $f_I(a(I,j))$ is the size of the jump of F_I at $a(I,j)$.

The proof of Theorem 4.1 is given in Section 5. The remainder of this section is devoted to several remarks related to Theorem 4.1.

REMARK 4.1. The sufficient part of Theorem 4.1(i) above is equivalent to the first part of Theorem 2.1.

REMARK 4.2. A series system is one example of a more general system in reliability known as a *coherent* system [see, for example, Barlow and Proschan (1975)]. A result similar to Theorem 4.1 also holds for arbitrary coherent systems. In Theorem 4.1, if we replace the life length T of a series system by the life length of a coherent system, then (i) and (ii) of Theorem 4.1 hold. For a proof of this remark, see the end of Section 5.

If $\mathbf{T}$ represents the vector of component life lengths of a system with system life length T, we say that failure pattern I is a *nonoccurring pattern* if $F_I(t) = P(T \leqq t, \xi(\mathbf{T}) = I) = 0$ for each $t \geqq 0$. For example, in a two-component series system, if $P(T_1 = T_2) = 0$, then failure pattern $\{1, 2\}$ is a nonoccurring pattern, since then $P(T \leqq t, \xi(\mathbf{T}) = \{1, 2\}) = P(T \leqq t, T_1 = T_2) = 0$ for each $t \geqq 0$.

REMARK 4.3. In (4.1) of Theorem 4.1, if $F_I(t) = 0$ for some $I \in \mathscr{I}$ and each $t \geqq 0$, then the corresponding survival probability $\bar{G}_I$ is identically one on the interval $[0, \alpha(F))$, where we define the product over an empty set as unity. Here, failure pattern I is a nonoccurring pattern. The corresponding random variable H_I in (4.1) is almost surely (a.s.) infinite, corresponding to a hammerman who never strikes, and, for our purposes, can be ignored. In the same way, every nonoccurring pattern can be associated with a hammerman who never strikes. It follows from Theorem 4.1 that if the original system has exactly r occurring failure patterns, $1 \leqq r \leqq 2^n - 1$, then the original system is equivalent in life length and patterns to a system involving the same number r of independent hammermen (random variables) $(H_i, 1 \leqq i \leqq r)$.

5. Proofs. The proof of Theorem 4.1 is based on four lemmas. Throughout this section we use the following notation. For every Lebesgue-Stieltjes measure Q, Q^C denotes the continuous part of Q, $\bar{Q}(t) = Q(t, \infty)$, and $\bar{Q}(t^-) = \bar{Q}(t) + q(t)$, where $q(t)$ is the size of the jump (possibly 0) of $\bar{Q}$ at t. Let $C(Q)$ and $D(Q)$ be the sets of continuity points and discontinuity points, respectively, of the nondecreasing right continuous function associated with Q.

LEMMA 5.1 *For every Lebesgue–Stieltjes measure Q such that $Q \leqq 1$ and $\bar{Q}(0^-) = 1$, and every $t \geqq 0$, the following holds*:

$$-\ln \bar{Q}(t) = \textstyle\int_0^t (dQ^C/\bar{Q}) + \sum_{a_i \leqq t} \ln [\bar{Q}(a_i^-)/\bar{Q}(a_i)], \tag{5.1}$$

where $\{a_i\}_i = D(Q)$.

For a proof of Lemma 5.1, see Lee and Thompson (1975), page 8, or Peterson (1975), page 118.

To prove the sufficiency in (i) of Theorem 4.1, it suffices to find a set of probability distributions $(G_I, I \in \mathscr{I})$ on $[0, \infty)$ for the random variables $(H_I, I \in \mathscr{I})$

which satisfy

$$\prod_{I \in \mathscr{I}} \bar{G}_I(t) = \bar{F}(t), \quad t \geqq 0, \tag{5.2}$$

and

$$\int_{0-}^{t} (dG_I/\bar{G}_I) = \int_{0-}^{t} (dF_I/\bar{F}), \quad I \in \mathscr{I}, \qquad 0 \leqq t < \alpha(F).$$

To see this, note that if, for each $I \in \mathscr{I}$, H_I has distribution G_I satisfying (5.2), and if the H_I's are independent, then

$$\begin{aligned} P(H > t, \xi^*(\mathbf{H}) = I) &= P(H > t, H_I < H_J \text{ for each } J \neq I) \\ &= \int_t^\infty \prod_{J \neq I} \bar{G}_J \, dG_I = \int_t^\infty (\bar{F}/\bar{G}_I) \, dG_I = \int_t^\infty dF_I = \bar{F}_I(t), \end{aligned}$$

so that $\mathbf{H} =_{LP} \mathbf{T}$.

LEMMA 5.2. *If the sets of discontinuities of the* F_I *are pairwise disjoint on* $[0, \alpha(F))$, *then* $(G_I, I \in \mathscr{I})$ *defined by* (4.1) *satisfy* (5.2).

PROOF. Suppose $0 \leqq t < \alpha(F)$. Then

$$\begin{aligned} \prod_I \bar{G}_I(t) &= \prod_I \prod_{a(I,j) \leqq t} \{\bar{F}(a(I,j))/[\bar{F}(a(I,j)) + f_I(a(I,j))]\} \exp[-\int_0^t (dF_I{}^C/\bar{F})] \\ &= \exp[-\int_0^t (dF^C/\bar{F})] \prod_{a_j \leqq t; a_j \in D(F)} [\bar{F}(a_j)/\bar{F}(a_j{}^-)] = \bar{F}(t) \end{aligned}$$

by Lemma 5.1.

To complete the proof, we must verify the second equality in (5.2). It follows from (4.1) that $\bar{G}_I(a)/\bar{G}_I(a^-) = \bar{F}(a)/[\bar{F}(a) + f_I(a)]$, $0 \leqq a < \alpha(F)$, $I \in \mathscr{I}$. Hence, $D(F_I) = D(G_I)$ for every $I \in \mathscr{I}$. Consequently,

$$\int_{[0,t] \cap D(F_I)} (dG_I/\bar{G}_I) = \int_{[0,t] \cap D(F_I)} (dF_I/\bar{F}).$$

By Lemma 5.1 again,

$$\begin{aligned} \int_0^t (dG_I{}^C/\bar{G}_I) &= -\ln \bar{G}_I(t) - \sum_{a(I,j) \leqq t} \ln [\bar{G}_I(a(I,j)^-)/\bar{G}_I(a(I,j))] \\ &= -\ln \bar{G}_I(t) + \sum_{a(I,j)) \leqq t} \ln \bar{F}(a(I,j))/[\bar{F}(a(I,j)) + f_I(a(I,j))]\} \\ &= \int_0^t (dF_I{}^C/\bar{F}). \end{aligned}$$ □

LEMMA 5.3. *There is at most one collection* $(G_I, I \in \mathscr{I})$ *of distributions which satisfies* (5.2) *on the interval* $[0, \alpha(F))$.

PROOF. For t in the interval $[0, \alpha(F))$, define

$$M([0, t]) = \int_{0-}^{t} (dG_I/\bar{G}_I).$$

We view M as a measure on the Borel field of $[0, \alpha(F))$. By (5.2), it suffices to show that the measure M uniquely determines G_I on the interval $[0, \alpha(F))$. This is immediate since by Lemma 5.1,

$$-\ln \bar{G}_I(t) = M([0, t] \cap C(M)) + \sum_{a(I,j) \leqq t} \ln M\{a(I,j)\} + 1)$$

for every t in the interval $[0, \alpha(F))$, where $\{a(I,j)]_j = D(F_I)$. □

LEMMA 5.4. *In order that a collection* $(G_I, I \in \mathscr{I})$ *of distributions satisfy* (5.2), *it is necessary and sufficient that the sets of discontinuities of the* F_I *be pairwise disjoint on* $[0, \alpha(F))$.

PROOF. The sufficiency was shown in Lemma 5.2. To prove the necessity, let $(G_I, I \in \mathscr{I})$ be a collection of distributions satisfying (5.2). Then

$$\prod_I \int_0^\infty dG_I = \bar{F}(0) = \sum_I \int_0^\infty \prod_{J \neq I} \bar{G}_J \, dG_I .$$

In the above expression, the term on the left represents the measure on the positive orthant in $(2^n - 1)$-dimensional space. The term on the right represents the measure of a subset of the positive orthant. Hence the remainder has measure zero, i.e.,

$$\int_{[0,\alpha(F))} \prod_{K \neq I; K \neq J} \bar{G}_K(t) G_J(t) \, dG_I(t) = 0$$

for every $J \neq I$, which implies that $[0, \alpha(F)) \cap D(G_I) \cap D(G_J) = \emptyset$ whenever $J \neq I$. The conclusion follows since $D(F_I) = D(G_I)$ for every $I \in \mathscr{I}$. □

PROOF OF THEOREM 4.1. In Lemmas 5.2, 5.3, and 5.4, note that $\bar{F} = \sum_I \bar{F}_I$ and define $\bar{F}_I(A) = P(T \in A, T = T_i \text{ for } i \in I, T \neq T_i \text{ for } i \notin I)$, $I \in \mathscr{I}$, for every subset A of the interval $[0, \alpha(F))$, where $T_1, \cdots, T_n$ and T are as specified in the hypotheses. Then part (i) of the theorem follows from Lemma 5.4, and part (ii) follows from Lemmas 5.2 and 5.3. □

To prove Remark 4.2, simply replace T in the proof of Theorem 4.1 by the life length of the coherent system.

REMARK 5.5. In Theorem 4.1, the possibility exists that for some I, $\lim_{t\to\infty} P(H_I > t) > 0$. Suppose $\mathbf{H} =_{LP} \mathbf{T}$. Then $(\min(H_I, I \in \mathscr{I}) =_{st} \min(T_i, 1 \leqq i \leqq n)$. Since each T_i is a life length, it follows that $\lim_{t\to\infty} P(H_I > t) = 0$ for *at least one* I.

In this remark we state three conditions each of which is sufficient for *all* of the random variables $(H_I, I \in \mathscr{I})$ in Theorem 4.1 to be almost surely finite.

(i) Clearly, if the distribution of at least one T_i has finite support, then the distribution of $\min(T_i, 1 \leqq i \leqq n)$ has finite support, and $\alpha(F) < \infty$ in Theorem 4.1. If we define $\bar{G}_I(t) = 0$ for every $t \geqq \alpha(F)$ and every $I \in \mathscr{I}$, then each of the random variables H_I in (4.1) is almost surely finite.

(ii) Assume now that the distribution of T_i has finite support and that $P(T_i = T_j) = 0$ whenever $i \neq j$. If $T_1, \cdots, T_n$ are exchangeable random variables, the survival probabilities represented in (4.1) are all equal. By Remark 4.3, the original system is equivalent in life length and failure patterns to a system involving n independent random variables H_i, $1 \leqq i \leqq n$. Since at least one H_i is almost surely finite, it follows that each H_i is.

(iii) In Theorem 4.1, assume that the functions F_I, $I \in \mathscr{I}$, have no common discontinuities in the interval $[0, \alpha(F))$, where $\alpha(F) = \infty$, and that $P(T_i = T_j) = 0$ whenever $i \neq j$. In addition, suppose that for each $I \in \mathscr{I}$, the indicator function $\chi_{[\xi(\mathbf{T})=I]}$ of the event $[\xi(\mathbf{T}) = I]$ and the random variable $T = \min(T_i, 1 \leqq i \leqq n)$ are independent. By Remark 4.3, the original system is equivalent in life length and failure patterns to a system involving n independent random variables H_i, $1 \leqq i \leqq n$. It was conjectured by Miller (1977) that under the above assumptions each H_i may be chosen to be almost surely finite. We prove his conjecture as follows.

Let $\pi_i = P(\xi(\mathbf{T}) = i)$, $1 \leqq i \leqq n$. Since T and χ are independent, $\bar{F}_i(t) = \pi_i \bar{F}(t)$ for every $t \geqq 0$ and every i, $1 \leqq i \leqq n$. If $n = 1$, then the conclusion holds since then there is exactly one random variable H_i and, by the argument above, it must be almost surely finite. Hence suppose $n \geqq 2$. Then F is continuous since the functions $F_i(\cdot)$, $1 \leqq i \leqq n$, have no common discontinuities. Hence, (4.1) can be expressed as

$$\bar{G}_i(t) = \exp[-\pi_i \textstyle\int_0^t (dF/\bar{F})] = [\bar{F}(t)]^{\pi_i},$$

which completes the proof.

REFERENCES

BARLOW, R. E. and PROSCHAN, F. (1975). *Statistical Theory of Reliability and Life Testing: Probability Models.* Holt, Rinehart, and Winston, New York.

ESARY, J. D. and MARSHALL, A. W. (1974). Multivariate distributions with exponential minimums. *Ann. Statist.* **2** 84–98.

LANGBERG, N., PROSCHAN, F., and QUINZI, A. J. (1977). Converting dependent models into independent ones, with applications in reliability. In *The Theory and Applications of Reliability, with Emphasis on Bayesian and Nonparametric Methods,* **1** (I. Shimi and C. P. Tsokos, eds.) 259–276. Academic Presss, New York.

LEE, L. and THOMPSON, W. A., Jr. (1975). Failure rate—a unified approach. University of Missouri—Columbia Technical Report No. 51.

MARSHALL, A. W. and OLKIN, I. (1967). A multivariate exponential distribution. *J. Amer. Statist. Assoc.* **62** 30–44.

MILLER, D. R. (1977). A note on independence of multivariate lifetimes in competeting risks models. *Ann. Statist.* **5** 576–579.

PETERSON, A. V., Jr. (1975). Nonparametric estimation in the competing risks problem. Stanford University Technical Report No. 13.

TSIATIS, A. (1975). A nonidentifiability aspect of the problem of competing risks. *Proc. Nat. Acad. Sci. USA.* **72.**

DEPARTMENT OF STATISTICS AND
STATISTICAL CONSULTING CENTER
THE FLORIDA STATE UNIVERSITY
TALLAHASSEE, FLORIDA 32306

DEPARTMENT OF STATISTICS
TEMPLE UNIVERSITY
PHILADELPHIA, PA 19122

DEPARTMENT OF STATISTICS AND
STATISTICAL CONSULTING CENTER
THE FLORIDA STATE UNIVERSITY
TALLAHASSEE, FLORIDA 32306