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PROCESSES THAT CAN BE EMBEDDED IN BROWNIAN MOTION[1]

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A process is equivalent to a time change of Brownian motion if and only if it is a local semimartingale.

1. In this paper, it is shown that a process $(X_s, \mathscr{F}_s)$ can be embedded in Brownian motion if and only if $(X_s, \mathscr{F}_s)$ is a local semimartingale. To embed a process in Brownian motion is to find a Wiener process $(W_t, \mathscr{G}_t)$ and an increasing family of $\mathscr{G}_t$ stopping times T_s such that W_{T_s} has the same joint distributions as X_s.

The embedding problem was treated first by Skorohod [16] who showed that if X_n is any sequence of independent random variables with mean zero and finite variance, then $Y_n = \sum_{i=1}^n X_i$ could be embedded in Brownian motion with stopping times T_n such that $E\{T_n\} = E\{Y_n^2\}$. A number of other methods of defining the stopping times were then proposed by Dubins [3], Root [14] and others. In [11], Monroe generalized the procedure somewhat by embedding the random walk in a symmetric stable process of index $\alpha > 1$ but the stopping times generally had infinite expectations in this case.

In the meantime the embedding procedure was used by several people to obtain either new results or elegant proofs of known results. See, for instance, Strassen's paper on the law of the iterated logarithm [17].

Röst [15] took up the problem for a general Markov process and obtained necessary and sufficient conditions on measures μ and ν to guarantee the existence of a stopping time T such that the process started with distribution μ would have distribution ν when stopped at time T.

As early as 1965, Dambis in the Soviet Union [2] and Dubins and Schwartz in the U. S. [4] had shown that every continuous martingale could be time-changed into Brownian motion. In [6], Huff showed that every process of pathwise bounded variation could be embedded in Brownian motion. Independently it was shown by Monroe in [12] that every right continuous martingale could be embedded in a Brownian motion process with sufficiently large σ-fields. The present paper is a rather natural extension of [12].

Extensive use will be made of what Föllmer [5] calls processes of bounded variation.

DEFINITION. A process X_s adapted to $\mathscr{F}_s$ is said to be of bounded variation

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if $V(X) = \sup \sum_{i=1}^n E\{|X_{s_{i-1}} - E\{X_{s_i} \mid \mathscr{F}_{s_{i-1}}\}|\} + E\{|X_{s_n}|\} < \infty$ where the supremum is over all partitions $s_0 < s_1 < \cdots < s_n < \infty$. The crucial fact is that a process of bounded variation is a semimartingale as defined by Meyer [8]. This was first demonstrated by Rao in [13] and is also presented very nicely in [5].

More generally, for a fixed a and b, $a < b$ and an $\mathscr{F}_s$ stopping time S, we will use

$$V(X\colon a, b, S) = \sup \sum_{i=1}^n E\{|E\{(X_{s_i} - X_{s_{i-1}})I_{\{s_i < S\}} \mid \mathscr{F}_{s_{i-1}}\}|\}$$

where the sup is over all partitions $a \leqq s_0 < s_1 < \cdots < s_n \leqq b$. For a discrete time process $(X_n, \mathscr{F}_n)$ and $\mathscr{F}_n$ stopping time S, $V(X\colon a, b, S) = \sum_{i=a+1}^b E\{|E\{(X_i - X_{i-1})I_{\{i<S\}} \mid \mathscr{F}_{i-1}\}|\}$.

Section 2 contains a few preliminary comments. In Section 3, it is shown that if X_s can be embedded in Brownian motion, then it is a local semimartingale with respect to the σ-fields generated by the process X_s. Then in Section 4, it is shown that all local semimartingales can be embedded in Brownian motion.

2. Let $\mathscr{F}$ be a σ-field on some probability space with measure P and let $\mathscr{F}_t$ be an increasing right continuous family of σ-fields contained in $\mathscr{F}$. All processes will be assumed to be right continuous and have left-hand limits and adapted to the σ-fields $\mathscr{F}_t$. For basic facts about martingales, see [8].

DEFINITION. A process $(M_t, \mathscr{F}_t)$ is a local martingale if there is a nondecreasing sequence of stopping times S_n such that $\lim_n S_n = \infty$ and $M_{t \wedge S_n}$ is a uniformly integrable martingale.

DEFINITION. A process $(A_t, \mathscr{F}_t)$ is said to be of pathwise bounded variation if almost surely the paths are of finite total variation on every closed interval $[0, n]$.

DEFINITION. A process $(X_t, \mathscr{F}_t)$ is said to be a local semimartingale if $X_t = M_t + A_t$ where M_t is a local martingale and A_t is of pathwise bounded variation.

It has been proposed by Kazamaki [7] that the definition of local martingale be weakened to require only that for each n, there exist some uniformly integrable martingale such that $M_t = M_t^{(n)}$ on the set $t < S_n$ rather than $t \leqq S_n$ as in the usual definition. This certainly leads to a wider class of local martingales but interestingly, it does not extend the class of local semimartingales.

LEMMA 1. *Let $(X_t, \mathscr{F}_t)$ be a stochastic process. Suppose there is a nondecreasing sequence of stopping times $S_n \uparrow \infty$, a sequence of uniformly integrable martingales $M_t^{(n)}$ adapted to $\mathscr{F}_t$ and a sequence of processes $A_t^{(n)}$ of pathwise bounded variation also adapted to $\mathscr{F}_t$. If for every n, $X_t = M_t^{(n)} + A_t^{(n)}$ on the interval $t < S_n$ then X_t is a local semimartingale.*

PROOF. Define $S_0 = 0$,

$$M_t = M_t^{(n)} - \sum_{k=1}^{n-1} (M_{S_k}^{(k+1)} - M_{S_k}^{(k)}) \quad S_{n-1} < t \leqq S_n$$

and

$$A_t = A_t^{(n)} + \sum_{k=1}^{n-1} (M_{S_k}^{(k+1)} - M_{S_k}^{(k)}) \quad S_{n-1} \leqq t < S_n\,.$$

Clearly the paths of the process A_t are of bounded variation and, just as clearly, $X_t = M_t + A_t$. In addition, $M_{t\wedge S_n}$ is uniformly integrable since $|M_{t\wedge S_n}| \leqq \sum_{k=1}^n |M^{(k)}_{t\wedge S_n}|$. The only problem is to show that M_t is a local martingale. By induction, it is enough to show that $M_{t\wedge S_2}$ is a martingale. But if $s < t$,

$$E\{M_{t\wedge S_2} \mid \mathscr{F}_s\} = E\{M_{t\wedge S_2} I_{\{s > S_1\}} \mid \mathscr{F}_s\} + E\{M_{t\wedge S_2} I_{\{s \leqq S_1 < t\}} \mid \mathscr{F}_s\} + E\{M_{t\wedge S_2} I_{\{t \leqq S_1\}} \mid \mathscr{F}_s\} .$$

Now

$$\begin{aligned} E\{M_{t\wedge S_2} I_{\{s \leqq S_1 < t\}} \mid \mathscr{F}_s\} &= E\{(M_t^{(2)} - M_{S_1}^{(2)} + M_{S_1}^{(1)}) I_{\{s \leqq S_1 < t\}} \mid \mathscr{F}_s\} \\ &= E\{(E\{M_t^{(2)} \mid \mathscr{F}_{S_1}\} - M_{S_1}^{(2)} + M_{S_1}^{(1)}) I_{\{s \leqq S_1 < t\}} \mid \mathscr{F}_s\} \\ &= E\{M_{S_1}^{(1)} I_{\{s \leqq S_1 < t\}} \mid \mathscr{F}_s\} \end{aligned}$$

and

$$E\{M_{t\wedge S_2} I_{\{t \leqq S_1\}} \mid \mathscr{F}_s\} = E\{M^{(1)}_{t\wedge S_1} I_{\{t \leqq S_1\}} \mid \mathscr{F}_s\}$$

so

$$\begin{aligned} E\{M_{t\wedge S_2} I_{\{s \leqq S_1\}} \mid \mathscr{F}_s\} &= E\{M^{(1)}_{t\wedge S_1} I_{\{s \leqq S_1\}} \mid \mathscr{F}_s\} \\ &= M_s^{(1)} I_{\{s \leqq S_1\}} = M_{s\wedge S_2} I_{\{s \leqq S_1\}} . \end{aligned}$$

The first term is more easily dealt with.

3.

THEOREM 1. *Let $(X_t, \mathscr{G}_t)$ be a local semimartingale and X_{T_s} be a time change of X_t. Let $\mathscr{F}_s$ be the family of right continuous σ-fields generated by X_{T_v}, $v \leqq s$. Then $(X_{T_s}, \mathscr{F}_s)$ is a local semimartingale.*

PROOF. It is enough to show that there is an increasing sequence of $\mathscr{F}_s$ stopping times S_n' such that the process

$$\begin{aligned} Z_s &= X_{T_s} \quad & s < S_n' \\ &= 0 & s \geqq S_n' \end{aligned}$$

is of bounded variation. Indeed by Rao's work [13], this would imply that $Z_s = M_s^{(n)} + A_s^{(n)}$ where $M_s^{(n)}$ is a uniformly integrable martingale and $A_s^{(n)}$ is a process whose paths are almost surely of bounded variation. Lemma 1 then asserts that X_{T_s} is a local semimartingale.

To this end let S_n be a sequence of $\mathscr{G}_t$ stopping times such that $\lim_n S_n = \infty$ and $X_t = M_t + A_t$ where $M_{t\wedge S_n}$ is a $\mathscr{G}_t$ uniformly integrable martingale for all n and A_t is a process of pathwise bounded variation adapted to $\mathscr{G}_t$. One can assume that for all $t < S_n$, $|M_t| \leqq n$, $|A|_t = \sup_{0 < s_0 < \cdots < s_n = t} \sum_{i=1}^n |A_{s_i} - A_{s_{i-1}}| \leqq n$, and $|X_t| \leqq n$. Define

$$S_n' = \inf\{s : P\{S_n \leqq T_s \mid \mathscr{F}_s\} \geqq \tfrac{1}{8}\} .$$

Before continuing with the proof of Theorem 1, we will prove some preliminary lemmas.

LEMMA 2. *If S is a $\mathscr{G}_t$ stopping time such that $S \leqq S_n$, then for any $s_0 < s_1 < \cdots < s_k$*

$$E\{\textstyle\sum_{i=1}^k |E\{(X_{T_{s_i}} - X_{T_{s_{i-1}}}) I_{\{T_{s_i} < S\}} \mid \mathscr{G}_{T_{s_{i-1}}}\}|\} \leqq 2n + E\{|M_{S_n}|\} .$$

PROOF. If $T_s < S$, $|M_{T_s}| < n$. Thus

$$\begin{aligned}
E\{|E\{(M_{T_{s_i}} - M_{T_{s_{i-1}}})I_{\{T_{s_i}<S\}} \mid \mathscr{G}_{T_{s_{i-1}}}\}|\} \\
&= E\{I_{\{T_{s_{i-1}}<S\}}|E\{(M_S - M_{T_{s_{i-1}}})I_{\{T_{s_i}<S\}} \mid \mathscr{G}_{T_{s_{i-1}}}\}|\} \\
&= E\{|E\{(M_S - M_{T_{s_{i-1}}})I_{\{T_{s_{i-1}}<S\leqq T_{s_i}\}} \mid \mathscr{G}_{T_{s_{i-1}}}\}|\} \\
&\leqq E\{|E\{M_S I_{\{T_{s_{i-1}}<S\leqq T_{s_i}\}} \mid \mathscr{G}_{T_{s_{i-1}}}\}|\} \\
&\quad + E\{|E\{M_{T_{s_{i-1}}} I_{\{T_{s_{i-1}}<S\leqq T_{s_i}\}} \mid \mathscr{G}_{T_{s_{i-1}}}\}|\} \\
&\leqq E\{|M_{S_n}|I_{\{T_{s_{i-1}}<S\leqq T_{s_i}\}}\} + nP\{T_{s_{i-1}} < S \leqq T_{s_i}\}\,.
\end{aligned}$$

Also

$$E\{|E\{(A_{T_{s_i}} - A_{T_{s_{i-1}}})I_{\{T_{s_i}<S\}} \mid \mathscr{G}_{T_{s_i}}\}|\} \leqq E\{|A_{T_{s_i}} - A_{T_{s_{i-1}}}| I_{\{T_{s_i}<S\}}\}\,.$$

Summing on i one obtains

$$\begin{aligned}
\textstyle\sum_{i=1}^k E\{|E\{(X_{T_{s_i}} - X_{T_{s_{i-1}}})I_{\{T_{s_i}<S\}} \mid \mathscr{G}_{T_{s_{i-1}}}\}|\} \\
&\leqq E\{|M_{S_n}|\} + n + E\{\textstyle\sum_{i=1}^k |A_{T_{s_i}} - A_{T_{s_{i-1}}}|I_{\{T_{s_i}<S\}}\} \\
&\leqq E\{|M_{S_n}|\} + 2n
\end{aligned}$$

by the choice of S_n.

LEMMA 3. *Let* $P_s = P\{S_n \leqq T_s \mid \mathscr{F}_s\}$. *Then*

$$\begin{aligned}
E\{\textstyle\sum_{i=1}^k |E\{(X_{T_{s_i}} - X_{T_{s_{i-1}}})P_{s_i}I_{\{S_n'>s_i\}} \mid \mathscr{F}_{s_{i-1}}\}|\} \\
&< 14n + (\tfrac{4}{5})E\{\textstyle\sum_{i=1}^k |E\{(X_{T_{s_i}} - X_{T_{s_{i-1}}})I_{\{S_n'>s_i\}} \mid \mathscr{F}_{s_{i-1}}\}|\}\,.
\end{aligned}$$

PROOF. Consider the function

$$F(x, p) = (x - a)p + 2[(10n^2 - x^2)(1 - p^2)]^{\frac{1}{2}}$$

with $a \leqq n$. One computes that $|\partial F/\partial x| \leqq \frac{4}{5}$, $|\partial F/\partial p| \leqq 3n$, $\partial^2F/\partial x^2 \leqq 0$, $\partial^2F/\partial p^2 \leqq 0$ and $(\partial^2F/\partial x^2)(\partial^2F/\partial p^2) - [\partial^2F/\partial x\,\partial p]^2 \geqq 0$ as long as $0 \leqq p \leqq \frac{1}{8}$ and $|x| \leqq n$. Thus

$$F(x, p) - F(a, p_0) - \frac{\partial F}{\partial x}(a, p_0)(x - a) - \frac{\partial F}{\partial p}(a, p_0)(p - p_0) \leqq 0$$

and letting $a = X_{T_{s_{i-1}}}$ and $p_0 = P_{s_{i-1}}$

$$\begin{aligned}
E\{(X_{T_{s_i}} - X_{T_{s_{i-1}}})P_{s_i}I_{\{S_n'>s_i\}} \mid \mathscr{F}_{s_{i-1}}\} \\
&\leqq 2E\{I_{\{S_n'>s_i\}}[(10n^2 - X^2_{T_{s_{i-1}}})(1 - P^2_{s_{i-1}})]^{\frac{1}{2}} \mid \mathscr{F}_{s_{i-1}}\} \\
&\quad - 2E\{I_{\{S_n'>s_i\}}[(10n^2 - X^2_{T_{s_i}})(1 - P^2_{s_i})]^{\frac{1}{2}} \mid \mathscr{F}_{s_{i-1}}\} \\
&\quad + (\tfrac{4}{5})|E\{(X_{T_{s_i}} - X_{T_{s_{i-1}}})I_{\{S_n'>s_i\}} \mid \mathscr{F}_{s_{i-1}}\}| \\
&\quad + 3n|E\{(P_{s_i} - P_{s_{i-1}})I_{\{S_n'>s_i\}} \mid \mathscr{F}_{s_{i-1}}\}|\,.
\end{aligned}$$

The same upper bound can be obtained for $-E\{(X_{T_{s_i}} - X_{T_{s_{i-1}}})P_{s_i}I_{\{S_n'>s_{i-1}\}} \mid \mathscr{F}_{s_{i-1}}\}$ by using the function $F(x, p) = -(x - a)p + 2[(10n^2 - x^2)(1 - p^2)]^{\frac{1}{2}}$. Summing on i, one has

$$
\begin{aligned}
E\{\textstyle\sum_{i=1}^{k} &|E\{(X_{T_{s_i}} - X_{T_{s_{i-1}}})P_{s_i} I_{\{S_n' > s_i\}} \mid \mathscr{F}_{s_{i-1}}\}|\} \\
&\leqq 2E\{[(10n^2 - X_{T_{s_0}}^2)(1 - P_{s_0}^2)]^{\frac{1}{2}} I_{\{S_n' > s_0\}}\} \\
&\quad + (\tfrac{4}{5})E\{\textstyle\sum_{i=1}^{k} |E\{(X_{T_{s_i}} - X_{T_{s_{i-1}}}) I_{\{S_n' > s_i\}} \mid \mathscr{F}_{s_{i-1}}\}|\} \\
&\quad + 3n \textstyle\sum_{i=1}^{k} E\{(P_{s_i} - P_{s_{i-1}}) I_{\{S_n' > s_{i-1}\}}\} + 3n \sum_{i=1}^{k} P\{s_{i-1} < S_n' \leqq s_i\} \\
&< 14n + (\tfrac{4}{5})E\{\textstyle\sum_{i=1}^{k} |E\{(X_{T_{s_i}} - X_{T_{s_{i-1}}}) I_{\{S_n' > s_i\}} \mid \mathscr{F}_{s_{i-1}}\}|\} .
\end{aligned}
$$

Returning to the proof of Theorem 1,

$$
\begin{aligned}
E\{\textstyle\sum_{i=1}^{k} &|E\{(X_{T_{s_i}} - X_{T_{s_{i-1}}}) I_{\{S_n' > s_i\}} \mid \mathscr{F}_{s_{i-1}}\}|\} \\
&\leqq E\{\textstyle\sum_{i=1}^{k} |E\{(X_{T_{s_i}} - X_{T_{s_{i-1}}}) I_{\{S_n > T_{s_i}, S_n' > s_i\}} \mid \mathscr{F}_{s_{i-1}}\}|\} \\
&\quad + E\{\textstyle\sum_{i=1}^{k} |E\{(X_{T_{s_i}} - X_{T_{s_{i-1}}}) I_{\{S_n \leqq T_{s_i}\}} I_{\{S_n' > s_i\}} \mid \mathscr{F}_{s_{i-1}}\}|\} \\
&\leqq E\{\textstyle\sum_{i=1}^{k} |E\{(X_{T_{s_i}} - X_{T_{s_{i-1}}}) I_{\{T_{s_i} < S_n \wedge T_{S_n'}\}} \mid \mathscr{G}_{T_{s_{i-1}}}\}|\} \\
&\quad + E\{\textstyle\sum_{i=1}^{k} |E\{(X_{T_{s_i}} - X_{T_{s_{i-1}}}) P_{s_i} I_{\{S_n' > s_i\}} \mid \mathscr{F}_{s_{i-1}}\}|\} \\
&\leqq 2n + E\{|M_{S_n}|\} + 14n + (\tfrac{4}{5})E\{\textstyle\sum_{i=1}^{k} |E\{(X_{T_{s_i}} - X_{T_{s_{i-1}}}) I_{\{S_n' > s_i\}} \mid \mathscr{F}_{s_{i-1}}\}|\} .
\end{aligned}
$$

Thus

$$
E\{\textstyle\sum_{i=1}^{k} |E\{(X_{T_{s_i}} - X_{T_{s_{i-1}}}) I_{\{S_n' > s_i\}} \mid \mathscr{F}_{s_{i-1}}\}|\} < 70n + 5E\{|M_{S_n}|\}
$$

and the proof of Theorem 1 is complete.

The above proof is easily altered to prove the following.

COROLLARY. Let $(X_t, \mathscr{G}_t)$ be a local semimartingale and S a $\mathscr{G}_t$ stopping time such that

$$S \leqq \inf\{t : |X_t| \geqq n\} .$$

Let $t_0 < t_1 < \cdots$ be any sequence of real numbers and $\mathscr{F}_{t_i}$ be the σ-fields generated by X_{t_j}, $j \leqq i$. Let

$$S' = \inf\{t_i : P\{S \leqq t_i \mid \mathscr{F}_{t_i}\} \geqq \tfrac{1}{8}\} .$$

Then

$$V(X_{t_i} : t_0, t_n, S') \leqq c\{V(X_t : t_0, t_n, S) + n\}$$

where c is a universal constant.

4. In this section, it will be shown that every local semimartingale is a time change of a Brownian motion process. The semimartingale $(X_s, \mathscr{F}_s)$ will be presumed defined on some probability space which will not be named. It seems to be necessary to introduce several other probability spaces which will be denoted by Ω, Ω^0 and Ω^*. Although the corresponding σ-fields will be denoted by $\mathscr{F}$, $\mathscr{F}^0$, and $\mathscr{F}^*$, for simplicity the probability on each space will simply be denoted by P and the expectation by $E\{\ \}$.

The first step is to embed a discrete time process in a Brownian motion process using Skorokhod's techniques. This procedure is well known but will be presented in order to be specific in the proofs that follow.

LEMMA 4. *Every discrete time process X_n is equivalent to a time change of Brownian motion.*

PROOF. Let $(W_t, \mathscr{F}_t^0)$ be a Wiener process defined on a measure space $(\Omega^0, \mathscr{F}^0, P)$. Let $\Omega^* = \Omega^0 \times \pi_{n=1}^{\infty} I_n$ where $I_n = [0, 1]$, $\mathscr{F}^* = \mathscr{F}^0 \times \mathscr{B}$ and $\mathscr{F}_t^* = \mathscr{F}_t^0 \times \mathscr{B}$ where $\mathscr{B}$ is the class of Borel on $\pi_{n=1}^{\infty} I_n$. The probability measure on Ω^*, again denoted by P, is the product of the measures P on Ω^0 and the ordinary Lebesgue measures on I_n. Note that this makes the process W_t independent of the evaluation map from $\Omega^* \to [0, 1]$, $(\omega, \theta_1, \theta_2, \cdots) \to \theta_n$ where $\theta_n \in I_n$. Suppose that $\mathscr{F}_t^*$ stopping times $T_1 \leqq T_2 \leqq \cdots \leqq T_{n-1}$ have been defined so that $X_1, \cdots, X_{n-1}$ and $W(T_1), \cdots, W(T_{n-1})$ have the same joint distributions. Define

$$
\begin{aligned}
F(x) &= P\{X_n - X_{n-1} \leqq x \mid X_1 = x_1, \cdots, X_{n-1} = x_{n-1}\} \\
a(\theta) &= -\infty && \text{if } \theta = 0 \\
&= \inf\{x : F(x) > \theta\} && \text{if } 0 < \theta < F(0) \\
b(\theta) &= \inf\{x : F(x) > \theta\} && \text{if } F(0) \leqq \theta < 1 \\
&= \infty && \text{if } \theta = 1 \\
A(\theta) &= \textstyle\int_\theta^{F(0)} |a(s)|\, ds && \text{if } \theta < F(0)
\end{aligned}
$$

and

$$B(\theta) = \textstyle\int_{F(0)}^{\theta} b(s)\, ds \qquad \text{if } \theta \geqq F(0)\,.$$

Observe that $b(\theta) > 0$ on the set $(F(0), 1]$ so that $B(\theta)$ is continuous and monotone increasing. Likewise, $A(\theta)$ is continuous and monotone decreasing on the set $\{\theta : a(\theta) < 0\}$ which in general is not $[0, F(0))$. Define

$$
\begin{aligned}
a(\theta) &= a(A^{-1}(B(\theta))) && \text{if } F(0) \leqq \theta \text{ and } B(\theta) \leqq A(0) \\
&= -\infty && \text{if } F(0) \leqq \theta \text{ and } B(\theta) > A(0)\,,
\end{aligned}
$$

and likewise

$$
\begin{aligned}
b(\theta) &= b(B^{-1}(A(\theta))) && \text{if } \theta < F(0) \text{ and } A(\theta) \leqq B(1) \\
&= \infty && \text{if } \theta < F(0) \text{ and } A(\theta) > B(1)\,.
\end{aligned}
$$

Now all the functions defined above are defined in terms of the conditional distribution and thus functions of $x_1, \cdots, x_{n-1}$ as well as θ. Define T_n on the set

$$\{W(T_1) = x_1, \cdots, W(T_{n-1}) = x_{n-1}\}$$

by

$$T_n(\omega, \theta_1, \theta_2, \cdots) = \inf\{s > T_{n-1} : W_s - W_{T_{n-1}} \notin (a(\theta_n), b(\theta_n))\}\,.$$

Then $W(T_n)$ has the desired distribution. As an example, let $x > 0$ and $\theta_0 = \inf\{\theta : b(\theta) > x\}$. If, for instance, $B(\theta_0) > A(0)$, then

$$
\begin{aligned}
&P\{0 < W(T_n) - W(T_{n-1}) \leqq x\} \\
&\quad = \textstyle\int_0^1 P\{0 < W(T_n) - W(T_{n-1}) \leqq x \mid \theta_n = \theta\}\, d\theta \\
&\quad = \textstyle\int_0^{F(0)} |a(s)|(|a(s)| + b(s))^{-1}\, ds \\
&\qquad + \textstyle\int_{F(0)}^{B^{-1}(A(0))} |a(\theta)|(|a(\theta)| + b(\theta))^{-1}\, d\theta + \int_{B^{-1}(A(0))}^{\theta_0} d\theta\,.
\end{aligned}
$$

Letting $s = A^{-1}(B(\theta))$ in the first integral and using the fact that $a(s) = a(\theta)$,

$b(s) = b(\theta)$ and $ds = D_\theta A^{-1}(B(\theta)) = -b(\theta)(|a(\theta)|)^{-1} d\theta$ we have $P\{0 < W(T_n) - W(T_{n-1}) \leqq x\} = \int_{F(0)}^{\theta_0} dt = \theta_0 - F(0) = P\{0 < X_n - X_{n-1} \leqq x\}$ since $\theta_0 = F(x)$. The lemma is considered proved.

LEMMA 5. *If the stopping times* T_n *are defined as in Lemma* 4, *then for any* $B > 3\lambda > 0$ *and any* $\mathscr{F}_n$ *stopping time* $S \leqq \inf\{n > a: |W_{T_n} - W_{T_a}| > \lambda\}$ *where* $\mathscr{F}_n$ *is the σ-field generated by* W_{T_i}, $i \leqq n$,

$$P\{\sup_{T_a \leqq t \leqq T_b} |W_t - W_{T_a}| > B, S > a\} \leqq (B - \lambda)^{-1} V(W_{T_n}: a, b, S) + 2P(a < S \leqq b)$$

and

$$P\{\sup_{T_a \leqq t \leqq T_b} |W_t - W_{T_a}| > B\} \leqq (B - \lambda)^{-1} V(W_{T_n}: a, b, S) + 2P(S \leqq b)\,.$$

PROOF. The second inequality follows easily from the first. In proving the first, it simplifies matters to assume that $W(T_a) = 0$. Let

$$\Gamma_n = \{S = n + 1\} \cup \{\sup_{T_n < t \leqq T_{n+1}} |W_t| > B\}\,.$$

Since $\{\sup_{T_a < t < T_b} |W_t| > B, S > a\} \subset \bigcup_{n=a}^{b-1} (\Gamma_n \cap \{S > n\})$ it is enough to show that

$$P(\Gamma_n \cap \{S > n\}) \leqq (B - \lambda)^{-1} E\{|E\{(W_{T_{n+1}} - W_{T_n}) I_{\{n+1<S\}} \,|\, \mathscr{F}_n\}|\} + 2P\{S = n + 1\}$$

because the sum on n of the terms on the right of this inequality are the right-hand terms of the first inequality in the lemma. Now the set $\{S > n\}$ is in $\mathscr{F}_n$ and $\{S > n + 1\} \cup \{S = n + 1\} \subset \{S > n\}$, of course, so taking conditional expectations with respect to $\mathscr{F}_n$ it is enough to show that on the set $\{S > n\}$

$$P\{\sup_{T_n < t \leqq T_{n+1}} |W_t| > B \text{ or } S = n + 1 \,|\, \mathscr{F}_n\} \leqq (B - \lambda)^{-1} |E\{(W_{T_{n+1}} - W_{T_n}) I_{\{n+1<S\}} \,|\, \mathscr{F}_n\}| + 2P\{S = n + 1 \,|\, \mathscr{F}_n\}\,.$$

But now the problem has been reduced to studying the probability that a Wiener process W_t started at $x = W_{T_n}$ (note that $|W_{T_n}| < \lambda$ on $\{S > n\}$) will exit from $[-B, B]$ before it exits from $(a(\theta) + x, b(\theta) + x)$ where $a(\theta)$ and $b(\theta)$ are the functions of Lemma 2. Of course $a(\theta) \leqq 0 \leqq b(\theta)$ but another important property is that either

$$\{\theta: a(\theta) + x < -\lambda, b(\theta) + x < \lambda\} = \phi\,,$$

or

$$\{\theta: a(\theta) + x > -\lambda, b(\theta) + x > \lambda\} = \phi\,.$$

Assume then that $\{\theta: a(\theta) + x < -\lambda, b(\theta) + x < \lambda\} = \phi$. It will be shown that in this case, taking conditional expectations with respect to $\mathscr{F}_n$ and θ,

$$P\{\sup_{T_n < t \leqq T_{n+1}} |W_t| > B \text{ or } S = n + 1 \,|\, \mathscr{F}_n, \theta\} \leqq -(B - \lambda)^{-1} E\{(W_{T_{n+1}} - W_{T_n}) I_{\{n+1<S\}} \,|\, \mathscr{F}_n, \theta\} + 2P\{S = n + 1 \,|\, \mathscr{F}_n, \theta\}$$

on the set $\{S > n\}$.

Consider cases. If $-B < a(\theta) + x < b(\theta) + x < B$, the left-hand side is $P\{S = n + 1 \mid \mathscr{F}_n, \theta\}$. It must be shown then that

$$(B - \lambda)^{-1}E\{W_{T_{n+1}} - W_{T_n}\}I_{\{n+1<S\}} \mid F_n, \theta\} \leqq P\{S = n + 1 \mid F_n, \theta\} .$$

But this certainly must be the case unless $S = n + 1$ on the set $W_{T_{n+1}} = a(\theta) + x$ but $S > n + 1$ on the set $W_{T_{n+1}} = b(\theta) + x$. (Recall that S depends only on the process W_{T_i}.) This would require that $b(\theta) + x < \lambda$ and then by the assumption above that $a(\theta) + x \geqq -\lambda$. Thus $|a(\theta)| < 2\lambda$. One then computes

$$\begin{aligned} P\{S = n + 1 \mid F_n, \theta\} &- (B - \lambda)^{-1}E\{(W_{T_{n+1}} - W_{T_n})I_{\{n+1<S\}} \mid F_n, \theta\} \\ &= b(\theta)(b(\theta) - a(\theta))^{-1} - (B - \lambda)^{-1}b(\theta)|a(\theta)|(b(\theta) - a(\theta))^{-1} \\ &= b(\theta)(b(\theta) - a(\theta))^{-1}(1 + (B - \lambda)^{-1}a(\theta)) \geqq 0 \end{aligned}$$

since $|a(\theta)| < 2\lambda$ and $B > 3\lambda$.

In the case $a(\theta) + x \leqq -\lambda$ and $b(\theta) + x \geqq \lambda$, the right-hand side of the inequality is at least 1.

Finally, in the case $a(\theta) + x > -\lambda$ and $b(\theta) + x \geqq B$, $S = n + 1$ at least on the set $W_{T_{n+1}} = b(\theta) + x$. If $S = n + 1$ also on the set $W_{T_{n+1}} = a(\theta) + x$, the inequality is obvious. If not, one computes

$$\begin{aligned} -(B - \lambda)^{-1}E\{(W_{T_{n+1}} &- W_{T_n})I_{\{n+1<S\}} \mid F_n, \theta\} \\ &= -(B - \lambda)^{-1}a(\theta)b(\theta)(b(\theta) - a(\theta))^{-1} \\ &\geqq (B - \lambda)^{-1}|a(\theta)|(B - x)(B - x - a(\theta))^{-1} \\ &\geqq |a(\theta)|(B - x - a(\theta))^{-1} \end{aligned}$$

where the first inequality is justified by the fact that $z(z - a(\theta))^{-1}$ is increasing in z and the second inequality follows from the fact that $B - x > B - \lambda$. On the other hand, in this case

$$P\{\sup_{T_n < t \leqq T_{n+1}} |W_t| > B \mid F_n, \theta\} = |a(\theta)|(B - x - a(\theta))^{-1} ,$$

so this last case is also settled. The lemma is proved.

Consider the local semimartingale X_s and write $X_s = M_s + A_s$ in the usual manner. By redefining the reducing sequence S_n if necessary, one can assume that if $s < S_n$, $|X_s| < n$, $|M_s| < n$ and $|A|_s < n$ where

$$|A|_t = \sup_{s_0 < \cdots < s_k = t} \textstyle\sum_{i=0}^{k-1} |A_{s_{i+1}} - A_{s_i}| .$$

Henceforth, the reducing sequence will *always* be chosen in this way.

LEMMA 6. *For any fixed t and any $\varepsilon > 0$, there is a $\lambda > 0$ such that for any $t_1 < t_2 < \cdots < t_k = t$, if T_i are stopping times chosen as in Lemma 4 so that X_{t_i} and $W(T_i)$ have the same joint distributions, $P\{T_k > \lambda\} < \varepsilon$.*

PROOF. Choose n such that $P\{S_n < t\} < \varepsilon/64$. Let $S_n' = \inf\{t_i : P\{S_n \leqq t_i \mid \mathscr{F}_{t_i}\} > \frac{1}{8}\}$ where $\mathscr{F}_{t_i}$ is the σ-field generated by X_{t_j}, $j \leqq i$. Then by the last inequality in the proof of Theorem 1, $V(X_{t_i} : 0, t, S_n') \leqq 70n + 5E\{|W_{S_n}|\}$. (One considers the time change $T_s = t_i$, $t_i \leqq s < t_{i+1}$.) Then by Lemma 5, for any

$B > 3n$

$$P\{\sup_{s \leqq T_k} |W_s| > B\} \leqq (B-n)^{-1}V(X_{t_i}: 0, t, S_n') + 2P(S_n' < t)$$
$$\leqq (B-n)^{-1}\{70n + 5E\{|W_{S_n}|\} + \varepsilon/4 .$$

Choose B large enough that $(B-n)^{-1}\{70n + 5E\{|W_{S_n}|\}\} < \varepsilon/4$. Then $P\{\sup_{s<T_k} |W_s| > B\} \leqq \varepsilon/2$. Finally choose λ large enough that $P\{\sup_{s \leqq \lambda} |W_s| \leqq B\} < \varepsilon/2$. Then $P\{T_k > \lambda\} < \varepsilon$.

THEOREM 2. *The local semimartingale X_s is equivalent to a time change of Brownian motion.*

PROOF. For each m, define on the space Ω^* of Lemma 2, a sequence of stopping times $T_i^{(m)}$ such that the processes $X_{i2^{-m}}$ and $W(T_i^{(m)})$ have the same joint distributions.

Let C be the set of all continuous maps from $[0, \infty)$ into R. The set C with the local sup norm metric is a complete separable metric space. Let $\mathscr{T}$ be the set of all nondecreasing, right continuous functions from $[0, \infty)$ into $[0, \infty)$. The set $\mathscr{T}$ also admits a metric which makes it into a complete separable metric space. In fact, one can define for $t_1(s)$ and $t_2(s) \in \mathscr{T}$

$$d(t_1, t_2) = \sum_{k=1}^{\infty} 2^{-k} \min \left(\int_0^k |t_1(s) - t_2(s)|\, ds, 1\right) .$$

Note that if α_n is any sequence of positive numbers, then the set $\{t(s) \in \mathscr{T}: t(n) \leqq \alpha_n\}$ is compact.

Let $\Omega = C \times \mathscr{T}$. Then with the product topology, Ω is a complete separable metric space. Define $f_m: \Omega^* \to \Omega$ by

$$f_m(\omega^*) = (x(s), t(s))$$

where

$$x(s) = W_s(\omega^*)$$

and

$$t(s) = T_i^{(m)}(\omega^*) \qquad \text{if} \quad i2^{-m} \leqq s < (i+1)2^{-m} .$$

The functions f_m are measurable. Let μ_m be the measure induced on Ω by the random variables f_m. It will be shown that the measures μ_m are tight and if μ is an accumulation point of μ_m, then Ω, $W_s(x, t) = x(s)$, $T_s(x, t) = t(s)$ and μ define a Wiener process W_s and a family of stopping times T_s on the probability space (Ω, μ) such that W_{T_s} is equivalent to X_s. (The σ-fields will be discussed later.)

To show that the μ_m are tight, it is enough to show that the projections onto Ω and $\mathscr{T}$ are tight. That the projections on Ω are tight is obvious since all the measures coincide there.

On the other hand, by Lemma 6, for each $n > 0$, there is a λ_n such that $P\{T_{n2^m}^{(m)} > \lambda_n\} < \varepsilon 2^{-n}$. Then

$$\mu_m\{\exists\, n \ni t(n) > \lambda_n\} \leqq \varepsilon .$$

Since the set $\bigcap_n \{t(n) \leqq \lambda_n\}$ is compact, the measures μ_m are tight.

Let μ be an accumulation point of the measures μ_m. For simplicity assume that μ_m converges to μ. Since for any open sets $U_1, U_2 \subset R$,

$$\begin{aligned}\mu\{W_{t_1} \in U_1, W_{t_2} \in U_2\} &= \mu\{x(t_1) \in U_1, x(t_2) \in U_2\} \\ &= P\{W^*_{t_1} \in U_1, W^*_{t_2} \in U_2\}\end{aligned}$$

it is clear that (Ω, W_t, μ) is a Wiener process.

Next it must be shown that the process W_{T_s} has the same finite joint distributions as X_s. Suppose $s_1 < s_2 < \cdots < s_n$ are of the form $s_i = k_i 2^{-m_0}$ for some m_0. Let C_i be a compact set for each i. Of course $P\{X_{s_i} \in C_i : i = 1, 2, \cdots, n\} = \mu_m\{x(t(s_i)) \in C_i : i = 1, 2, \cdots, n\}$ for all $m > m_0$. Choose compact sets C_i' such that C_i is contained in the interior of C_i'. Now for any $\delta > 0$, the set

$$\Gamma_\delta = \{(x(t), t(s)) : x(t(s)) \in C_i' \text{ for } s_i \leqq s < s_i + \delta \text{ and } i = 1, 2, \cdots, n\}$$

is closed and by right continuity of paths

$$\begin{aligned}&\{(x(t), t(s)) : x(t(s_i)) \in C_i \text{ for } i = 1, 2, \cdots, n\} \\ &\qquad \subset \textstyle\bigcup_\delta \Gamma_\delta \subset \{(x(t), t(s)) : x(t(s_i)) \in C_i' \text{ for } i = 1, 2, \cdots, n\} .\end{aligned}$$

Thus

$$\begin{aligned}&\mu\{(x(t), t(s)) : x(t(s_i)) \in C_i'\} \\ &\qquad \geqq \lim_{\delta \to 0} \lim_{m \to \infty} \mu_m(\Gamma_\delta) \\ &\qquad = \lim_{\delta \to 0} \lim_{m \to \infty} P\{X_{k2^{-m}} \in C_i' : s_i 2^m \leqq k < (s_i + \delta)2^m, i = 1, 2, \cdots, n\} \\ &\qquad \geqq P\{X_{s_i} \in C_i : i = 1, 2, \cdots, n\} .\end{aligned}$$

Since C_i' is arbitrary,

$$\mu\{(x(t), t(s)) : x(t(s_i)) \in C_i\} \geqq P\{X_{s_i} \in C_i : i = 1, 2, \cdots, n\} .$$

The inequality can easily be extended to all Borel sets and since μ and P are probability measures, the inequality is in fact an equality.

We have not yet discussed σ-fields. The σ-field we have implicitly been using on Ω is the collection of Borel sets generated by the product topology on Ω. Denote this field by $\mathscr{G}$. If $\mathscr{G}_t'$ is the σ-field generated by the functions $W_v, v \leqq t$, then $\mathscr{G}_t' \subset \mathscr{G}$ and of course W_v is Markovian with respect to $\mathscr{G}_t'$. But T_s (for fixed s) is not in general a stopping time with respect to $\mathscr{G}_t'$. Let $\mathscr{G}_t$ be the σ-field generated by $\mathscr{G}_t'$ and the sets of the form $\{T_s \leqq v; v \leqq t\}$. It must be shown that W_t is still Markovian with respect to $\mathscr{G}_t$.

It must be shown that the σ-field generated by the functions $W_t - W_{t_*}, t \geqq t_*$, is independent of the σ-field generated by the sets $\{T_s \leqq v\}$ with $v \leqq t_*$ and the functions W_t with $t \leqq t_*$. However, it is enough to show that for any $t_0 > t_*$ the σ-field generated by the functions $W_t - W_{t_0}$ is independent of the σ-field generated by the sets $\{T_s \leqq v\}$ with $v < t_0$ and the functions W_t with $t \leqq t_*$.

First observe that if a is any real number then $\{W_t - W_{t_0} < a\}$ and $\{W_t < a\}$ are continuity sets of μ since W_t is normally distributed with respect to μ. If a_i, $i \leqq k$, and b_j, $j \leqq l$, are real numbers, if $t_i \leqq t_*$, $i \leqq k$, and $t_j \geqq t_0$, $j \leqq l$, and if $\{T_{s_i} < v_i\}$ is a continuity set for each $i \leqq k$ where $v_i < t_0$, $i \leqq k$, then one computes

$$\begin{aligned}
&\mu[(\textstyle\bigcap_{i=1}^k \{T_{s_i} < v_i\} \cap \{W_{t_i} < a_i\}) \cap (\bigcap_{j=1}^l \{W_{t_j} - W_{t_0} < b_j\})] \\
&\quad = \lim_{m\to\infty} \mu_m[(\textstyle\bigcap_{i=1}^k \{t_{s_i} < v_i\} \cap \{x(t_i) < a_i\}) \cap (\bigcap_{j=1}^l \{x(t_j) - x(t_0) < b_j\})] \\
&\quad = \lim_{m\to\infty} P[T^{(m)}_{[s_i 2^m]} < v_i,\ W_{t_i}(\omega^*) < a_i,\ \forall\, i \leqq k, \\
&\qquad\qquad W_{t_j}(\omega^*) - W_{t_0}(\omega^*) < b_j,\ \forall\, j \leqq l] \\
&\quad = \lim_{m\to\infty} P[T^{(m)}_{[s_i 2^m]} < v_i,\ W_{t_i}(\omega^*) < a_i,\ \forall\, i \leqq K] \\
&\qquad\qquad \times P[W_{t_j}(\omega^*) - W_{t_0}(\omega^*) < b_j,\ \forall\, j \leqq l] \\
&\quad = \mu[\textstyle\bigcap_{i=1}^k \{T_{s_i} < v_i\} \cap \{W_{t_i} < a_i\}]\mu[\bigcap_{j=1}^l \{W_{t_j} - W_{t_0} < b_j\}]
\end{aligned}$$

since the $T^{(m)}_{[s_i 2^m]}$ are stopping times for the Wiener process W_t on Ω^*. (Here $[s_i 2^m]$ denotes the largest integer smaller than $s_i 2^m$.)

A monotone class argument (see Theorem 2.2 of the Preliminaries of [1]) applied twice finishes the proof if it can be shown that the σ-field generated by the sets $\{T_s \leqq v\}$ with $v \leqq t_*$ is contained in the σ-field generated by the sets $\{T_s < v\}$ with $v < t_0$ which are continuity sets of μ. Fix $v_0 < t_0$ and s_0. If $s_i \downarrow s_0$ and $v_j \downarrow v_0$ then

$$\{T_{s_0} \leqq v_0\} = \textstyle\bigcap_{j=1}^\infty \bigcup_{i=1}^\infty \{T_{s_i} < v_j\}$$

since T_s is right continuous and nondecreasing. Thus it must only be shown that there are dense sets $\mathscr{V}$ and $\mathscr{S}$ such that for every $s \in \mathscr{S}$ and $v \in \mathscr{V}$, the set $\{T_s < v\}$ is a continuity set of μ.

The boundary of the set $\{T_{s_0} < v\}$ is the set $\{t: \sup\{s: t(s) < v\} \leqq s_0 \leqq \inf\{s: t(s) > v\}\}$. For a fixed s_i, the sets depending on v

$$\{t: \sup\{s: t(s) < v\} < s_i < \inf\{s: t(s) > v\}\}$$

are all disjoint. Thus, if

$$\mathscr{V}' = \textstyle\bigcup_{s_i \in Q} (v: \mu\{t: \sup\{s: t(s) < v\} < s_i < \inf\{s: t(s) > v\}\} \neq 0)$$

where Q is the set of rational numbers, then $\mathscr{V}'$ is countable. One checks that if $v \notin \mathscr{V}'$, then for every $s_0 > 0$

$$\mu\{t: \sup\{s: t(s) < v\} < s_0 < \inf\{s: t(s) > v\}\} = 0\,.$$

In particular, if $v \in \mathscr{V}'$ and $\{T_{s_0} < v\}$ is not a continuity set of μ then

$$\mu\{t: \sup\{s: t(s) < v\} = s_0 = \inf\{s: t(s) > v\}\} > 0\,.$$

But for a fixed v, these sets are disjoint so one concludes that for each $v \notin \mathscr{V}'$, the set

$$\mathscr{S}_v = \{s_0: \{T_{s_0} < v\} \text{ is a continuity set of } \mu\}$$

is the complement of a countable set. All one needs to do now is let $\mathscr{V}$ be any countable dense set disjoint from $\mathscr{V}'$ and let $\mathscr{S} = \bigcap_{\mathscr{V} \in \mathscr{V}'} \mathscr{S}_v$. The proof is complete.

The inequality of Lemma 5 shows that in some sense the Skorokhod stopping times are "good." A similar inequality can be obtained for the stopping times of Theorem 2. No attempt will be made to get the sharpest form possible however. In addition, the form given here is designed for use in the study of stochastic integrals [10]. The following lemma is needed.

LEMMA 7. *Let F be an event in the σ-field* $\mathscr{H}$ *generated by the function* $(x(t), t(s)) \to x(t(s_i))$ *for some fixed* $s_0 < s_1 < \cdots < s_n$ *of the form* $s_i = k_i 2^{-m_0}$. *Let* μ_m' *and* μ' *be the restrictions of the measures* μ_m *and* μ *of Theorem 2 to F. Then* $\mu_m' \to \mu'$.

PROOF. Let $\mathscr{H}'$ denote the collection of sets $F \in \mathscr{H}$ such that for any $\varepsilon > 0$, there is a continuous function, $0 \leqq f \leqq 1$, such that

$$\limsup \int_F 1 - f\,d\mu_m < \varepsilon\,, \qquad \limsup \int_{F^c} f\,d\mu_m < \varepsilon$$
$$\int_F 1 - f\,d\mu < \varepsilon\,, \qquad \text{and} \qquad \int_{F^c} f\,d\mu < \varepsilon\,.$$

Clearly $\mathscr{H}'$ is a σ-field. The lemma will be proved if it can be shown that $\mathscr{H}' = \mathscr{H}$.

As in the proof of Theorem 2, let $C_0, C_1, \cdots, C_n$ be compact sets of real numbers and choose compact sets C_i' such that C_i is contained in the interior of C_i' and

$$P\{X_{s_i} \in C_i', 0 \leqq i \leqq n\} < P\{X_{s_i} \in C_i, 0 \leqq i \leqq n\} + \varepsilon/2\,.$$

Let $\Gamma_\delta = \{(x(t), t(s)) \colon x(t(s)) \in C_i', s_i \leqq s < s_i + \delta, 0 \leqq i \leqq n\}$ and select δ so small that $\mu_m(\Gamma_\delta) > P\{X_{s_i} \in C_i, 0 \leqq i \leqq n\} - \varepsilon/2$. Since Γ_δ is closed and the measures μ_m are tight, there is a compact set $K \subset \Gamma_\delta$ such that for all m, $\mu_m(K) \geqq P\{X_{s_i} \in C_i, 0 \leqq i \leqq n\} - \varepsilon$. Since $K \subset \{x(t(s_i)) \in C_i', 0 \leqq i \leqq n\}$,

$$\mu(K) \leqq P\{X_{s_i} \in C_i, 0 \leqq i \leqq n\} + \varepsilon/2\,.$$

Choose an open set $U \supset K$ such that $\mu(\bar{U}) \leqq \mu(K) + \varepsilon/2$ where $\bar{U}$ is the closure of U. Note that

$$\begin{aligned} \limsup \mu_m(\bar{U}) &\leqq \mu(\bar{U}) \\ &\leqq P\{X_{s_i} \in C_i', 0 \leqq i \leqq n\} + \varepsilon/2 \\ &\leqq P\{X_{s_i} \in C_i, 0 \leqq i \leqq n\} + \varepsilon\,. \end{aligned}$$

Let f be a continuous function, $0 \leqq f \leqq 1$, which takes the value one on K and zero outside of U. Then if

$$F = \{x(t(s_i)) \in C_i, 0 \leqq i \leqq n\}\,,$$
$$\limsup \int_F 1 - f\,d\mu_m \leqq \limsup \mu_m(F \backslash K) \leqq 2\varepsilon\,,$$
$$\limsup \int_{F^c} f\,d\mu_m \leqq \limsup \mu_m(\bar{U} \backslash F) \leqq \varepsilon\,,$$
$$\int_F 1 - f\,d\mu \leqq \mu(F \backslash K) \leqq 2\varepsilon$$

and

$$\int_{F^c} f\,d\mu \leqq \mu(F \backslash K) \leqq 2\varepsilon\,.$$

One need only observe that the sets F of this form generate $\mathscr{H}$ to finish the proof.

Fix $s_0 < s_1 < \cdots < s_n$ and for fixed m_0 let α_i be discrete $\mathscr{F}_{s_i}^{(m_0)}$-measurable random variables for each $i = 0, 1, \cdots, n-1$ where $\mathscr{F}_{s_i}^{(m_0)}$ is the σ-field generated by $X_{j2^{-m_0}}$, $j2^{-m_0} < s_i$. Define on the space Ω of Theorem 2 where

$$X_s = W(T_s)\ ,$$
$$Y_s = \textstyle\sum_{i=1}^k \alpha_{i-1}(X_{s_i} - X_{s_{i-1}}) + \alpha_k(X_s - X_{s_k}) \qquad \text{for} \quad s_k < s \leqq s_{k+1}$$
$$Z_t = \textstyle\sum_{i=1}^k \alpha_{i-1}(W(T_{s_i}) - W(T_{s_{i-1}})) + \alpha_k(W_t - W(T_{s_k})) \qquad \text{for} \quad T_{s_k} < t \leqq T_{s_{k+1}}$$

so that for instance if $T_{s_k} < t \leqq T_{s_{k+1}}$, then

$$Z_t - Y_{s_k} = \alpha_k(W_t - W(T_{s_k})) = \alpha_k(W_t - X_{s_k})\ .$$

Let $S = \inf\{s > s_0 : |Y_s| > \lambda/2\}$ and $T = \inf\{t > T_{s_0} : |Z_t| > 2B\}$.

PROPOSITION. *If $B > 3\lambda > 0$ and $P(T_s = T_{s-}) = 1$ for all $s \in \{s_0, s_1, \cdots, s_n\}$, then*

$$P\{T < T_{s_n}\} < c\{(B-\lambda)^{-1}V(Y_s : s_0, s_n, S) + \lambda\} + 3P\{S < s_n\}$$

where c is a constant independent of X_s.

PROOF. Let $\Gamma = \{(x(t), t(s)) : T(x(t), t(s)) < T_{s_n} = t(s_n)\}$ and for $a \in R^n$, $a = (a_0, a_1, \cdots, a_{n-1})$ let $\Lambda(a)$ be the set

$$\{(x(t), t(s)) : \sup_{t(s_k) < t \leqq t(s_{k+1})} |\textstyle\sum_{i=1}^k a_{i-1}(x(t(s_i)) - x(t(s_{i-1}))) + a_k(x(t) - x(t(s_k)))| > 2B \ \text{for some} \ k < n\}\ .$$

Then since the random variables α_i are discrete,

$$P(\Gamma) = \textstyle\sum_{a \in R^n} \mu(\Gamma \cap \{\alpha_i = a_i, 0 \leqq i < n\}) = \textstyle\sum_{a \in R^n} \mu(\Lambda(a) \cap \{\alpha_i = a_i, 0 \leqq i < n\})\ .$$

Now one can check that $\overline{\Lambda(a)^c} \cap \Lambda(a)$ is contained in the set of points $(x(t), t(s))$ with discontinuities at at least one of the points $s_0, s_1, \cdots, s_n$ which by assumption has μ measure zero. Thus by Lemma 7,

$$P(\Gamma) \leqq \textstyle\sum_{a \in R^n} \limsup_m \mu_m(\Lambda(a) \cap \{\alpha_i = a_i, 0 \leqq i < n\}) \leqq \limsup \mu_m(\Gamma)\ .$$

Now μ_m is the measure induced on Ω by the map f_m from Ω^* so $\mu_m(\Gamma) = P(\Delta)$ where Δ is the set of Ω^* for which there is a t, $T^{(m)}_{[s_k 2^m]} < t < T^{(m)}_{[s_{k+1}2^m]}$ for some $k < n$ such that $|Z_t^{(m)}| > 2B$ where

$$Z_t^{(m)} = |\textstyle\sum_{i=1}^k \alpha_{i-1}(W(T^{(m)}_{[s_i 2^m]}) - W(T^{(m)}_{[s_{i-1}2^m]})) + \alpha_k(W_t - W(T^{(m)}_{[s_k 2^m]}))|\ .$$

Again let T be the infimum of all such t. Let (for fixed $m \geqq m_0$)

$$S' = \inf\{k2^{-m} : P(S \leqq k2^{-m} \mid X_{j2^{-m}}, j \leqq k) \geqq \tfrac{1}{8}\}\ .$$

Since S' depends only on $X_{j2^{-m}}$, S' can be considered a function on Ω^*. Let $j_0' = [2^m s_0]$, $j_n' = [2^m s_n]$ and for other $k < n$, $j_k = [2^m s_k]$ and $j_k' = [2^m s_k + 1]$.

Write

$$\begin{aligned}P(\Delta) &= \textstyle\sum_{k=1}^{n} P\{T^{(m)}_{j'_{k-1}} < T < T^{(m)}_{j_k};\ |Z(T^{(m)}_{j'_{k-1}})| < \lambda;\ S' > s_n\} + P\{S' \leqq s_n\} \\ &\quad + \textstyle\sum_{k=0}^{n} P\{T^{(m)}_{j_k} < T < T^{(m)}_{j'_k};\ |Z^{(m)}(T_{j'_k})| < \lambda\} \\ &\quad + \textstyle\sum_{k=0}^{n} P\{S' > s_n;\ |Z^{(m)}(T^{(m)}_{j'_k})| \geqq \lambda\} \\ &= P_1 + P_2 + P_3 + P_4 .\end{aligned}$$

We treat the limit supremum of each of these probabilities in turn.

First

$$S' < \min\{j2^{-m} : j'_{k-1} < j < j_k,\ \alpha_{k-1}|W(T_j{}^{(m)}) - W(T^{(m)}_{j'_{k-1}})| > \lambda\}$$

since S satisfies this inequality and the events depend only on $X_{j2^{-m}}$. Moreover, if $|Z^{(m)}(T^{(m)}_{j'_{k-1}})| < \lambda$ and $T^{(m)}_{j'_{k-1}} < T \leqq T^{(m)}_{j_k}$ then $\alpha_{k-1}|W(T) - W(T^{(m)}_{j'_{k-1}})| > B$ since $B > 3\lambda$. This is impossible if $\alpha_{k-1} = 0$, of course, and otherwise a conditional version of Lemma 5 gives

$$\begin{aligned}P_1 &< \textstyle\sum_{k=1}^{n} \{(B-\lambda)^{-1}V(Z^{(m)}(T_i{}^{(m)}) : j'_{k-1}, j_k, 2^m S') + 2P(s_{k-1} < S' \leqq s_k)\} \\ &< (B-\lambda)^{-1}V(Y_{i2^{-m}} : s_0, s_n, S') + 2P(S' < s_n') \\ &< c(B-\lambda)^{-1}\{V(Y_t : s_0, s_n, S) + \lambda\} + 2P(S' \leqq s_n')\end{aligned}$$

by the corollary to Theorem 1.

To get a bound for $\limsup_m (P_1 + P_2)$, observe that $P\{S \leqq s_n \mid X_{j2^{-m}}, j2^{-m} \leqq s_n\}$ converges to zero on $\{S > s_n\}$ except for a set of probability zero. Thus

$$\limsup P\{S' \leqq s_n\} \leqq P\{S \leqq s_n\} \qquad \text{as} \quad m \to \infty .$$

For P_3, recall that by assumption, with probability one, T_s is continuous at $s_0, s_1, \cdots, s_n$ and that therefore X_s is also. By the nature of the definition of the Skorokhod stopping times then,

$$\mu_m\{|Z(T^{(m)}_{j'_k})| < \lambda;\ T^{(m)}_{j'_k} < T < T^{(m)}_{j_k}\} \to 0 .$$

Finally observe that

$$\begin{aligned}&P\{S' > s_n;\ |Z^{(m)}(T^{(m)}_{j'_k})| \geqq \lambda\} \\ &\quad = P\{S' > s_n;\ |\textstyle\sum_{i=1}^{k} \alpha_{i-1}(X_{j_i 2^{-m}} - X_{j_{i-1}2^{-m}}) + \alpha_k(X_{j'_k 2^{-m}} - X_{j_k 2^{-m}})| > \lambda\} .\end{aligned}$$

Moreover

$$\begin{aligned}Y_{j'_k 2^{-m}} - \textstyle\sum_{i=1}^{k} \alpha_{i-1}(X_{j_i 2^{-m}} - X_{j_{i-1}2^{-m}}) - \alpha_k(X_{j'_k 2^{-m}} - X_{j_k 2^{-m}}) \\ = \textstyle\sum_{i=1}^{k} (\alpha_i - \alpha_{i-1})(X_{s_i} - X_{j_i 2^{-m}})\end{aligned}$$

which goes to zero with probability one by the continuity of X_s at s_i, $i = 1, 2, \cdots, n$. But on $\{S' > s_n\}$, $|Y_{j'_i 2^{-m}}| < \lambda/2$ for all i so $\limsup P_4 = 0$. Thus

$$\begin{aligned}P(\Gamma) &< \limsup_m \mu_m(\Gamma) \\ &< \limsup_m P(\Delta) \\ &< c(B-\lambda)^{-1}\{V(Y_t : 0, s_n, S) + \lambda\} + 3P\{S < s_n\} .\end{aligned}$$

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REFERENCES

[1] BLUMENTHAL, R. M. and GETOOR, R. K. (1968). *Markov Processes and Potential Theory.* Academic Press, New York.

[2] DAMBIS, K. E. (1965). On the decomposition of continuous submartingales. *Theor. Probability Appl.* **10** 401-410.

[3] DUBINS, L. (1968). On a theorem of Skorokhod. *Ann. Math. Statist.* **39** 2094-2097.

[4] DUBINS and SCHWARTZ (1965). On continuous martingales. *Proc. Nat. Acad. Sciences U.S.A.* **53** 913-916.

[5] FÖLLMER, H. (1973). On the representation of semimartingales. *Ann. Probability* **1** 580-589.

[6] HUFF, B. W. (1969). The loose subordination of differential processes to Brownian motion. *Ann. Math. Statist.* **40** 1603-1609.

[7] KAZAMAKI, N. (1972). Changes of time, stochastic integrals, and weak martingales. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **22** 25-32.

[8] MEYER, P. A. (1966). *Probability and Potentials.* Blaisdell, Waltham, Mass.

[9] MEYER, P. A. (1967). Integrales stochastiques. Séminaire de Probabilités I, *Lecture Notes in Mathematics* **39** 72-163. Springer-Verlag, New York.

[10] MONROE, I. Stochastic integrals. Unpublished.

[11] MONROE, I. (1972). Using additive functionals to embed preassigned distributions in symmetric stable processes. *Trans. Amer. Math. Soc.* **163** 131-146.

[12] MONROE, I. (1972). On embedding right continuous martingales in Brownian motion. *Ann. Math. Statist.* **43** 1293-1311.

[13] RAO, K. M. (1969). Quasi—martingales. *Math. Scand.* **24** 79-92.

[14] ROOT, D. H. (1969). The existence of certain stopping times on Brownian motion. *Ann. Math. Statist.* **40** 715-718.

[15] RÖST, H. (1971). The stopping distributions of a Markov process. *Invent. Math.* **14** 1-16.

[16] SKOROKHOD, A. (1965). *Studies in the Theory of Random Processes.* Addison-Wesley, Reading.

[17] STRASSEN, V. (1965). Almost sure behavior of sums of independent random variables and martingales. *Proc. Fifth Berkeley Symp. Math. Statist. Prob.* **2** 315-343.

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