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THE RANGE OF STOCHASTIC INTEGRATION

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Every measurable real-valued function f on the space of Wiener process paths with $E(|f|^p) < \infty$ (where $0 < p < 1$) can be represented as a stochastic integral $f = \int \varphi\, dX$, where $E(\int \varphi^2(t)\, dt)^{p/2} < \infty$. A similar result holds for $1 < p < \infty$ if and only if $E(f) = 0$.

Let $\Omega = C_0[0, \infty)$ be the space of continuous functions on $[0, \infty)$ which vanish at 0, let W be the Wiener measure on Ω, let $X(t, \omega)$ be the Wiener process, let F be the collection of W-measurable sets and let $(F_t)_{t \geqq 0}$ be the usual filtration of F.

Let $WL^0(L^2)$ denote the space of all real-valued well-measurable functions φ on $R^+ \times \Omega$ for which $\int_0^\infty \varphi^2(t, \omega)\, dt < \infty$ almost surely. If $\varphi \in WL^0(L^2)$ the stochastic integral $I_\infty(\varphi) = \int_0^\infty \varphi\, dX$ can be defined in the usual way (cf. McKean (1969)). Recently Dudley (1977) has shown that any W-measurable f can be written as $I_\infty(\varphi)$, for some $\varphi \in WL^0(L^2)$. (In fact, Dudley worked with the time interval $[0, 1]$, but a (nonrandom) change of time scale shows that his result is equivalent to the one stated above.)

What happens if we place some conditions on f? Can we impose corresponding conditions on φ? If the condition is that $f \in L^p$, we can (and indeed must) do so if $1 < p < \infty$: we shall see that, if $0 < p < 1$, similar conditions can be placed on φ; the ideas here are essentially those introduced by Dudley. We shall not deal with the important and interesting case where $p = 1$.

There is some advantage in considering the indefinite stochastic integral. To this end, let $WL^0(C_0)$ denote the space of all real-valued well-measurable functions f on $[0, \infty] \times \Omega$ for which the maps $t \to f(t, \omega)$: $[0, \infty] \to R$ are continuous and vanish at 0 for almost all ω; if $f \in WL^0(C_0)$, let $f^*(\omega) = \sup\{|f(t, \omega)| : 0 \leqq t \leqq \infty\}$. If $0 < p < \infty$, let $WL^p(L^2) = \{\varphi \in WL^0(L^2) : E((\int_0^\infty \varphi^2(t, \omega)\, dt)^{p/2}) < \infty\}$, and let $WL^p(C_0) = \{f \in WL^0(C_0) : E(f^{*p}) < \infty\}$. We give $WL^p(L^2)$ and $WL^p(C_0)$ their natural metrizable topologies (Banach space topologies, when $p \geqq 1$). If $1 \leqq p < \infty$, let $ML^p(C_0)$ denote the closed linear subspace of $WL^p(C_0)$ consisting of all closed martingales in $WL^p(C_0)$. Finally if $0 \leqq t \leqq \infty$, let π_t be the coordinate projection of $WL^p(C_0)$ (or $ML^p(C_0)$, if $1 \leqq p < \infty$) into $L^p = L^p(\Omega, W)$. If $1 < p < \infty$, π_∞ is an isomorphism of $ML^p(C_0)$ onto $L_0{}^p = \{f \in L^p : E(f) = 0\}$.

Now let I denote the indefinite stochastic integral: $I(\varphi)(t, \omega) = \int_0^t \varphi(s, \omega) X(ds, \omega)$. Clearly $I_t = \pi_t I$. As I_∞ is an isometry from $WL^2(L^2)$ into L^2 and π_∞ is an isomorphism from $ML^2(C_0)$ into L^2, it follows that I is an isomorphism of $WL^2(L^2)$ into $ML^2(C_0)$. It now follows immediately from the continuous time version of

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the Burkholder-Davis-Gundy inequalities (cf. Meyer (1972), Appendix 2, but note that the proof given there can be greatly simplified in the present case, as there are no jumps, which are the major source of difficulties) that I is an isomorphism of $WL^p(L^2)$ into $WL^p(C_0)$ for $0 < p < \infty$. Finally the usual truncation argument extends the result to the case $p = 0$. Summing up, we get

THEOREM 1. *I is an isomorphism of $WL^p(L^2)$ into $WL^p(C_0)$ for $0 \leqq p < \infty$.*

It is by now classical that $I_\infty(WL^2(L^2)) = L_0^2$ (see Meyer (1976), page 41 for an elegant proof due to Dellacherie). As $I_\infty = \pi_\infty I$ is an isomorphism of $WL^p(L^2)$ into L^p for $1 < p < \infty$, it follows easily (by restriction for $p > 2$, by approximation for $1 < p < 2$) that

THEOREM 2. *$I_\infty(WL^p(L^2)) = L_0^p$, for $1 < p < \infty$.*

We now turn to the case where $0 < p < 1$. We shall prove

THEOREM 3. *$I_\infty(WL^p(L^2)) = L^p$, for $0 < p < 1$.*

From the remarks above, I_∞ is certainly continuous from $WL^p(L^2)$ into L^p. To show that it is onto it is sufficient to show that it has dense range and is a homomorphism.

Let A_s denote the space of simple F_s measurable functions, and let $A = \bigcup_{0 \leqq s < \infty} A_s$. A is a dense linear subspace of L^p. Suppose now that $a = \sum_{i=1}^n \alpha_i \chi_{E_i} \in A_s$, where the E_i are disjoint sets in F_s. Let $\tau(\omega) = \inf\{t \geqq s: \omega(t) - \omega(s) = a\}$. τ is a stopping time, and conditional on E_i, $\tau - s$ has Laplace transform

$$L(u) = e^{-(2u)^{\frac{1}{2}}|\alpha_i|} .$$

Now if $0 < r < 1$,

$$\int_0^\infty \frac{1 - e^{-uv}}{u^{r+1}} du = v^r \int_0^\infty \frac{1 - e^{-u}}{u^{r+1}} du = C_r v^r , \quad \text{say},$$

so that if X is any nonnegative random variable with Laplace transform $L(u)$,

$$E(X^r) = C_r^{-1} \int_0^\infty \frac{1 - L(u)}{u^{r+1}} du \qquad \text{for} \quad 0 < r < 1 .$$

Applying this to the present situation,

$$E((\tau - s)^r \mid E_i) = 2^{r+1} C_r^{-1} C_{2r} |\alpha_i|^{2r} \qquad \text{for} \quad 0 < r < \tfrac{1}{2} .$$

Thus if we set $\varphi(t) = 1$ for $s \leqq t < \tau$, and $\varphi(t) = 0$ otherwise, then $I_\infty(\varphi) = a$ (so that I_∞ has dense range) and

$$E((\int_0^\infty \varphi^2(t)\, dt)^{p/2}) = 2^{(p/2)+1} C_{p/2}^{-1} C_p E(|a|^p) \qquad \text{for} \quad 0 < p < 1 .$$

This shows that I_∞ is a homomorphism, and completes the proof.

Let us conclude by remarking that the usual truncation argument now shows that I_∞ is a homomorphism of $WL^0(L^2)$ onto L^0, giving Dudley's result.

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