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BOREL-PROGRAMMABLE FUNCTIONS

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A new class of functions, the BP (Borel-programmable) functions, is defined. It is strictly larger than the class of Borel functions, but has some similar properties, including closure under composition. All BP functions are absolutely measurable. The class of BP sets (those with BP indicators) is a Borel field and is closed under operation A. The relation of BP sets to the R-sets of Kolmogorov is not treated.

1. Definitions and summary. Think of the set N of positive integers as the set of memory locations of a computer. A function $x: N \to \{0, 1\}$ will be called a *state* of the computer: $x(n)(= 0 \text{ or } 1)$ is the number stored in the location n. A function $p: X \to X$, where X is the set of all states, will be called a *program* if it is *decreasing*, i.e., if $p(x) \leqq x$ for all x. If our computer is now in state x and has program p, its next state will be $x' = p(x)$. That p is decreasing means just that $x(n) = 0$ implies $x'(n) = 0$; p never replaces 0 by 1. Associated with each program p and countable ordinal α is the program p_α, defined by

$$p_0(x) = x\,, \qquad p_\alpha(x) = p(\inf_{\beta<\alpha} p_\beta(x)) \quad \text{for} \quad \alpha > 0\,.$$

So $x_\alpha = p_\alpha(x)$ is the state after α steps. Since x_α decreases as α increases, it eventually stabilizes; there is an α_0, depending on p and x, with $x_{\alpha_0} = p(x_{\alpha_0})$. Define p_* by $p_*(x) = x_{\alpha_0}$, so that $p_*(x)$is the final state of our computer, if it has the initial state x and program p.

Let U and V be any spaces, let e and d be functions, with $e: U \to X$ and $d: X \to V$, and let p be any program. We associate with the triple (e, p, d) a function $f: U \to V$, denoted by $J(e, p, d)$ and defined by $f = d \circ p_* \circ e$. We think of e, p, d as describing the calculation of f. To evaluate $f(u)$, encode u as $x = e(u)$, use program p on x until the computer stabilizes at $x_* = p_*(x)$, and decode x_* as $v = d(x_*)$. Then $v = f(u)$.

For polish spaces U and V, the functions $J(e, p, d)$ as e, p, d vary over all Borel functions will be called *Borel-programmable functions*, or *BP functions*. So the BP functions are just those that can be evaluated with Borel encoders, programs, and decoders, if we can wait for the computer to stabilize. A subset of a polish space will be called a *BP set* if its indicator is a BP function.

Our results, proved in Section 2, are: The class of BP sets is a Borel field that includes all Borel sets and is closed under operation A, so that it is strictly larger than the Borel field determined by the analytic sets. The class of BP functions

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is closed under composition. If $f_1, f_2, \cdots$ are BP functions on U, so is $f = (f_1, f_2, \cdots)$. All BP functions are absolutely measurable, i.e., if f is a BP function on U and Q is a probability measure on the Borel sets of U, there is a Borel function f_1 on U and a Borel subset of U with Q-measure 1 on which $f_1 = f$. It is this latter property that may make BP functions of interest to probabilists.

I do not know whether the BP sets are a subclass of the R-sets introduced by Kolmogorov and studied by Kantorovich and Livenson (1933) and by Lyapunov (1953a), (1953b). The theory of BP sets seems simpler than that of R-sets.

For general BP functions e, p, d, I do not know whether $J(e, p, d)$ is a BP function, or whether it is absolutely measurable: the class of BPP functions is not studied here.

2. Proofs.

A. *For any Borel program p and any probability measure Q on (the Borel sets of) X, there is an α_0 such that, if the initial state has distribution Q, p stabilizes at or before step α_0 with probability* 1:

$$Q\{x : p_{\alpha_0}(x) \in F\} = 1 , \qquad \text{where} \quad F = \{x : p(x) = x\} .$$

PROOF. Put $B_\alpha(n) = \{x : x_\alpha(n) = 1\}$, where $x_\alpha = p_\alpha(x)$. The Borel sets $B_\alpha(n)$ decrease as α increases, so the numbers $Q(B_\alpha(n))$ are eventually constant: there is an α_0 with $Q(B_\alpha(n)) = Q(B_{\alpha_0}(n))$ for all $\alpha > \alpha_0$ and all n. Since

$$\{x : p_{\alpha_0}(x) \notin F\} = \bigcup_n (B_{\alpha_0}(n)/B_{\alpha_0+1}(n))$$

and $Q(B_{\alpha_0}(n)/B_{\alpha_0+1}(n)) = 0$ for all n, this α_0 works.

B. *Every* BP *function is absolutely measurable.*

PROOF. Say $f = J(e, p, d)$ is a BP function on U and Q is a probability measure on the Borel sets of U. Let Q_1 be the distribution of e, and (using A) choose α_0 with $Q_1\{x : p_{\alpha_0}(x) \in F\} = 1$, so that $Q\{u : p_{\alpha_0} \circ e(u) \in F\} = 1$. Define $f_1 = d \circ p_{\alpha_0} \circ e$. Then f_1 is Borel and agrees with f on $\{u : p_{\alpha_0} \circ e(u) \in F\}$.

C. *Every Borel function is* BP.

PROOF. For any polish U, these are Borel functions $e : U \to X$ and $d : X \to U$, with $d \circ e$ the identity on U. (For instance if $B_1, B_2, \cdots$ are Borel sets in U that separate points and e_n is the indicator of B_n, the function $e = (e_1, e_2, \cdots)$ is Borel and $1 - 1$ and has a Borel inverse e^{-1} on the Borel set eU. Take $d = e^{-1}$ on eU and constant elsewhere.) Thus for any $f : U \to V$ we have $f = J(e, I, f \circ d)$, where I is the identity program.

D. *The composition g of two* BP *functions is* BP.

PROOF. $g = d_2 \circ p_{2*} \circ e_2 \circ d_1 \circ p_{1*} \circ e_1$, where e_i and d_i are Borel functions and p_1 and p_2 are Borel programs. We now describe the Borel calculation (e, p, d)

for g. We choose a special location $a \in N$, and divide the remaining locations into two infinite sets N_1 and N_2. The encoding e stores $e_1(u)$ in N_1, and puts $x = 1$ elsewhere. If $x(a) = 1$, p replaces the x' now stored in N_1 by $p_1(x')$. If $p_1(x') \neq x'$, p leaves x as is off N_1. But if $p_1(x') = x'$, p sets $x(\mathrm{a}) = 0$ and stores $e_2 \circ d_1(x')$ in N_2, if possible (in conjunction with our e, it will always be possible, since N_2 contains all 1's at this stage). If $x(a) = 0$, p replaces the x'' now stored in N_2 by $p_2(x'')$, and leaves other locations unchanged. Then $p_* \circ e$ will have $p_{1*} \circ e_1(u)$ stored in N_1, 0 in a, and $p_{2*} \circ e_2 \circ d_1 \circ p_{1*} \circ e(u)$ stored in N_2, so that our decoder d is just d_2 applied to the content of N_2.

E. *If* $f_1, f_2, \cdots$ *are* BP *functions on* U, *so is* $f = (f_1, f_2, \cdots)$.

PROOF. Say $f_n = J(e_n, p_n, d_n)$. Divide N into a sequence of disjoint infinite sets $N_1, N_2, \cdots$. Our encoder e stores $e_n(u)$ in N_n. The program p replaces the content x_n of N_n by $p_n(x_n)$. And the decoder d has nth coordinate $d_n(x_n)$.

F. *The* BP *sets are a Borel field.*

PROOF. If I is the indicator of a set, then $r \circ I$ is the indicator of its complement, where $r(t) = 1 - t$. So, from D, the complement of a BP set is BP. If $I_1, I_2, \cdots$ are indicators of sets then $m \circ (I_1, I_2, \cdots)$ is the indicator of the union, where $m(x) = \max_n x(n)$. So, from D and E, the countable union of BP sets is BP.

G. *The class of* BP *sets is closed under operation* A.

PROOF. We shall replace N by the set S of finite sequences $s = (n_1, \cdots, n_k)$ of positive integers, and X by the set Y of functions $y\colon S \to \{0, 1\}$. Let D associate with each $s \in S$ a corresponding BP-set $D(s)$. The operation A associates with the family D the set H of all u for which there is an infinite sequence $n_1, n_2, \cdots$ of positive integers with $u \in D(n_1, \cdots, n_k)$ for all k, and our claim is that H is a BP set.

Define $f\colon U \to Y$ by $f(u) = y$ with $y(\mathrm{s}) = 1$ if $u \in D(s)$ and 0 if $u \notin D(s)$. Then (using E, with N replaced by S), f is a BP function. The indicator of H is just $I_{A_1} \circ f$, where A_1 is the set of all y: $\exists\, n_1, n_2 \cdots$ with $y(n_1, \cdots n_k) = 1$ for all k. So it suffices to show that the special set A_1 is a BP set. It was noted by Hausdorff (1957, page 213) that the program p that reduces $y(s)$ from 1 to 0 if s is *final*, i.e., $y(sn) = 0$ for all n where, if $s = (n_1, \cdots, n_k)$, then $(sn) = (n_1, \cdots, n_k, n)$, or *unreachable*, i.e., $y(t) = 0$ for some segment t of s, will have $p_*(y) = 0$ (the zero function) iff $y \notin A_1$.

To see this, first take any y with $p_*(y) = y_*$ not the zero function. There is an s with $y_*(s) = 1$. Since $p(y_*) = y_*$, this s is reachable, i.e., $y_*(t) = 1$ for all segments t of s, and is not final, i.e., there is an n with $y_*(sn) = 1$. So there is an infinite sequence $(n_1, n_2, \cdots)$, with s as a segment, with $y_*(n_1, \cdots, n_k) = 1$ for all k. Since $y_* \leqq y$, we have $y(n_1, \cdots, n_k) = 1$ for all k, and $y \in A_1$. Conversely, if $y \in A_1$, there is a sequence $(n_1, n_2, \cdots)$ with $y(n_1, \cdots, n_k)$ for all k. If $y_\alpha = p_\alpha(y)$, we see by induction on α that $y_\alpha(n_1, \cdots, n_k) = 1$ for all k and α, so that $p_*(y)$ is not the zero function. Thus $I_{A_1} = J(I, p, d)$ where $d(0) = 0$ and $d = 1$ otherwise.

It is known (Kunugui (1937)) that the Borel field determined by the analytic sets is not closed under operation A, so that the class of BP sets is a strictly larger class.

As noted by the referee, the following facts are easy consequences of the results above.

(a) The set of BP functions from one polish space to another is closed under taking pointwise limits.

(b) If U and V are polish spaces and f is a function from U to V, then the following are equivalent:

(i) f is a BP function;

(ii) f is measurable with respect to the BP σ-algebras on U and V;

(iii) f is measurable with respect to the BP σ-algebra on U and the Borel σ-algebra on V.

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