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THE SURVIVAL OF CONTACT PROCESSES[1]

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A new proof is given that a contact process on Z^d has a nontrivial stationary measure if the birth rate is sufficiently large. The proof is elementary and avoids the use of percolation processes, which played a key role in earlier proofs. It yields upper bounds for the critical birth rate which are significantly better than those available earlier. In one dimension, these bounds are no more than twice the actual value, and they are no more than four times the actual critical value in any dimension. A lower bound for the particle density of the largest stationary measure is also obtained.

1. Introduction. A contact process on Z^d is a particular type of continuous time Markov process whose state space is the set of all subsets of Z^d. In a contact process, each element in the set which is the state of the process at a given time is removed at a uniform rate independently of everything else, and each point not in the set is added to it at a rate which is an increasing function of the number of its neighbors which are in the set. Moreover, the rate at which an element of Z^d is added to the set is zero if none of its neighbors is already in it. This guarantees that the empty set is absorbing for the process, and one of the basic problems concerning a contact process is to determine when the point-mass on the empty set is the only stationary measure for it.

It is the purpose of this paper to give a new proof of a theorem due to Harris ([1] and [3]) to the effect that certain contact processes have more than that one stationary measure. This proof is more elementary than previous proofs in that it does not require results concerning percolation processes, and in that it is purely distributional in character. It also gives specific upper bounds for the critical values which are several orders of magnitude better than those which apparently can be deduced from the earlier proofs. The primary reason for the improvement in these bounds is that our technique treats the continuous time process directly, instead of comparing it to a discrete time process as was done earlier.

In order to be more specific, we begin by describing the contact processes with which we will be concerned. Let E_d be the set of all subsets of Z^d. Arbitrary elements of E_d will be denoted by lower case Greek letters, and finite

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elements of E_d will be denoted by capital Roman letters. For $x \in Z^d$, $\eta \in E_d$, and $\alpha > 0$, define $c_x^{\alpha}(\eta)$ to be one if $x \in \eta$ and to be α times the cardinality of $\eta \cap \{y : |y - x| = 1\}$ if $x \notin \eta$. If f is a cylinder function on E_d (i.e., $f(\eta)$ depends only on $\eta \cap A$ for some fixed finite set $A \subset Z^d$), put

$$\begin{aligned} \mathscr{L}^{(d,\alpha)}f(\eta) &= \textstyle\sum_{x \in Z^d} c_x^{\alpha}(\eta)[f(\eta \triangle \{x\}) - f(\eta)] \\ &= \textstyle\sum_{x \in \eta} [f(\eta \backslash \{x\}) - f(\eta)] \\ &\quad + \alpha \textstyle\sum_{x \in \eta} \sum_{|y-x|=1} [f(\eta \cup \{y\}) - f(\eta)] \,. \end{aligned} \tag{1.1}$$

The d-dimensional linear contact process is defined to be the Markov process on E_d whose infinitesimal generator, when restricted to the cylinder functions, is given by (1.1). Processes of this general type have been studied extensively (see for example, [6] for a review and list of references). In particular, it is known that there is a unique such process.

Our main result is the following.

(1.2) THEOREM. *If $\alpha \geqq 2/d$, then the d-dimensional linear contact process with parameter α has a stationary measure m which is invariant under translations in Z^d for which*

$$m\{\eta : \eta \ni 0\} \geqq \tfrac{1}{2} + \left(\tfrac{1}{4} - \frac{1}{2d\alpha}\right)^{\frac{1}{2}} .$$

In particular, m is not concentrated on the empty set.

Harris has proved (see Theorem 7.7 in [1]) that if $\alpha < 1/(2d - 1)$, then the d-dimensional linear contact process with parameter α has only the trivial stationary measure. Moreover, by standard monotonicity arguments (see [1]), it can be shown that there is a number α_d such that if $\alpha < \alpha_d$, then there is only one stationary measure, while if $\alpha > \alpha_d$, there is more than one stationary measure. Thus combining our Theorem 1.2 with Harris' Theorem 7.7 in [1] yields $1/(2d - 1) \leqq \alpha_d \leqq 2/d$.

Since a linear contact process is self-associate (see Section 7 of [2]), it also follows from Theorem 1.2 that if $\alpha \geqq 2/d$, then

$$P_{\{0\}}^{(d,\alpha)}(A(t) = \varnothing \quad \text{for some} \quad t) \leqq \tfrac{1}{2} - \left(\tfrac{1}{4} - \frac{1}{2d\alpha}\right)^{\frac{1}{2}} ,$$

where $P_\eta^{(d,\alpha)}$ denotes the probability measure corresponding to the Markov process with generator $\mathscr{L}^{(d,\alpha)}$ and initial state η.

Theorem 1.2 is proved first in case $d = 1$, from which the general case is deduced in Section 3. In Section 2, we prove a somewhat more general result in case $d = 1$. In order to state that theorem, let $\beta > 0$ and $0 \leqq p \leqq 1$, and define $d_x^{(\beta,p)}(\eta)$ to be one if $x \in \eta$ and to be $\beta(pI_\eta(x - 1) + (1 - p)I_\eta(x + 1))$ if $x \notin \eta$, where I_η is the indicator function of η.

(1.3) THEOREM. *If $\beta \geqq 4$, then the contact process on E_1 with generator $\Omega^{(\beta,p)}$ given by*

$$\Omega^{(\beta,p)}f(\eta) = \textstyle\sum_{x \in Z} d_x^{(\beta,p)}(\eta)[f(\eta \triangle \{x\}) - f(\eta)]$$

for cylinder functions f, has a translation invariant stationary measure m with $m\{\eta:\eta\ni 0\}\geqq \frac{1}{2}+(\frac{1}{4}-(1/\beta))^{\frac{1}{2}}$.

Since $\Omega^{(2\alpha,\frac{1}{2})}=\mathscr{L}^{(1,\alpha)}$, Theorem 1.2 in one dimension is a particular case of Theorem 1.3. Also, since the contact process with generator $\Omega^{(\beta,p)}$ has the contact process with generator $\Omega^{(\beta,1-p)}$ as an associate, as will as seen in Section 2, it follows from Theorem 1.3 that if $\beta\geqq 4$, then the contact process with generator $\Omega^{(\beta,p)}$ and probability measure $Q_A^{(\beta,p)}$ satisfies

$$Q_{\{0\}}^{(\beta,p)}(A(t)=\varnothing \text{ for some } t)\leqq \frac{1}{2}-\left(\frac{1}{4}-\frac{1}{\beta}\right)^{\frac{1}{2}}.$$

The totally asymmetric case obtained by taking $p=0$ or 1 in Theorem 1.3 is quite interesting in its own right, and plays an important role in the results of [3], for example.

2. Proof of Theorem 1.3. If A is a finite subset of Z and $\eta\in E\equiv E_1$, let

$$\begin{aligned}\chi_A(\eta)=\chi_\eta(A)&=1 \quad \text{if} \quad \eta\cap A=\varnothing\\ &=0 \quad \text{if} \quad \eta\cap A\neq\varnothing .\end{aligned}$$

Fix $\beta>0$ and $p\in[0,1]$, and let T_t be the semigroup on $C\{0,1\}^Z$ which corresponds to the contact process with generator $\Omega^{(\beta,p)}$ given in Theorem 1.3. The idea of the proof is to find a nontrivial translation invariant measure μ on E with the property that for all finite $A\subset Z$,

$$\int T_t\chi_A(\eta)\mu(d\eta)=\int\chi_A(\eta)(T_t{}^*\mu)(d\eta)$$

is a nonincreasing function of t. E is endowed with a compact metric topology via the obvious identification with $\{0,1\}^Z$ with the product topology. Since $\{\chi_A: A\subset Z, A \text{ finite}\}$ is a convergence determining class for the topology of weak convergence of measures on E, and since $\{T_t: t\geqq 0\}$ is a Feller semigroup (see [6]), it would then follow that $T_t{}^*\mu$ converges weakly to a measure m which is stationary for the semigroup. Moreover, since each $T_t{}^*\mu$ is translation invariant, the same is true of m. Finally, the monotonicity of the convergence would give

$$\begin{aligned}m\{\eta:\eta\ni 0\}&=\lim_{t\to\infty}\{1-\int\chi_{\{0\}}(\eta)(T_t{}^*\mu)(d\eta)\}\\ &\geqq 1-\int\chi_{\{0\}}(\eta)\mu(d\eta).\end{aligned}\tag{2.1}$$

Since χ_A is a cylinder function, and hence in the domain of $\Omega^{(\beta,p)}$, it follows from the Hille–Yosida theorem that

$$\frac{d}{dt}\int T_t\chi_A(\eta)\mu(d\eta)=\int\Omega^{(\beta,p)}T_t\chi_A(\eta)\mu(d\eta),$$

and thus it suffices to show that for all $t\geqq 0$ and finite $A\subset Z$,

$$\int\Omega^{(\beta,p)}T_t\chi_A(\eta)\mu(d\eta)\leqq 0.\tag{2.2}$$

Now one easily checks

$$
\begin{aligned}
\Omega^{(\beta,p)}\chi_A(\eta) &= \textstyle\sum_{x\in\eta}[\chi_A(\eta\backslash\{x\}) - \chi_A(\eta)] \\
&\quad + \beta \textstyle\sum_{x\in\eta}[p(\chi_A(\eta \cup \{x+1\}) - \chi_A(\eta)) \\
&\quad + (1-p)(\chi_A(\eta \cup \{x-1\}) - \chi_A(\eta))] \\
&= \textstyle\sum_{x\in A}[\chi_{A\backslash\{x\}}(\eta) - \chi_A(\eta)] \\
&\quad + \beta \textstyle\sum_{x\in A}[p(\chi_{A\cup\{x-1\}}(\eta) - \chi_A(\eta)) \\
&\quad + (1-p)(\chi_{A\cup\{x+1\}}(\eta) - \chi_A(\eta))] \\
&= \Omega^{(\beta,1-p)}\chi_\eta(A) .
\end{aligned} \tag{2.3}
$$

This equation shows that the contact process with generator $\Omega^{(\beta,p)}$ is the associate of (or dual to) the contact process with generator $\Omega^{(\beta,1-p)}$. Thus for all $t \geqq 0$ and all finite $A \subset Z$,

$$T_t\chi_A(\eta) = \textstyle\sum_{B\subset Z} Q_A{}^{(\beta,1-p)}(A(t) = B)\chi_B(\eta) , \tag{2.4}$$

where the summation is over all finite subsets of Z. For more information on associate processes and a proof of (2.4) see [2], [4], [5], or [6]. All that we need from (2.4) is that $T_t\chi_A$ is a linear combination with positive coefficients of other χ_B's, so that in order to prove (2.2), it suffices to prove

$$\textstyle\int \Omega^{(\beta,p)}\chi_A \, d\mu \leqq 0 \tag{2.5}$$

for all finite $A \subset Z$.

It appears to be difficult to compute efficiently the expression on the left side of (2.5) unless μ is particularly simple. It is easy to check that there is no translation invariant product measure on $\{0, 1\}^Z$ with positive particle density which satisfies (2.5) for all finite A. In view of the results in [7], it is natural then to look among the renewal measures for one which will satisfy (2.5). In order to simplify the notation, let $\eta_k = I_\eta(k)$ for $\eta \in E$ and $k \in Z$. If $\{f(k): k = 0, 1, 2, \cdots\}$ is a probability density with

$$a^{-1} = 1 + \textstyle\sum_{k=0}^\infty kf(k) < \infty ,$$

then the renewal measure ν_f determined by f is the probability measure on E with

$$\nu_f(A^k_{k_1,\cdots,k_j}) = a \textstyle\prod_{i=1}^j f(k_i - 1) , \tag{2.6}$$

where $A^k_{k_1,\cdots,k_j} = \{\eta \in E: \eta_k = \eta_{k+k_1} = \cdots = \eta_{k+k_1+\cdots+k_j} = 1$ and $\eta_i = 0$ for all other $i \in [k, k + k_1 + \cdots + k_j]\}$, $k \in Z$, and $k_i \geqq 1$. Note that ν_f is translation invariant and $\nu_f\{\eta: \eta \ni 0\} = a$.

In order to find a good candidate for a renewal measure which will satisfy (2.5), consider finding a density f such that for all $n \geqq 1$,

$$\textstyle\int \Omega^{(\beta,p)}\chi_{\{0,1,\cdots,n-1\}} \, d\nu_f = 0 . \tag{2.7}$$

By (2.3), (2.7) is equivalent to

$$
\begin{aligned}
\textstyle\sum_{k=0}^{n-1} \int [\chi_{\{0,1,\cdots,n-1\}\backslash\{k\}} - \chi_{\{0,1,\cdots,n-1\}}] \, d\nu_f \\
= \beta\{(1-p) \textstyle\int [\chi_{\{0,\cdots,n-1\}} - \chi_{\{0,\cdots,n\}}] \, d\nu_f \\
+ p \textstyle\int [\chi_{\{0,\cdots,n-1\}} - \chi_{\{-1,\cdots,n-1\}}] \, d\nu_f\} .
\end{aligned} \tag{2.8}
$$

Letting $F(n) = \sum_{k=n}^{\infty} f(k)$, (2.8) becomes

(2.9) $$\beta F(n) = \sum_{k=0}^{n-1} F(k)F(n-1-k)\,, \quad n \geqq 1 \quad \text{and} \quad F(0) = 1\,.$$

To solve this for F, let $\varphi(x) = \sum_{n=0}^{\infty} x^n F(n)$ and conclude from (2.9) that

(2.10) $$\varphi(x) = [\beta - (\beta^2 - 4\beta x)^{\frac{1}{2}}]/2x\,.$$

Thus $F(n) = (2n)!/(n!\,(n+1)!)\beta^{-n}$, which is a decreasing summable sequence for $\beta \geqq 4$. In fact, if $\beta \geqq 4$, then

(2.11) $$a^{-1} = \sum_{k=0}^{\infty} F(k) = \varphi(1) = [\beta - (\beta^2 - 4\beta)^{\frac{1}{2}}]/2\,.$$

Now let $f(n) = F(n) - F(n+1)$ for this choice of F, and $\mu = \nu_f$. Then by construction, (2.5) holds with equality for A of the form $\{k, k+1, \cdots, k+n\}$, and it remains to prove (2.5) for arbitrary finite A. The first step is contained in the following lemma.

(2.12) LEMMA. *Let* $B(n, 0) = \{\eta : \eta_0 = 1, \eta_{n+1} = 0\}$ *and* $B(n, j; k_1, \cdots, k_j) = \{\eta : \eta_0 = 1, \eta_{n+1} = \eta_{n+1+k_1} = \eta_{n+1+k_1+k_2} = \cdots = \eta_{n+1+k_1+\cdots+k_j} = 0\}$ *for* $j \geqq 1$, $k_1 \geqq 1, \cdots, k_j \geqq 1$. *Then* $\mu\{B(n, j; k_1, \cdots, k_j) \mid \eta_0 = 1\}$ *is a nondecreasing function of* n.

PROOF. The proof is in two steps, using induction in each one. The first case is $j = 0$, and we will prove by induction on n that

(2.13) $$\mu\{B(n, 0) \mid \eta_0 = 1\} = \frac{1}{\beta} \sum_{k=0}^{n} F(k)\,,$$

which will give the required result. From (2.6) and (2.9),

$$\mu\{B(0, 0) \mid \eta_0 = 1\} = F(1) = \frac{1}{\beta} F(0)\,,$$

so that (2.13) holds for $n = 0$. Assuming (2.13) for $n \leqq N - 1$, and using the translation invariance of μ and (2.9) gives

$$\begin{aligned}\mu\{B(N, 0) \mid \eta_0 = 1\} &= F(N+1) + \sum_{k=0}^{N-1} f(k)\mu\{B(N-1-k, 0) \mid \eta_0 = 1\} \\ &= \frac{1}{\beta} \sum_{k=0}^{N} F(k)F(N-k) + \sum_{k=0}^{N-1} f(k) \frac{1}{\beta} \sum_{j=0}^{N-1-k} F(j) \\ &= \frac{1}{\beta} \sum_{k=0}^{N} F(k)[F(N-k) + \sum_{j=0}^{N-1-k} f(j)] = \frac{1}{\beta} \sum_{k=0}^{N} F(k)\,,\end{aligned}$$

and therefore (2.13) holds for $n = N$ as well. We now do an induction on j. The lemma has just been proved for $j = 0$, so assume now that it is true for $j \leqq J - 1$. Then

$$\begin{aligned}\mu\{B(n, J; k_1, \cdots, k_J) \mid \eta_0 = 1\} &= \mu\{B(n + k_1, J - 1; k_2, \cdots, k_J) \mid \eta_0 = 1\} \\ &\quad - \mu\{\eta_{n+1} = 1 \mid \eta_0 = 1\}\mu\{B(k_1 - 1, J - 1; k_2, \cdots, k_J) \mid \eta_0 = 1\} \\ &= \mu\{B(n + k_1, J - 1; k_2, \cdots, k_J) \mid \eta_0 = 1\} \\ &\quad - [1 - \mu\{B(n, 0) \mid \eta_0 = 1\}]\mu\{B(k_1 - 1, J - 1; k_2, \cdots, k_J) \mid \eta_0 = 1\}\,,\end{aligned}$$

which is nondecreasing in n by the inductive hypothesis, thus completing the proof of the lemma.

Now fix a finite subset $A = \bigcup_{i=1}^k A_i$ of Z, where $A_1, \cdots, A_k$ are the ordered maximal connected components of A, so that there are integers l_i and r_i such that $A_i = [l_i + 1, r_i - 1]$ and $r_i \leqq l_{i+1} < r_{i+1} - 1$ for all i. Define

$$\rho(x) = \mu\{\eta : \eta = 0 \text{ on } A \cap (x, \infty) \mid \eta_x = 1\}$$

and

$$\lambda(x) = \mu\{\eta : \eta = 0 \text{ on } A \cap (-\infty, x) \mid \eta_x = 1\}$$

for $x \notin A$. By the translation invariance of μ and Lemma 2.12, $\rho(x)$ is a nonincreasing function on each connected component of A^c, and since μ is invariant under reflection about the origin, $\lambda(x)$ is a nondecreasing function on each connected component of A^c.

Applying (2.3) to the set A under consideration gives

$$\begin{aligned} &\int \Omega^{(\beta,p)}\chi_A \, d\mu \\ &\quad = \mu\{\eta_0 = 1\}[\textstyle\sum_{x\in A} \sum_{y,z\in A^c; y<x<z} \rho(z)\lambda(y)f(x-y-1)f(z-x-1) \\ &\qquad - \beta \textstyle\sum_{i=1}^k \{p\rho(l_i)\lambda(l_i) + (1-p)\rho(r_i)\lambda(r_i)\}] \, . \end{aligned} \tag{2.14}$$

The monotonicity of ρ gives

$$\begin{aligned} &\textstyle\sum_{x\in A} \sum_{y,z\in A^c; y<x<z} \rho(z)\lambda(y)f(x-y-1)f(z-x-1) \\ &\quad \leqq \textstyle\sum_{i=1}^k \sum_{x\in A_i} \sum_{j=i}^k \rho(r_j)F(r_j - x - 1) \sum_{y\in A^c; y<x} \lambda(y)f(x-y-1) \, , \end{aligned} \tag{2.15}$$

while the monotonicity of λ gives

$$\begin{aligned} &\textstyle\sum_{x\in A} \sum_{y,z\in A^c; y<x<z} \rho(z)\lambda(y)f(x-y-1)f(z-x-1) \\ &\quad \leqq \textstyle\sum_{i=1}^k \sum_{x\in A_i} \sum_{j=1}^i \lambda(l_j)F(x - l_j - 1) \sum_{z\in A^c; z>x} \rho(z)f(z-x-1) \, . \end{aligned} \tag{2.16}$$

Substituting (2.15) and (2.16) into (2.14) yields

$$\begin{aligned} &[\mu\{\eta_0 = 1\}]^{-1} \int \Omega^{(\beta,p)}\chi_A \, d\mu \\ &\quad \leqq p\{\textstyle\sum_{i=1}^k \sum_{x\in A_i} \sum_{j=1}^i \lambda(l_j)F(x-l_j-1) \sum_{z\in A^c; z>x} \rho(z)f(z-x-1) \\ &\qquad - \beta \textstyle\sum_{i=1}^k \rho(l_i)\lambda(l_i)\} \\ &\qquad + (1-p)\{\textstyle\sum_{i=1}^k \sum_{x\in A_i} \sum_{j=i}^k \rho(r_j)F(r_j - k - 1) \\ &\qquad \times \textstyle\sum_{y\in A^c; y<x} \lambda(y)f(x-y-1) - \beta \sum_{i=1}^k \rho(r_i)\lambda(r_i)\} \, . \end{aligned} \tag{2.17}$$

In order to complete the proof of (2.5), it then suffices to show that

$$\beta\rho(l_j) = \textstyle\sum_{x\in A; z\in A^c; l_j<x<z} F(x - l_j - 1)\rho(z)f(z-x-1) \, , \tag{2.18}$$

and

$$\beta\lambda(r_j) = \textstyle\sum_{x\in A; y\in A^c; y<x<r_j} F(r_j - x - 1)\lambda(y)f(x-y-1) \, . \tag{2.19}$$

We will prove (2.18) only, since the proof of (2.19) is similar. The proof of (2.18) is based on the identity

$$\rho(x) = \textstyle\sum_{z\in A^c; z>x} f(z-x-1)\rho(z) \, , \tag{2.20}$$

which holds for all $x \in A^c$ and is an immediate consequence of definition of $\rho(\cdot)$. It follows from (2.9) that

$$\beta f(n) = \sum_{k=0}^{n-1} F(k)f(n-k-1) - F(n)\,, \qquad n \geqq 1\,. \tag{2.21}$$

Thus, since $z \in A^c$ and $z < l_j$ imply that $z - l_j - 1 \geqq 1$, (2.20) and (2.21) yield

$$\beta\rho(l_j) = \sum_{z\in A^c; z>l_j} \rho(z)[\sum_{l_j<x<z} F(x-l_j-1)f(z-x-1) - F(z-l_j-1)]\,. \tag{2.22}$$

Identity (2.20) also gives

$$\sum_{z\in A^c; z>l_j} \rho(z)F(z-l_j-1) = \sum_{x,z\in A^c; l_j<x<z} F(x-l_j-1)f(z-x-1)\rho(z)\,, \tag{2.23}$$

and (2.22) and (2.23) together imply (2.18). The proof of Theorem 1.3 is now completed by using (2.1) and (2.11).

3. **Proof of Theorem 1.2.** A computation similar to that in (2.3) shows that the d-dimensional linear contact process with parameter α is self-associate, which means that

$$E_\eta^{(d,\alpha)}[\chi_A(\eta(t))] = E_A^{(d,\alpha)}[\chi_\eta(A(t))] \tag{3.1}$$

for all $\eta \in E_d$ and all finite A in E_d. Setting $\eta = Z^d$ and noting that $\chi_{Z^d}(A) = 0$ unless $A = \varnothing$, one sees from (3.1) that

$$E_{Z^d}^{(d,\alpha)}[\chi_A(\eta(t))] = P_A^{(d,\alpha)}\{A(t) = \varnothing\} \to P_A^{(d,\alpha)}\{A(t) = \varnothing \text{ for some } t\}\,.$$

Thus there is a translation invariant stationary measure m such that

$$\int \chi_A\, dm = P_A^{(d,\alpha)}\{A(t) = \varnothing \text{ for some } t\}\,. \tag{3.2}$$

In order to prove Theorem (1.2), it therefore suffices to show that if $\alpha \geqq 2/d$, then

$$P_{\{0\}}^{(d,\alpha)}\{A(t) = \varnothing \text{ for some } t\} \leqq \tfrac{1}{2} - \left(\tfrac{1}{4} - \frac{1}{2d\alpha}\right)^{\frac{1}{2}}. \tag{3.3}$$

First set $d = 1$ and $\alpha \geqq 2$ and let m be the stationary measure obtained in Theorem 1.3 (take $\beta = 2\alpha \geqq 4$ and $p = \frac{1}{2}$ in that theorem). Integrating both sides of (2.4) with respect to m gives

$$\int \chi_A\, dm = \int T_t\chi_A\, dm = \sum_B P_A^{(1,\alpha)}\{A(t) = B\} \int \chi_B\, dm \geqq P_A^{(1,\alpha)}\{A(t) = \varnothing\}\,.$$

Setting $A = \{0\}$, it then follows from Theorem 1.3 that if $\alpha \geqq 2$, then

$$P_{\{0\}}^{(1,\alpha)}\{A(t) = \varnothing \text{ for some } t\} \leqq \tfrac{1}{2} - \left(\tfrac{1}{4} - \frac{1}{2\alpha}\right)^{\frac{1}{2}}. \tag{3.4}$$

We will prove (3.3) by showing that

$$P_{\{0\}}^{(d,\alpha)}\{A(t) = \varnothing \text{ for some } t\} \leqq P_{\{0\}}^{(1,d\alpha)}\{A(t) = \varnothing \text{ for some } t\}\,. \tag{3.5}$$

Define $\pi_d : Z^d \to Z$ by

$$\pi_d(x_1, \cdots, x_d) = x_1 + \cdots + x_d ,$$

and let $\pi_d(\xi) = \{\pi_d(x) : x \in \xi\} \in E_1$ for $\xi \in E_d$.

The quickest way to see that (3.5) is true is via a coupling argument. The idea is to couple $A_1(t)$ and $A_d(t)$ together in such a way that $A_1(t)$ and $A_d(t)$ are separately Markovian with laws $P_{\{0\}}^{(1,d\alpha)}$ and $P_{\{0\}}^{(d,\alpha)}$ respectively, and so that $A_1(t) \subset \pi_d(A_d(t))$ with probability one for each t. While this makes (3.5) clear, the coupling is somewhat difficult to write out formally, so we will give an analytic proof instead. Let $F_d = \{A \in E_d,\ A \text{ finite}\}$, and note that for any bounded function f on F_d, (1.1) defines a function $\mathscr{L}^{(d,\alpha)}f$ on F_d such that $|\mathscr{L}^{(d,\alpha)}f(A)| \leqq 2|A|\|f\|_\infty(1 + 2\alpha d)$. A function ψ on F_d is said to be increasing if $A \subset B$ implies that $\psi(A) \leqq \psi(B)$. Let H be the mapping from functions on F_1 to functions on F_d defined by $H\psi(A) = \psi(\pi_d(A))$. Note that H maps bounded functions to bounded functions, and that $|\psi(A)| \leqq$ constant $|A|$ implies $|H\psi(A)| \leqq$ constant $|A|$, since $|\pi_d(A)| \leqq |A|$. The proof of the following lemma is a straightforward computation, which is left to the reader.

LEMMA 3.6. *Let ψ be a bounded increasing function on F_1. Then for all $A \in F_d$,*

$$\mathscr{L}^{(d,\alpha)}H\psi(A) \geqq H\mathscr{L}^{(1,\alpha d)}\psi(A) .$$

Let $T_t^{(d,\alpha)}$ be the semigroup on $C(E_d)$ corresponding to the linear contact process. Note that $T_t^{(d,\alpha)}$ extends naturally to functions on F_d which satisfy $|\psi(A)| \leqq \text{const}\,(1 + |A|)$, since $E_A^{(d,\alpha)}|A(t)| < \infty$ for $A \in F_d$.

LEMMA 3.7. *Let ψ be a bounded increasing function on F_1. Then $T_t^{(d,\alpha)}H\psi \geqq HT_t^{(1,\alpha d)}\psi$ on F_d.*

PROOF. This follows immediately from Lemma 3.6 and

$$T_t^{(d,\alpha)}H\psi - HT_t^{(1,\alpha d)}\psi = \int_0^t T_{t-s}^{(d,\alpha)}(\mathscr{L}^{(d,\alpha)}H - H\mathscr{L}^{(1,\alpha d)})T_s^{(1,\alpha d)}\psi\, ds \tag{3.8}$$

on F_d, since $T_s^{(1,\alpha d)}$ maps bounded increasing functions into bounded increasing functions ([1]) and $T_{t-s}^{(d,\alpha)}$ maps nonnegative functions into nonnegative functions. Observe that the derivative with respect to s of $T_{t-s}^{(d,\alpha)}HT_s^{(1,\alpha d)}\psi(A)$ is $-T_{t-s}^{(d,\alpha)}(\mathscr{L}^{(d,\alpha)}H - H\mathscr{L}^{(1,\alpha d)})T_s^{(1,\alpha d)}\psi(A)$, so that (3.8) is obtained by integration. In justifying the differentiation, $E_A^{(d,\alpha)}|A(t)| < \infty$ for $A \in F_d$ is used.

To complete the proof of (3.5), apply Lemma 3.7 to the monotone function $\psi(\eta) = 1 - \chi_B(\eta)$ and let $B \uparrow Z$ to obtain

$$P_{\{0\}}^{(d,\alpha)}\{A(t) = \emptyset\} \leqq P_{\{0\}}^{(1,\alpha d)}\{A(t) = \emptyset\}$$

for all $t \geqq 0$. Now use the fact that $\emptyset$ is absorbing for the contact process.

REMARK. If α_d is as in the introduction, then Lemma 3.7 shows that $d\alpha_d \leqq \alpha_1$. The proof can be easily modified to show that $mn\alpha_{mn} \leqq m\alpha_m$ for all positive integers m and n. Using this, together with the easily obtained fact that α_d is nonincreasing in d, it is not hard to show that $\lim_{d\to\infty} d\alpha_d$ exists. It would be interesting to know the value of this limit. Our results show that it is between $\frac{1}{2}$ and 2.

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