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A SIGNED MEASURE ON PATH SPACE RELATED TO WIENER MEASURE[1]

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The connection between the heat equation and Brownian motion is generalized to a process related to the equation $\partial u/\partial t = (-1)^{n+1}\,\partial^{2n}u/\partial x^{2n}$, $n \geqq 2$. The associated measure is of unbounded variation and signed; the process cannot be realized in the space of continuous functions. Stochastic integrals $\int_0^t \varphi(x(s))(dx)^j(s)$, $j = 1, 2, \cdots, 2n$, are defined, and an analogue of Itô's lemma for the Brownian integral is proven. Specifically, one gets $2n$ independent differentials $(dx)^j$, with $(dx)^{2n} = (-1)^{n+1}(2n)!\,dt$. Applications include the derivation of the analogue of the Brownian exponential martingale $\exp\{\alpha x - \alpha^2 t/2\}$ and a class of orthogonal functions which generalize the Hermite polynomials. These are followed by the Feynmann–Kac formula, distribution of the maximum function, arc-sine law, and distribution of eigenvalues. Finally, central limit theorems are proven for convergence of sums of independent random variables identically distributed by a signed measure, normalized to have first $2n - 1$ moments equal to zero and $2n$th moment equal to $(-1)^{n+1}(2n)!$.

1. Introduction. The well-known connections between the theory of Markov stochastic processes and the study of second-order elliptic operators and associated parabolic equations are herein extended to relate some higher-order elliptic operators to processes determined by signed measures defined on function spaces. Specifically, the connection between the Brownian motion (Wiener) process and the heat equation $\partial u/\partial t = (\frac{1}{2})\,\partial^2 u/\partial x^2$ is generalized to a process corresponding to the even-order parabolic partial differential equation

$$\frac{\partial u}{\partial t} = (-1)^{n+1}\,\frac{\partial^{2n} u}{\partial x^{2n}}\,, \qquad n \geqq 2\,.$$

The fundamental solution $p(t, x)$ of the equation, which is the Fourier transform of $\exp\{-\xi^{2n}t\}$, is taken to be the density of a measure which we associate with the process. The major difficulties arise from the fact that this measure is of unbounded variation and, unlike Wiener measure, is signed. Nevertheless, it is shown that many Brownian results do have analogues in the general case, though frequently with modified proofs. First, the fourth-order equation is studied in much detail; the final section contains generalizations of these results for arbitrary integral values of n.

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Section 2 covers the construction of the signed measure P from the density $p(t, x)$. The total measure of the real line is one, and hence P resembles a probability measure; moreover, the Markov property is satisfied. The asymptotic decay of $p(t, x)$ in x is analyzed, and the function $x(t)$, distributed by the given signed measure, is seen to satisfy certain inversion and scaling properties similar to those of Brownian motion.

Section 3 covers the notions of weak convergence and stochastic integration. We show that there are $2n$ independent stochastic differentials $(dx)^j$, $j = 1, 2, \cdots, 2n$, and, correspondingly, $2n$ types of stochastic integrals $\int_0^t \varphi(x(s))(dx)^j(s)$. The analogue of $(dx)^2 = dt$ for the Brownian integral is $(dx)^{2n} = (-1)^{n+1}(2n)!\, dt$. For elementary functions such as polynomials and exponentials, and for functions of the Schwarz class $C_\downarrow^\infty$ of infinitely differentiable rapidly decreasing functions, we prove a generalization of the Itô lemma which, in differential notation, says that

$$df = \sum_{k=1}^{2n-1} \frac{1}{k!} f^{(k)}(dx)^k + (-1)^{n+1} f^{(2n)}\, dt .$$

Several applications of the Itô lemma follow. In particular, we show that

$$y(t) = \exp\left\{\sum_{k=1}^{2n-1} (-1)^{k+1} \frac{\alpha^k}{k} \int_0^t (dx)^k(s) + (-1)^n (2n-1)!\, \alpha^{2n} t\right\}$$

satisfies the initial-value problem $dy = \alpha y\, dx$, $y(0) = 1$, and hence is the analogue of the customary exponential $e^{\alpha x(t)}$. Similarly, formulas for the functions defined by

$$h_k(t, x) = k! \int_0^t dx(t_1) \int_0^{t_1} dx(t_2) \cdots \int_0^{t_{k-1}} dx(t_k)$$

(the analogues of the customary powers $[x(t)]^k$) are derived, and the h_k are shown to form a class of orthogonal functions in t, x, $\int (dx)^2$, and $\int (dx)^3$ and are, therefore, generalizations of the Hermite polynomials, which play the same role in the theory of the Brownian integral.

In Section 4, the purely combinatorial identity of Spitzer [14] and the Feynmann-Kac formula [6, 8] are used to derive the distribution of the maximum function and to show that the arc-sine law [7, 14] holds. In particular, for the process $x(t)$ related to the fourth-order equation, we show that the Laplace transform of the density of the maximum function is given by

$$\int_0^\infty e^{-uT} \frac{\partial}{\partial \alpha} P\{\max_{s \leq T} x(s) \leqq \alpha\}\, dT = 2^{\frac{1}{2}} u^{-\frac{3}{4}} \exp\left\{-\frac{u^{\frac{1}{4}}\alpha}{2^{\frac{1}{2}}}\right\} \sin \frac{u^{\frac{1}{4}}\alpha}{2^{\frac{1}{2}}} .$$

From these results we conclude that the trajectories of the signed process are not continuous; i.e., the process cannot be realized in the space of continuous functions. A heuristic approach is then used to obtain the asymptotic distribution of the eigenvalues for the operator $L = -\Delta^2$.

The fifth section contains two central limit theorems for independent random variables x_i, identically distributed by a signed measure μ and normalized to have first $2n - 1$ moments equal to zero and $2n$th moment equal to $(-1)^{n+1}(2n)!$, where the moments exist as absolutely convergent integrals. Under appropriate conditions on μ and for appropriate functions $\varphi \in C_{\downarrow}^{\infty}$, we have

$$\lim_{k\to\infty} E\left\{\varphi\left(\frac{x_1 + x_2 + \cdots + x_k}{k^{1/2n}}\right)\right\} = \int_{-\infty}^{\infty} \varphi(x)p(1, x)\,dx .$$

2. The density and signed measure for $n = 2$. We begin by considering the initial value problem for the partial differential equation

$$\frac{\partial u}{\partial t} = -\frac{\partial^4 u}{\partial x^4}, \qquad -\infty < x < \infty,\ 0 \leqq t < \infty \tag{2.1}$$

$$u(0, x) = f(x) .$$

Let

$$\hat{g}(\xi) = (2\pi)^{-1} \int_{-\infty}^{\infty} e^{ix\xi} g(x)\,dx$$

denote the Fourier transform of a function $g(x)$. Then the solution of (2.1) for reasonable initial functions $f(x)$ is easily seen to be

$$u(t, x) = \int_{-\infty}^{\infty} p(t, x - y)f(y)\,dy \tag{2.2}$$

in which the fundamental solution $p(t, x)$ is expressed by

$$p(t, x) = [\exp(-\xi^4 t)]^{\wedge} = (2\pi)^{-1} \int_{-\infty}^{\infty} e^{ix\xi} \exp\{-\xi^4 t\}\,d\xi . \tag{2.3}$$

We note the following properties of $p(t, x)$:

(2.4) (i) $p(t, x) = p(1, t^{-\frac{1}{4}}x)t^{-\frac{1}{4}}$,

(ii) $p(t, x)$ is symmetric in x,

(iii) $p(t, x)$ belongs to the Schwarz class $C_{\downarrow}^{\infty}$ of infinitely differentiable functions f with $\lim_{|x|\to\infty} x^k(d^j/dx^j)f(x) = 0$ for each $k, j \geqq 0$.

To study the asymptotic behavior of $p(1, x)$ for large x, let $\xi = x^{\frac{1}{3}}s$, so

$$\int_{-\infty}^{\infty} e^{ix\xi} \exp\{-\xi^4\}\,d\xi = x^{\frac{1}{3}} \int_{-\infty}^{\infty} \exp\{x^{\frac{4}{3}}(is - s^4)\}\,ds .$$

The steepest descent and ascent curves for the resulting integral are those which pass through the critical points of $f(s) = is - s^4$ and along which the imaginary part of $f(s)$ remains constant. The three critical points of $f(s)$ are $s = (4)^{-\frac{1}{3}}e^{i\theta}$, where $\theta = \pi/6, 5\pi/6, 3\pi/2$; for $s = x + iy$, $\operatorname{Im} f(s) = x(1 - 4x^2y + 4y^3)$ must be a constant c. For the curve to pass through $s_1 = (4)^{-\frac{1}{3}}e^{\pi i/6}$, the constant c must be $3\cdot 3^{\frac{1}{2}}/8^3 4^{\frac{1}{3}}$; thus, $y^3 - x^2y = \frac{1}{4}(c/x - 1)$, from which we can conclude that $y \to 0$ as $x \to +\infty$ and $y \to +\infty$ as $x \to 0^+$. A similar asymptotic calculation for the critical point $s_2 = (4)^{-\frac{1}{3}}e^{5\pi i/6}$ leads to the conclusion that the steepest descent contour is that of Figure 1.

Using this contour, the leading term in the asymptotic expansion of the integral is obtained by evaluating the contribution about the saddle points s_1 and s_2. Expanding $f(s)$ about s_1 and s_2, we get

$$f(s_1) \approx \frac{3}{4^{\frac{4}{3}}} e^{2\pi i/3} - \frac{6}{4^{\frac{2}{3}}} e^{\pi i/3}(s - s_1)^2$$

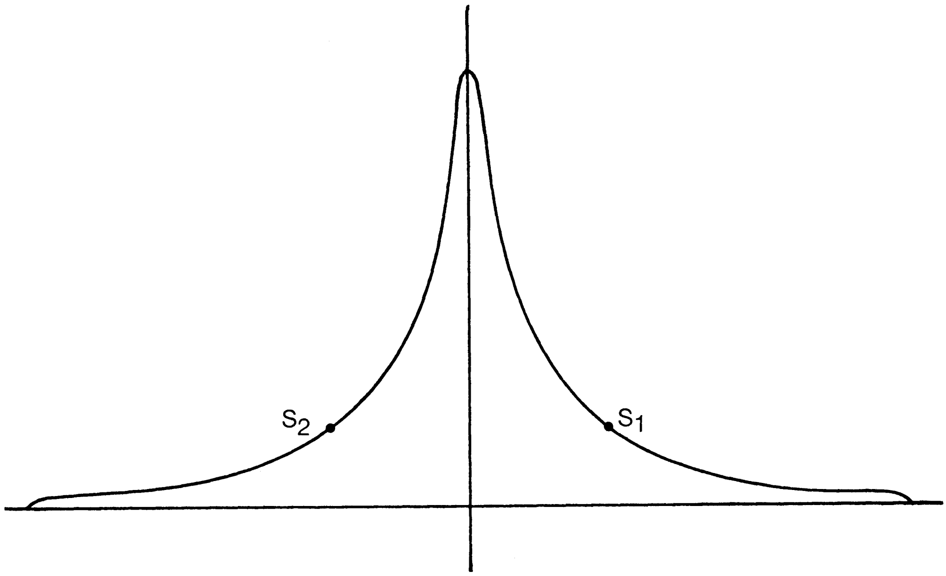

FIG. 1.

and

$$f(s_2) \approx \frac{3}{4^{\frac{4}{3}}} e^{4\pi i/3} - \frac{6}{4^{\frac{2}{3}}} e^{5\pi i/3}(s - s_2)^2 .$$

Thus, the contribution near s_1 is

$$\left(\frac{\pi}{6}\right)^{\frac{1}{2}} \left(\frac{4}{x}\right)^{\frac{1}{3}} \exp\left\{\frac{5\pi i}{6} + \frac{3}{4^{\frac{4}{3}}} e^{2\pi i/3} x^{\frac{4}{3}}\right\},$$

and near s_2 it is

$$\left(\frac{\pi}{6}\right)^{\frac{1}{2}} \left(\frac{4}{x}\right)^{\frac{1}{3}} \exp\left\{\frac{\pi i}{6} + \frac{3}{4^{\frac{4}{3}}} e^{4\pi i/3} x^{\frac{4}{3}}\right\}.$$

Therefore, for large x,

$$p(1, x) = kx^{-\frac{1}{3}} \exp\{-ax^{\frac{4}{3}}\} \cos(bx^{\frac{4}{3}}) + \text{lower order terms} \quad (k = \text{constant}) \tag{2.5}$$

where $a = (\frac{3}{8})(4)^{-\frac{1}{3}}$ and $b = 3^{\frac{1}{2}}a$.

REMARK. The expression for the leading term of the asymptotic expansion confirms the result of Pólya [11] that $p(1, x)$ is an integral function of order $\frac{4}{3}$ with infinitely many real zeros. Burwell [2] proved that these zeros are given asymptotically by

$$\lambda_k = \pm 4\left[\frac{2(3k - 1)\pi}{[illegible]}\right]^{\frac{3}{4}} + O(k^{-\frac{1}{4}}) . \tag{2.6}$$

Clearly, $\int_{-\infty}^{\infty} p(t, x)\, dx = 2\pi\hat{p}(t, 0) = 1$, so it might be conjectured that $p(t, x)$ is a probability density like the Gauss kernel $(2\pi t)^{-\frac{1}{2}} \exp\{-x^2/2t\}$ associated with the equation $\partial u/\partial t = (\frac{1}{2})(\partial^2 u/\partial x^2)$. This, however, is not the case; $p(t, x)$ *changes sign infinitely often*, as the asymptotics show. Notice in this connection that

$$\int_{-\infty}^{\infty} x^4 p(t, x)\, dx = \frac{d^4}{d\xi^4} e^{-\xi^4 t}\Big|_{\xi=0} = -24t < 0 . \tag{2.7}$$

Note also for future use that

$$\int_{-\infty}^{\infty} x^j p(t, x)\, dx = 0 \qquad \text{for } j = 1, 2, 3 . \tag{2.8}$$

We now use the function $p(t, x, y) = p(t, y - x)$ to construct a signed measure on path space satisfying the Markov property. For $s > 0$ and $t > 0$, the Chapman–Kolmogorov equation

$$p(t + s, x, y) = \int_{-\infty}^{\infty} p(t, x, dx')p(s, x', y) \tag{2.9}$$

is satisfied; this is obvious from $\hat{p}(t, \xi) = \exp\{-\xi^4 t\}/2\pi$. Also, we have seen that $p(t, x)$ resembles a "signed probability density." Therefore, adapting the language and techniques of the theory of Markov stochastic processes to our situation, we treat $p(t, x)$ as a "signed Markov transition density" (see, for example, [3], page 255) with which we can build up a signed measure on the space Ω of real-valued measurable functions $x: t \in [0, \infty) \to x(t)$, called *paths* (see, e.g., [10], page 52), in the usual way. Let $C \subset \Omega$ by a cylinder set

$$C = \{x: a_i \leqq x(t_i) \leqq b_i, i = 1, 2, \cdots, n\} , \qquad 0 < t_1 < t_2 < \cdots < t_n .$$

Then, a signed finitely additive measure P may be consistently defined on Ω by the rule

$$P(C) = \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \prod_{i=1}^{n} p(t_i - t_{i-1}, x_i - x_{i-1})\, dx_i , \tag{2.10}$$

where $x_0 = 0$ and $t_0 = 0$. Clearly, P is countably additive on the field of sets generated by $x(t_i)$ $(i = 1, 2, \cdots, n)$ for *fixed* $t_1 < t_2 < \cdots < t_n$ and $n < \infty$. (No claim of countable additivity is made on the field generated by the class of *all* cylinder sets.) E denotes the "expectation": $E(f) = \int f\, dP$, when the latter makes sense.

THEOREM 2.1. *P has unbounded total variation on paths $x(t)$.*

PROOF. We divide the interval $[0, 1]$ into n pieces, each of length $1/n$. The total variation of P on paths $x(t): 0 \leqq t \leqq 1$ is underestimated by

$$\begin{aligned} \lim_{n\to\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{i=1}^{n} \left| p\left(\frac{1}{n}, x_i - x_{i-1}\right)\right| dx_i \\ = \lim_{n\to\infty} \left[\int_{-\infty}^{\infty} \left| p\left(\frac{1}{n}, x\right)\right| dx\right]^n \\ = \lim_{n\to\infty} [n^{\frac{1}{4}} \int_{-\infty}^{\infty} |p(1, n^{\frac{1}{4}}x)|\, dx]^n \\ = \lim_{n\to\infty} [\int_{-\infty}^{\infty} |p(1, y)|\, dy]^n . \end{aligned}$$

But $\int_{-\infty}^{\infty} p(t, x)\, dx = 1$, while $p(t, x)$ assumes negative values for some values of x; thus, $\int_{-\infty}^{\infty} |p(1, y)|\, dy > 1$, so the limit above is infinite. □

P is Markovian in the following sense. Let $P_a(x \in E) = P(x + a \in E)$ and let $\mathscr{F}$ be the field of $x(t_i)$ $(i = 1, 2, \cdots, n)$ with fixed $n < \infty$ and $0 \leqq t_1 \leqq \cdots \leqq t_n = T$. Then

$$P_a\{x(t + T) \in dy \mid \mathscr{F}\} = P_b\{x(t) \in dy\}, \qquad b = x(T). \tag{2.11}$$

It would be nice to show that the signed measure P is concentrated on paths which satisfy the Hölder condition

$$|x(t_2) - x(t_1)| \leqq |t_2 - t_1|^{1/4-\varepsilon} \tag{2.12}$$

for arbitrary positive ε, by analogy with the Brownian case. Krylov [8] showed that if one subdivides the interval $0 \leqq t \leqq 1$ into pieces of length 2^{-n}, then the total variation of the restriction of the signed measure P to the field of $x(m2^{-n})$ $(m = 1, \cdots, 2^n)$, computed over the set of functions with $|x(t_2) - x(t_1)| > |t_2 - t_1|^{1/4-\varepsilon}$ for some $t_2 = i/2^n$ and $t_1 = j/2^n$ with $|t_2 - t_1|$ sufficiently small, tends to zero as $n \to \infty$. This is about as good as can be expected.

We now note the following proposition, which presents the analogues of the transformations which carry Brownian motion into Brownian motion.

PROPOSITION 2.2. *Each of these has the same distribution with respect to* P:

(i) $x(t)$, $t \geqq 0$
(ii) $-x(t)$, $t \geqq 0$ *(symmetry)*
(iii) $x(t + s) - x(s)$, $t \geqq 0$, $s \geqq 0$ *(additive property)*
(iv) $t^{\frac{1}{4}}x(1/t)$, $t > 0$ *(inversion)*
(v) $cx(t/c^4)$, $t \geqq 0$, $c > 0$ *(scaling)*.

The proof is elementary, making use of (2.3) and (2.4).

3. Stochastic integration.

3.1. *Weak convergence; existence.* We begin by introducing a notion of weak convergence. A function is said to be *tame* if it is a Borel function of a finite number of observations $x(t_i)$. A sequence of tame functions $\{f_n\}$ is said to have a *weak limit* if $\lim_{n\to\infty} E(f_n\psi)$ exists for every tame $\psi = \psi(x(t_1), x(t_2), \cdots, x(t_m))$ with $\psi \in C_\downarrow^\infty$ and defines a function $L(\psi)$ on $C_\downarrow^\infty$. In that case, we write $L(\psi) = E(f\psi)$ and say that $f = \lim_{n\to\infty} f_n$, though this is purely formal: we have no general way of proving the existence of f as a bona fide measurable function of x.

Now we define the stochastic integrals

$$\textstyle\int_0^t \varphi(x(s))(dx)^j(s), \qquad j = 1, 2, 3, 4, \ 0 \leqq t \leqq 1 \tag{3.1}$$

as the weak limits of the Riemann sums

$$\sum_{k=1}^{l} \varphi\left(x\left(\frac{k-1}{n}\right)\right)\left[x\left(\frac{k}{n}\right) - x\left(\frac{k-1}{n}\right)\right]^j + \varphi\left(x\left(\frac{l}{n}\right)\right)\left[x(t) - x\left(\frac{l}{n}\right)\right]^j, \tag{3.2}$$

where l is the greatest integer less than or equal to nt.

To simplify matters, we will at first let $t = 1$. In this case, demonstrating the existence of a weak limit of the sums (3.2) is equivalent to proving that

$$\lim_{n\to\infty} E\left\{\sum_{k=1}^{n} \varphi\left(x\left(\frac{k-1}{n}\right)\right)\left[x\left(\frac{k}{n}\right) - x\left(\frac{k-1}{n}\right)\right]^j \times \psi(x(t_1), x(t_2), \cdots, x(t_m))\right\} = L(\psi) \tag{3.3}$$

exists as a functional of $\psi \in C_\downarrow^\infty$; we then denote $L(\psi)$ by

$$E\{\textstyle\int_0^1 \varphi(x(t))(dx)^j(t)\psi(x(t_1), x(t_2), \cdots, x(t_m))\}\,.$$

The existence proof is now carried out for integrands $\varphi \in C_\downarrow^\infty$.

For $j = 1, 2, 3, 4$, to prove the existence of the limit (3.3), we first break up the sum in the expectation into m blocks, with the ith block consisting of those terms for which $t_{i-1} < k/n \leqq t_i$, $i = 1, 2, \cdots, m$, where $t_0 = 0$. For k such that $(k-1)/n > t_m$, the contribution to the expectation is zero by independence of $x(k/n) - x((k-1)/n)$ and $x(i/n): i < k$. If $(k-1)/n < t_m < k/n$, then the contribution to the expectation is

$$\begin{aligned}
&E\left\{\varphi\left(x\left(\frac{k-1}{n}\right)\right)\left[x\left(\frac{k}{n}\right) - x\left(\frac{k-1}{n}\right)\right]^j\psi(x(t_1), \cdots, x(t_m))\right\}\\
&= E\left\{\varphi\left(x\left(\frac{k-1}{n}\right)\right)\left[x(t_m) - x\left(\frac{k-1}{n}\right)\right]^j\psi(x(t_1), \cdots, x(t_m))\right\}\\
&= \int\int \cdots \int d\lambda\, d\mu_1\, d\mu_2 \cdots d\mu_m\, \hat\varphi(\lambda)\hat\psi(\mu_1, \mu_2, \cdots, \mu_m)\\
&\qquad \times E\left\{\exp\left\{i\lambda x\left(\frac{k-1}{n}\right)\right\}\left[x(t_m) - x\left(\frac{k-1}{n}\right)\right]^j \exp\{i \textstyle\sum_{l=1}^{m} \mu_l x(t_l)\}\right\}\\
&= (-i)^j \int\int \cdots \int \hat\varphi\hat\psi E\left\{\exp\left\{i(\lambda+\mu_m)x\left(\frac{k-1}{n}\right)\right\}\cdot \exp\{i \textstyle\sum_{l=1}^{m-1} \mu_l x(t_l)\}\right\}\\
&\qquad \times \frac{\partial^j}{\partial{\mu_m}^j} E\left\{\exp\left\{i\mu_m\left[x(t_m) - x\left(\frac{k-1}{n}\right)\right]\right\}\right\},
\end{aligned} \tag{3.4}$$

where, say, $t_{m-1} < (k-1)/n < t_m < k/n$. (If $t_{m-p} < (k-1)/n < t_{m-p+1}$, $p = 2, \cdots, m$, the adjustment is obvious.) But the last factor in (3.4) equals

$$\frac{\partial^{j-1}}{\partial{\mu_m}^{j-1}}(-4{\mu_m}^3)\left(t_m - \frac{k-1}{n}\right)\exp\left\{-{\mu_m}^4\left(t_m - \frac{k-1}{n}\right)\right\} = o(1)\,,$$

so the entire term is $o(1)$ as $n \to \infty$.

Thus, we need only consider the m blocks of the form

$$E\left\{\sum_{t_{i-1}<k/n\leqq t_i} \varphi\left(x\left(\frac{k-1}{n}\right)\right)\left[x\left(\frac{k}{n}\right) - x\left(\frac{k-1}{n}\right)\right]^j \psi(x(t_1), \cdots, x(t_m))\right\}\,.$$

If we now integrate this last expression over the future times $t_{i+1}, t_{i+2}, \cdots, t_m$ conditional upon $x(t): t \leqq t_i$, then the $C_\downarrow^\infty$ function $\psi(x(t_1), x(t_2), \cdots, x(t_m))$ is replaced by some other $C_\downarrow^\infty$ function of only the first t_i observations; hence, we can assume that $i = m$. Thus, we have reduced the problem to showing the existence of the limit of functions of the form

$$E\left\{\sum_{t_{m-1}\leqq k/n\leqq t_m} \varphi\left(x\left(\frac{k-1}{n}\right)\right)\left[x\left(\frac{k}{n}\right) - x\left(\frac{k-1}{n}\right)\right]^j \psi(x(t_1), \cdots, x(t_m))\right\}$$
$$= \int\int \cdots \int d\lambda\, d\mu_1\, d\mu_2 \cdots d\mu_m\, \hat{\varphi}(\lambda)\hat{\psi}(\mu_1, \mu_2, \cdots, \mu_m)$$
$$\times E\left\{\sum_{t_{m-1}<k/n\leqq t_m} \exp\left\{i\lambda x\left(\frac{k-1}{n}\right)\right\}\left[x\left(\frac{k}{n}\right) - x\left(\frac{k-1}{n}\right)\right]^j\right.$$
$$\left.\times \exp\{i \textstyle\sum_{l=1}^m \mu_l x(t_l)\}\right\}.$$

Since φ and ψ are both $C_\downarrow^\infty$, so are $\hat{\varphi}$ and $\hat{\psi}$. Therefore, this last integral will converge if the expectation inside converges under a polynomial bound. The term arising from the case $(k-1)/n \leqq t_{m-1} < k/n$ is easily seen to be $o(1)$. (The computation is similar to that in (3.4).) The remaining expression can be simplified by taking a preliminary expectation conditional upon $x(t): t \leqq t_{m-1}$,

$$E\left\{\sum_{t_{m-1}<(k-1)/n<k/n\leqq t_m} \exp\left\{i\lambda x\left(\frac{k-1}{n}\right)\right\}\left[x\left(\frac{k}{n}\right) - x\left(\frac{k-1}{n}\right)\right]^j\right.$$
$$\left.\times \exp\{i\mu_m x(t_m) \mid x(t): t \leqq t_{m-1}\right\},$$

and it is enough to show that the latter converges under a polynomial bound. Performing the conditional expectation leads to an expression of the form

$$[\text{a fixed function of } x(t_1), \cdots, x(t_{m-1})]$$
$$\times E\{\textstyle\sum_{k=1}^N \exp\{i\lambda x(s_{k-1})\}[x(s_k) - x(s_{k-1})]^j \exp\{i\mu x(t)\}\}$$

in which $\mu = \mu_m$, $t = t_m - t_{m-1}$, and $s_0, s_1, \cdots, s_N$ is a subdivision of the interval $[0, t]$. Now

$$E\{\textstyle\sum_{k=1}^N \exp\{i\lambda x(s_{k-1})\}[x(s_k) - x(s_{k-1})]^j \exp\{i\mu x(t)\}\}$$
$$= E\{\textstyle\sum_{k=1}^N \exp\{i(\lambda + \mu)x(s_{k-1})\}[x(s_k) - x(s_{k-1})]^j$$
$$\times \exp\{i\mu[x(s_k) - x(s_{k-1})]\} \exp\{i\mu[x(t) - x(s_k)]\}\}$$
$$= \sum_{k=1}^N [\exp\{-s_{k-1}(\lambda + \mu)^4\}]\left[(-i)^j \frac{\partial^j}{\partial\mu^j} \exp\{-(s_k - s_{k-1})\mu^4\}\right]$$
$$\times [\exp\{-(t - s_k)\mu^4\}],$$

which converges to

$$\frac{4!\,(-1)^{j+1}(i\mu)^j}{(4-j)!} \exp\{-t\mu^4\} \int_0^t \exp\{-s[(\lambda + \mu)^4 - \mu^4]\}\, ds$$
$$= \frac{4!}{(4-j)!}\,\frac{(-1)^{j+1}(i\mu)^j}{\mu^4 - (\lambda + \mu)^4}[\exp\{-t(\lambda + \mu)^4\} - \exp\{-t\mu^4\}]$$

under the polynomial bounds

$$\begin{array}{ll} 4|\mu|^3 t & \text{if } j = 1\,, \\ 12\mu^2 t + 16\mu^6 t^2 & \text{if } j = 2\,, \\ 24|\mu| t + 144|\mu|^5 t^2 + 64|\mu|^9 t^3 & \text{if } j = 3\,, \\ 24t + 816\mu^4 t^2 + 1152\mu^8 t^3 + 256\mu^{12} t^4 & \text{if } j = 4\,. \end{array}$$

The calculation of the limit (3.3) in the case $j = 4$ is now presented for $\psi = \psi(x(t_1), x(t_2))$, where, to cover a general type of situation, we let $(k-1)/n < t_1 < k/n < t_2$. It should be clear that if ψ is a function of *any* finite collection of observations, the calculation would be totally analogous, with some observation times lying in the interval $((k-1)/n, k/n)$ and others outside this interval.

For $j = 4$,

$$\begin{aligned} &E\left\{\sum_{k=1}^{n} \varphi\left(x\left(\frac{k-1}{n}\right)\right)\left[x\left(\frac{k}{n}\right) - x\left(\frac{k-1}{n}\right)\right]^4 \psi(x(t_1), x(t_2))\right\} \\ &= \iiint d\lambda\, d\mu_1\, d\mu_2\, \hat{\varphi}(\lambda)\hat{\psi}(\mu_1, \mu_2) E\left\{\sum_{k=1}^{n} \exp\left\{i\lambda x\left(\frac{k-1}{n}\right)\right\}\right. \\ &\qquad \left.\times \left[x\left(\frac{k}{n}\right) - x\left(\frac{k-1}{n}\right)\right]^4 \exp\{i[\mu_1 x(t_1) + \mu_2 x(t_2)]\}\right\} \\ &= \iiint d\lambda\, d\mu_1\, d\mu_2\, \hat{\varphi}(\lambda)\hat{\psi}(\mu_1, \mu_2) \\ &\qquad \times \sum_{k=1}^{n} \frac{\partial^4}{\partial \mu^4} E\left\{\exp\left\{i\left\{\lambda x\left(\frac{k-1}{n}\right) + \mu\left[x\left(\frac{k}{n}\right) - x\left(\frac{k-1}{n}\right)\right]\right.\right.\right. \\ &\qquad \left.\left.\left.+ \mu_1 x(t_1) + \mu_2 x(t_2)\right\}\right\}\right\}\Big|_{\mu=0} \end{aligned} \tag{3.5}$$

The expectation in the last expression equals

$$\exp\left\{-(\mu_1 + \mu_2 + \lambda)^4\left(\frac{k-1}{n}\right) - (\mu_1 + \mu_2 + \mu)^4\left(t_1 - \frac{k-1}{n}\right)\right. \\ \left. - (\mu_2 + \mu)^4\left(\frac{k}{n} - t_1\right) - \mu_2^4\left(t_2 - \frac{k}{n}\right)\right\},$$

whose fourth derivative with respect to μ evaluated at $\mu = 0$ is

$$\exp\left\{-(\mu_1 + \mu_2 + \lambda)^4\left(\frac{k-1}{n}\right) - (\mu_1 + \mu_2)^4\left(t_1 - \frac{k-1}{n}\right)\right. \\ \left. - \mu_2^4\left(\frac{k}{n} - t_1\right) - \mu_2^4\left(t_2 - \frac{k}{n}\right)\right\} \\ \times \left[-\frac{24}{n} + \text{ terms of order } \frac{1}{n^2} \text{ or better}\right].$$

When summed on k from 1 to n, this converges under polynomial domination to

$$-24 \int_0^1 \exp\{-(\mu_1 + \mu_2 + \lambda)^4 s - (\mu_1 + \mu_2)^4(t_1 - s) - \mu_2^4(t_2 - t_1)\}\, ds\,,$$

and, therefore, (3.5) converges to

$$
\begin{aligned}
&-24 \iiint d\lambda\, d\mu_1\, d\mu_2\, \hat{\varphi}(\lambda)\hat{\psi}(\mu_1, \mu_2) \\
&\qquad \times \int_0^1 \exp\{-(\mu_1 + \mu_2 + \lambda)^4 s - (\mu_1 + \mu_2)^4(t_1 - s) - \mu_2^4(t_2 - t_1)\}\, ds \\
&\qquad = -24 \iiint d\xi_1\, d\xi_2\, d\xi_3\, \hat{\varphi}(\xi_1 - \xi_2)\hat{\psi}(\xi_2 - \xi_3, \xi_3) \\
&\qquad\qquad \times \int_0^1 \exp\{-\xi_1^4 s - \xi_2^4(t_1 - s) - \xi_3^4(t_2 - t_1)\}\, ds \\
&\qquad = -24\, E\{\textstyle\int_0^1 \varphi(x(s))\, ds \cdot \psi(x(t_1), x(t_2))\}\,.
\end{aligned} \tag{3.6}
$$

Note that here, for the first time, we are able to "evaluate" the weak limit of the Riemann sums defining the integral, at least as another weak limit: from (3.5) and (3.6) we have

$$\int_0^1 \varphi(x(t))(dx)^4(t) = -24 \int_0^1 \varphi(x(t))\, dt\,, \tag{3.7}$$

or, in the notation of stochastic differentials,

$$(dx)^4(t) = -24\, dt\,. \tag{3.8}$$

This result is the analogue of the relationship $(dx)^2(t) = dt$ for the Brownian motion (Wiener) process.

Clearly, the integrals $I(\varphi) = \int_0^t \varphi(x(s))(dx)^j(s)$, $j = 1, 2, 3, 4$, have all the usual properties of Riemann integrals, e.g.,

$$I(\varphi_1 + \varphi_2) = I(\varphi_1) + I(\varphi_2) \tag{3.9}$$

and

$$I(k\varphi) = kI(\varphi) \tag{3.10}$$

for constant k.

3.2. *The Itô lemma.* We are now able to state an analogue of Itô's lemma [4, 5] for stochastic integrals.

THEOREM 3.1. *Let f be a function of class $C_\downarrow^\infty$ or an elementary function such as a polynomial or an exponential. Then,*

$$
\begin{aligned}
f(x(b)) - f(x(a)) &= \int_a^b f'(x(t))\, dx(t) + \tfrac{1}{2} \int_a^b f''(x(t))(dx)^2(t) \\
&\quad + \tfrac{1}{6} \int_a^b f'''(x(t))(dx)^3(t) - \int_a^b f''''(x(t))\, dt
\end{aligned} \tag{3.11}
$$

in the weak sense; i.e., *for any tame $\psi \in C_\downarrow^\infty$,*

$$
\begin{aligned}
&E\{[f(x(b)) - f(x(a))]\psi(x(t_1), x(t_2), \cdots, x(t_m))\} \\
&\qquad = \lim_{n\to\infty} E\{\textit{approximating Riemann sums of right-hand side} \\
&\qquad\qquad \times \psi(x(t_1), x(t_2), \cdots, x(t_m))\}\,.
\end{aligned} \tag{3.12}
$$

(3.13) NOTATION. $df = f'\, dx + \frac{1}{2}f''(dx)^2 + \frac{1}{6}f'''(dx)^3 - f''''\, dt$.

PROOF. Without loss of generality, we may assume that $a = 0$ and $b = 1$.

Letting $\Delta_k x = x(k/n) - x((k-1)/n)$ and using Taylor's theorem with remainder, we have

$$f(x(1)) - f(x(0)) = \sum_{k=1}^n \left[f\left(x\left(\frac{k}{n}\right)\right) - f\left(x\left(\frac{k-1}{n}\right)\right)\right]$$

$$= \sum_{k=1}^n \left\{ f'\left(x\left(\frac{k-1}{n}\right)\right)\Delta_k x + \tfrac{1}{2} f''\left(x\left(\frac{k-1}{n}\right)\right)(\Delta_k x)^2 \right.$$

$$+ \tfrac{1}{6} f'''\left(x\left(\frac{k-1}{n}\right)\right)(\Delta_k x)^3 + \tfrac{1}{24} f''''\left(x\left(\frac{k-1}{n}\right)\right)(\Delta_k x)^4$$

$$\left. + \tfrac{1}{24}\int_{x((k-1)/n)}^{x(k/n)} \left[x\left(\frac{k}{n}\right) - x\right]^4 f^{(5)}(x)\,dx \right\}.$$

For $f \in C_\downarrow^\infty$, we multiply by any $C_\downarrow^\infty$ function ψ of finitely many observations, take expectations, and let $n \to \infty$. Using (3.7) we get

$$E\{[f(x(1)) - f(x(0))]\psi(x(t_1), x(t_2), \cdots, x(t_m))\}$$
$$= E\{[\int_0^1 f'(x(t))\,dx(t) + \tfrac{1}{2}\int_0^1 f''(x(t))(dx)^2(t)$$
$$+ \tfrac{1}{6}\int_0^1 f'''(x(t))(dx)^3(t) - \int_0^1 f''''(x(t))\,dt]\psi(x(t_1), x(t_2), \cdots, x(t_m))\}$$
$$+ \lim_{n\to\infty} \tfrac{1}{24} E\left\{\sum_{k=1}^n \int_{x((k-1)/n)}^{x(k/n)} \left[x\left(\frac{k}{n}\right) - x\right]^4 f^{(5)}(x)\,dx \cdot \psi\right\},$$

and to complete the proof we must show that this last limit is zero. Since neither the number nor the relative positions of the observation times t_i affect the weak limit itself, we will assume, without loss of generality, that $\psi = \psi(x(t_1))$, $t_1 > 1$.

For $f \in C_\downarrow^\infty$ we may take Fourier transforms of f and its derivatives, so

$$E\left\{\sum_{k=1}^n \int_{x((k-1)/n)}^{x(k/n)} \left[x\left(\frac{k}{n}\right) - x\right]^4 f^{(5)}(x)\,dx \cdot \psi(x(t_1))\right\}$$
$$= \iint d\mu\, d\lambda [f^{(5)}]^\wedge(\lambda)\hat{\psi}(\mu)\frac{\partial^4}{\partial\zeta^4} E\{\sum_{k=1}^n \int_{x((k-1)/n)}^{x(k/n)} dx\, e^{i\lambda x} e^{i\zeta[x(k/n)-x]} e^{i\mu x(t_1)}\}\Big|_{\zeta=0}$$

Since the last expectation equals

$$-\sum_{k=1}^n i(\lambda - \zeta)^{-1}\Big[\exp\{-(\mu+\lambda)^4(k/n) - \mu^4(t_1 - k/n)\}$$
$$- \exp\left\{-(\mu+\lambda)^4((k-1)/n) - (\mu+\zeta)^4\frac{1}{n} - \mu^4\left(t_1 - \frac{k}{n}\right)\right\}\Big]$$
$$= -\sum_{k=1}^n i(\lambda-\zeta)^{-1}\exp\left\{-(\mu+\lambda)^4\left(\frac{k-1}{n}\right) - \mu^4\left(t_1 - \frac{k}{n}\right)\right\}$$
$$\times\left[\exp\left\{-(\mu+\lambda)^4\frac{1}{n}\right\} - \exp\left\{-(\mu+\zeta)^4\frac{1}{n}\right\}\right],$$

its fourth derivative with respect to ζ evaluated at $\zeta = 0$ equals

$$- \sum_{k=1}^n i(\lambda - \zeta)^{-1} \exp\left\{-(\mu + \lambda)^4 \left(\frac{k-1}{n}\right) - \mu^4\left(t_1 - \frac{k}{n}\right)\right\}$$
$$\times \left\{\frac{1}{n}\left[\frac{24}{\lambda^5}(\mu^4 - (\mu+\lambda)^4) + \frac{96\mu^3}{\lambda^4} + \frac{144\mu^2}{\lambda^3} + \frac{96\mu}{\lambda^2} + \frac{24}{\lambda}\right] e^{-\mu^4/n}\right.$$
$$\left. + \text{ terms of order } \frac{1}{n^2} \text{ or better}\right\},$$

which converges under a polynomial bound as $n \to \infty$ to zero. Thus, the entire remainder converges to zero as claimed.

For polynomial functions $f(x) = x^l$, where l is a positive integer, existence of the limits as n becomes infinite of expressions of the form

$$E\left\{\sum_{k=1}^n x^l\left(\frac{k-1}{n}\right)(\Delta_k x)^j \psi(x(t_1), \cdots, x(t_m))\right\} \quad (j = 1, 2, 3, 4;\ l = 0, 1, 2, \cdots)$$
$$= i^{-(j+l)} \int d\mu_1, \cdots, d\mu_m\, \hat{\psi}(\mu_1, \cdots, \mu_m) \frac{\partial^l}{\partial \lambda^l} \frac{\partial^j}{\partial \xi^j}$$
$$\times \sum_{k=1}^n E\left\{\exp\left\{i\lambda x\left(\frac{k-1}{n}\right)\right\} \exp\{i\xi(\Delta_k x)\} \exp\{i \sum \mu_i x(t_i)\}\right\}\Big|_{\lambda=0,\xi=0}$$

follows from the convergence under polynomial domination of

$$\sum_{k=1}^n \frac{\partial^l}{\partial \lambda^l} \frac{\partial^j}{\partial \xi^j} E\left\{\exp\left\{i\lambda x\left(\frac{k-1}{n}\right)\right\} \exp\{i\xi(\Delta_k x)\} \exp\{i \sum \mu_i x(t_i)\}\right\}\Big|_{\lambda=0,\xi=0},$$

analogous to the proof for $f \in C_\downarrow^\infty$. For $l > 5$ we must also show that the remainder term vanishes in the limit. Convergence of that term to zero follows exactly as it does for $f \in C_\downarrow^\infty$, with the exception that we write

$$f^{(5)}(x) = \frac{d^5}{dx^5} x^l = (-i)^{l-1} \frac{l!}{(l-5)!} \frac{\partial^{l-5}}{\partial \lambda^{l-5}} e^{i\lambda x}\Big|_{\lambda=0}$$

instead of

$$f^{(5)}(x) = \int e^{i\lambda x}[f^{(5)}]^\wedge(\lambda)\, d\lambda .$$

To carry out the proof for an exponential function f, say $f = e^{\alpha x}$, instead of introducing exponentials through the use of transforms or derivatives we leave f as is, and the only change from the previous cases is the calculation $E\{e^{\alpha x(t)}\} = e^{-\alpha^4 t}$. To see this, note that in (2.5) we showed that $p(t, x)$ behaves asymptotically like $|x|^{-\frac{1}{3}} \exp(-a|x|^{\frac{4}{3}})$, which decays more rapidly than $e^{\alpha x}$, thus assuring the analyticity of $\int e^{i\alpha x} p(t, x)\, dx$ in α and permitting an analytic continuation to imaginary values of α. Now existence of the limits

$$E\left\{\sum_{k=1}^n \exp\left\{\alpha x\left(\frac{k-1}{n}\right)\right\}(\Delta_k x)^j \psi(x(t_1), \cdots, x(t_m))\right\}$$
$$= (i)^{-j} \int \hat{\psi}(\mu_1, \cdots, \mu_m) \frac{\partial^j}{\partial \xi^j} \sum_{k=1}^n E\left\{\exp\left\{\alpha x\left(\frac{k-1}{n}\right)\right\} \exp\{i\xi(\Delta_k x)\}\right.$$
$$\left. \times \exp\{i \sum \mu_l x(t_l)\}\right\}\Big|_{\xi=0} d\mu_1 \cdots d\mu_m \qquad j = 1, 2, 3, 4$$

follows as before. To see that the remainder vanishes in the limit, note that

$$E\left\{ \sum_{k=1}^{n} \int_{x((k-1)/n)}^{x(k/n)} \left[x\left(\frac{k}{n}\right) - x \right]^4 \alpha^5 e^{\alpha x}\, dx \cdot \psi(x(t_1)) \right\}$$

$$= \sum_{k=1}^{n} \int d\mu\, \hat{\psi}(\mu) \frac{\partial^4}{\partial \alpha^4} E\left\{ \alpha^4 \left[\exp\left\{ \alpha x\left(\frac{k}{n}\right)\right\} - \exp\left\{\alpha x \left(\frac{k-1}{n}\right)\right\}\right] \exp\{i\mu x(t_1)\}\right\}$$

$$= \sum_{k=1}^{n} \int d\mu\, \hat{\psi}(\mu) \frac{\partial^4}{\partial \alpha^4} \left\{ \alpha^4 \left[\exp\left\{ -\frac{k}{n}(\alpha+\mu)^4 - \left(t_1 - \frac{k}{n}\right)\mu^4 \right\} \right.\right.$$

$$\left.\left. - \exp\left\{ -\frac{k-1}{n}(\alpha+\mu)^4 - \left(t_1 - \frac{k-1}{n}\right)\mu^4 \right\} \right] \right\}, \qquad t_1 \geqq 1$$

which converges under polynomial domination to zero. □

The basic feature of our stochastic calculus is the existence of four independent infinitesimals $(dx)^j(t)$, $j = 1, 2, 3, 4$, with $(dx)^4(t) = -24\, dt$. Letting $\Delta_k x = x(k/n) - x((k-1)/n)$, one gets the following related calculation:

$$\begin{aligned}
E\{\textstyle\sum_{k=1}^{n} (\Delta_k x)^j\} &= (-i)^j \textstyle\sum_{k=1}^{n} \dfrac{\partial^j}{\partial \mu^j} E\{\exp\{i\mu(\Delta_k x)\}\}\Big|_{\mu=0} \\
&= (-i)^j \textstyle\sum_{k=1}^{n} \dfrac{\partial^j}{\partial \mu^j} \exp\{-\mu^4/n\}\Big|_{\mu=0} \\
&= \textstyle\sum_{k=1}^{n} 0 \qquad \text{if } j \neq 4p, \quad p = 1, 2, 3, \cdots \\
&= \textstyle\sum_{k=1}^{n} -4!/n \qquad \text{if } j = 4 \qquad (3.14) \\
&= \textstyle\sum_{k=1}^{n} O\left(\dfrac{1}{n^2}\right) \qquad \text{if } j = 4p, \quad p = 2, 3, 4, \cdots \\
&\to 0 \qquad \text{if } j \neq 4 \text{ as } n \to \infty \\
&\to -4! \qquad \text{if } j = 4 \text{ as } n \to \infty .
\end{aligned}$$

The differentials dx, $(dx)^2$, and $(dx)^3$ have some interesting properties. For example,

$$E\{\exp\{i\alpha \textstyle\int_0^1 dx(t)\}\} = E\{\exp\{i\alpha x(1)\}\} = \exp\{-\alpha^4\} \tag{3.15}$$

while

$$E\{\exp\{i\beta \textstyle\int_0^1 (dx)^2(t)\}\} = \lim_{n\to\infty} E\{\exp\{i\beta \sum_{k=1}^{n} (\Delta_k x)^2\}\}, \tag{3.16}$$

which, by independence of the $\Delta_k x$, is

$$\begin{aligned}
\lim_{n\to\infty} [E\{\exp\{i\beta(\Delta_1 x)^2\}\}]^n &= \lim_{n\to\infty} [E\{\exp\{i\beta x^2(1)/n^{\frac{1}{2}}\}\}]^n \\
&= \lim_{n\to\infty} [E\{1 + i\beta x^2(1) n^{-\frac{1}{2}} - \beta^2 x^2(1)/2n + \cdots\}]^n \\
&= \lim_{n\to\infty} [1 + 12\beta^2/n]^n = \exp\{12\beta^2\},
\end{aligned}$$

and

$$\begin{aligned}
E\{\exp\{i\gamma \textstyle\int_0^1 (dx)^3(t)\}\} &= \lim_{n\to\infty} E\{\exp\{i\gamma \textstyle\sum_{k=1}^{n} (\Delta_k x)^3\}\} \\
&= \lim_{n\to\infty} [E\{\exp\{i\gamma(\Delta_1 x)^3\}\}]^n \qquad (3.17) \\
&= \lim_{n\to\infty} [E\{\exp\{i\gamma x^3(1)/n^{\frac{3}{4}}\}\}]^n = 1 .
\end{aligned}$$

Especially curious is the following:

$$
\begin{aligned}
(3.18)\qquad & E\{\exp\{i\alpha \textstyle\int_0^1 dx(t) + i\beta \int_0^1 (dx)^2(t) + i\gamma \int_0^1 (dx)^3(t)\}\} \\
&= \lim_{n\to\infty} E\{\exp\{i\alpha \textstyle\sum_{k=1}^n (\Delta_k x) + i\beta \sum_{k=1}^n (\Delta_k x)^2 + i\gamma \sum_{k=1}^n (\Delta_k x)^3\}\} \\
&= \lim_{n\to\infty} [E\{\exp\{i\alpha x(1)/n^{\frac14} + i\beta x^2(1)/n^{\frac12} + i\gamma x^3(1)/n^{\frac34}\}\}]^n \\
&= \lim_{n\to\infty}\left[E\left\{1 - \left(\alpha\gamma - \frac{\alpha^4}{24} + \frac{i\alpha^2\beta}{2} + \frac{\beta^2}{2}\right) x^4(1)/n\right\}\right]^n \\
&= \lim_{n\to\infty}\left[1 + 24\left(\alpha\gamma - \frac{\alpha^4}{24} + \frac{i\alpha^2\beta}{2} + \frac{\beta^2}{2}\right)\Big/ n\right]^n \\
&= \exp\left\{24\left(\alpha\gamma - \frac{\alpha^4}{24} + \frac{i\alpha^2\beta}{2} + \frac{\beta^2}{2}\right)\right\}.
\end{aligned}
$$

Note the presence of the γ in this last result, despite (3.17). Note as well that in (3.18), when $\alpha = 0$ the effect of the γ vanishes. Clearly, the behavior of these differentials is quite strange.

For a simple example of Itô's lemma, we apply (3.11) to the functions x^j, $j = 2, 3, 4$, and obtain, in differential notation,

$$(3.19)\qquad d(x^j) = jx^{j-1}\,dx + \tfrac{1}{2}j(j-1)x^{j-2}(dx)^2 + \cdots + (dx)^j,$$

which, combined with $\int dx = x$, yields the formula

$$(3.20)\qquad \int (dx)^j = x^j - \int [(x + dx)^j - x^j - (dx)^j], \qquad j = 1, 2, 3, 4,$$

or, equivalently,

$$(3.21)\qquad x^j = \int [(x + dx)^j - x^j]$$

for the stochastic integrals $\int_0^t (dx)^j(s)$, $j = 1, 2, 3, 4$.

For example,

$$d(x^4) = 4x^3\,dx + 6x^2(dx)^2 + 4x(dx)^3 + (dx)^4,$$

and

$$\int (dx)^3 = x^3 - 3\int x^2\,dx - 3\int x(dx)^2.$$

For $\psi \in C_\downarrow^\infty$ and φ a $C_\uparrow^\infty$ function or an elementary function such as a polynomial, we may alternatively expand $\varphi(x(t))$ and $\psi(x(t))$ in Taylor series. If $x(0) = 0$, we get

$$
\begin{aligned}
(3.22)\qquad E(\varphi\cdot\psi) = E\{&\varphi(0)\psi(0) - \varphi(0)\textstyle\int \psi'''' - \psi(0)\int \varphi'''' - 4\int \varphi'\psi''' \\
&- 4\textstyle\int \psi'\varphi''' - 6\int \varphi''\psi''\},
\end{aligned}
$$

where by φ we mean $\varphi(x(t))$ and $\int \varphi$ means $\int_0^t \varphi(x(s))\,ds$. Similarly, approximating $\int_0^t \varphi(x(s))(dx)^j(s)$ by its Riemann sum and expanding $\psi(x(t))$ in a Taylor series yields

$$
\begin{aligned}
(3.23)\qquad E\{\psi\cdot \textstyle\int \varphi(dx)^j\} &= -4E\{\textstyle\int \psi'''\varphi\}, & j &= 1 \\
(3.24)\qquad &= -12E\{\textstyle\int \psi''\varphi\}, & j &= 2 \\
(3.25)\qquad &= -24E\{\textstyle\int \psi'\varphi\}, & j &= 3 \\
(3.26)\qquad &= -24E\{\psi(0)\textstyle\int \varphi\}, & j &= 4.
\end{aligned}
$$

This procedure is equivalent to applying Itô's lemma directly. For example, from (3.22), (3.25), (3.23), and (3.24), one gets

$$\begin{aligned} E\{x^3 \cdot \psi\} &= E\{-24 \textstyle\int \psi' - 12 \int \psi''' x^2 - 36 \int \psi'' x\} \\ &= E\{\psi \cdot \textstyle\int (dx)^3 + 3\psi \cdot \int x^2\, dx + 3\psi \cdot \int x (dx)^2\}\,, \end{aligned}$$

which agrees with (3.19) for $j = 3$.

3.3. *Elementary functions.* Analogous to the appearance of the Hermite polynomials in the theory of the Brownian integral, let us introduce the iterated integrals

$$h_n(t) = n! \textstyle\int_0^t dx(t_1) \int_0^{t_1} dx(t_2) \cdots \int_0^{t_{n-1}} dx(t_n)\,, \qquad 0 \leqq t \leqq 1 \tag{3.27}$$

where, for example, $\int_0^1 dx(t_1) \int_0^{t_1} dx(t_2) \int_0^{t_2} dx(t_3)$ is defined as the weak limit as n becomes infinite of

$$\begin{aligned} &\textstyle\sum_{n \geqq i > j > k \geqq 1} \left[x\left(\frac{i}{n}\right) - x\left(\frac{i-1}{n}\right)\right] \\ &\qquad \times \left[x\left(\frac{j}{n}\right) - x\left(\frac{j-1}{n}\right)\right]\left[x\left(\frac{k}{n}\right) - x\left(\frac{k-1}{n}\right)\right]. \end{aligned}$$

The demonstration of the existence of this limit is just like that of the existence of the stochastic integrals $\int f(x(t))(dx)^j(t)$ and hence will not be repeated here.

Clearly, the h_n are the counterparts of the customary powers $[x(t)]^n$. Direct application of Itô's lemma to polynomial functions leads to explicit formulas for the h_n, the first few of which are:

$$\begin{aligned} h_0 &= 1 \\ h_1 &= \textstyle\int dx = x \\ h_2 &= -\textstyle\int (dx)^2 + \{\int dx\}^2 = -\int (dx)^2 + x^2 \\ h_3 &= 2 \textstyle\int (dx)^3 - 3x \int (dx)^2 + x^3 \\ h_4 &= 144t + 8x \textstyle\int (dx)^3 + 3\{\int (dx)^2\}^2 - 6x^2 \int (dx)^2 + x^4\,. \end{aligned} \tag{3.28}$$

For example, applying Itô's lemma to x^2 gives

$$h_2 = 2 \textstyle\int_0^t dx(t_1) \int_0^{t_1} dx(t_2) = 2 \int_0^t x(s)\, dx(s) = x^2(t) - \int_0^t (dx)^2(s)\,.$$

THEOREM 3.2. *The functions $h_n(t)$ form an orthogonal set in the sense that $E\{h_l(t) \cdot h_m(t)\} = 0$ for $l \neq m$. Also, $E\{h_n\} = 0$ for $n = 1, 2, \cdots$.*

EXAMPLE. Before proceeding with the proof, we illustrate the result for $E\{h_1 \cdot h_3\}$. We have

$$\begin{aligned} \text{(i)} \quad E\{x \cdot 2 \textstyle\int (dx)^3\} &= \lim_{n\to\infty} \textstyle\sum_{i,j=1}^n E\{\Delta_i x \cdot 2(\Delta_j x)^3\} \\ &= 2E\{\textstyle\int (dx)^4\} \quad (\text{from } i = j)\,, \\ \text{(ii)} \quad E\{x \cdot (-3) x \textstyle\int (dx)^2\} &= \lim_{n\to\infty} \textstyle\sum_{i,j,k=1}^n E\{\Delta_i x \cdot (-3)(\Delta_j x)(\Delta_k x)^2\} \\ &= -3E\{\textstyle\int (dx)^4\} \quad (\text{from } i = j = k) \\ &\quad -3E\{\textstyle\int (dx)^2 \int (dx)^2\} \quad (\text{from } i = j \neq k)\,, \end{aligned}$$

$$\text{(iii)} \qquad \begin{aligned} E\{x \cdot x^3\} &= \lim_{n\to\infty} \textstyle\sum_{i,j,k,l=1}^{n} E\{\Delta_i x \cdot (\Delta_j x)(\Delta_k x)(\Delta_l x)\} \\ &= E\{\textstyle\int (dx)^4\} \quad (\text{from } i = j = k = l) \\ &\quad + E\{[\textstyle\int (dx)^2]^2\} \quad (\text{from } i = j \neq k = l) \\ &\quad + E\{[\textstyle\int (dx)^2]^2\} \quad (\text{from } i = k \neq j = l) \\ &\quad + E\{[\textstyle\int (dx)^2]^2\} \quad (\text{from } i = l \neq j = k) . \end{aligned}$$

Thus,

$$\begin{aligned} E\{h_1 \cdot h_3\} &= E\{x \cdot [2 \textstyle\int (dx)^3 - 3x \textstyle\int (dx)^2 + x^3]\} \\ &= \text{(i)} + \text{(ii)} + \text{(iii)} = 0 . \end{aligned}$$

PROOF OF THEOREM 3.2. Without loss of generality, let $t = 1$. From (3.27),

$$\begin{aligned} h_l(1) &= l! \textstyle\int_0^1 dx(t_1) \int_0^{t_1} dx(t_2) \cdots \int_0^{t_{l-1}} dx(t_l) , \\ h_m(1) &= m! \textstyle\int_0^1 dx(t_1) \int_0^{t_1} dx(t_2) \cdots \int_0^{t_{m-1}} dx(t_m) . \end{aligned}$$

Each term of the approximating Riemann sum for h_l is a product of l nonoverlapping increments Δx. Similarly, each term of the Riemann approximation for h_m is a product of m such increments. Now take the product of the Riemann sums and consider its expectation. Since no two different increments overlap, and since each increment is independent of all previous increments, for the expectation of a product of terms of the Riemann sums to be nonzero it is necessary that no increment appear only once in that product. Clearly, for this to occur, each increment in the term from the Riemann sum for h_l must agree with an increment in the term from the sum for h_m, so l must equal m. Thus $E\{h_l \cdot h_m\} = 0$ for $l \neq m$. The fact that $E\{h_n\} = 0$ for any integer $n \geqq 1$ is now obvious as well. □

For a stochastic integral A, we define the exponential e^A or $\exp\{A\}$ as the formal power series $\sum (A^n/n!)$. From the definition it is clear that $e^A e^B = e^{A+B}$.

Now consider the function

$$\text{(3.29)} \qquad \begin{aligned} y(t) &= \exp\left\{\alpha \textstyle\int_0^t dx(s) - \frac{\alpha^2}{2} \int_0^t (dx)^2(s) + \frac{\alpha^3}{3} (dx)^3(s) - \frac{\alpha^4}{4} \int_0^t (dx)^4(s)\right\} \\ &= \exp\{\xi(t)\} = \sum \frac{[\xi(t)]^n}{n!} . \end{aligned}$$

A formal application of the Itô lemma leads to

$$\text{(3.30)} \qquad \begin{aligned} dy &= y\, d\xi + \tfrac{1}{2} y(d\xi)^2 + \tfrac{1}{6} y(d\xi)^3 + \tfrac{1}{24} y(d\xi)^4 \\ &= y \left\{\left[\alpha\, dx - \frac{\alpha^2}{2} (dx)^2 + \frac{\alpha^3}{3} (dx)^3 - \frac{\alpha^4}{4} (dx)^4\right]\right. \\ &\qquad + \tfrac{1}{2}[\alpha^2 (dx)^2 - \alpha^3 (dx)^3 + \tfrac{11}{12} \alpha^4 (dx)^4] \\ &\qquad \left. + \tfrac{1}{6}[\alpha^3 (dx)^3 - \tfrac{3}{2} \alpha^4 (dx)^4] + \tfrac{1}{24} \alpha^4 (dx)^4 \right\} \\ &= \alpha y\, dx ; \end{aligned}$$

also, $y(0) = 1$, showing that y is an analogue of the customary exponential

$e^{\alpha x(t)}$. As is the case with the Brownian exponential $\exp\{\alpha x(t) - \alpha^2 t/2\}$, y is a martingale, since for any tame $\psi \in C_\downarrow^\infty$ and $t_n < t$, $E\{y(t)\psi(x(t_1), \cdots, x(t_n))\} = E\{y(t_n)\psi(x(t_1), \cdots, x(t_n))\}$.

Note that if we consider the differential equation

$$dy = \alpha y \, dx \,, \qquad y_0 = y(0) = 1 \,,$$

and formally iterate toward a solution, we obtain

$$y_n(t) = y_{n-1} + \alpha^n \int_0^t dx(t_1) \int_0^{t_1} dx(t_2) \cdots \int_0^{t_{n-1}} dx(t_n) \,;$$

thus it appears that h_n should agree with the nth derivative of y with respect to α evaluated at $\alpha = 0$. But the solution y is the analogue of the customary exponential $\exp\{\alpha x(t)\}$ given in (3.29). Equating the formal power series (3.29) in α with the formal power series $\sum_n (\alpha^n/n!)h_n$, we get

$$\begin{aligned}\sum \frac{\alpha^n h_n}{n!} &= 1 + \alpha \int dx - \frac{\alpha^2}{2} \int (dx)^2 + \frac{\alpha^3}{3} \int (dx)^3 - \frac{\alpha^4}{4} \int (dx)^4 \\ &\quad + \tfrac{1}{2}\left[\alpha^2\{\textstyle\int dx\}^2 - \alpha^3 \int dx \int (dx)^2 + \dfrac{\alpha^4}{4} \{\textstyle\int (dx)^2\}^2 \right. \\ &\quad \left. + \frac{2\alpha^4}{3} \textstyle\int dx \int (dx)^3 \right] + \tfrac{1}{6}[\alpha^3\{\textstyle\int dx\}^3 - \tfrac{3}{2}\alpha^4\{\int dx\}^2 \int (dx)^2] \\ &\quad + \tfrac{1}{24}\alpha^4\{\textstyle\int dx\}^4 + \text{terms involving} \quad \alpha^k \,, \quad k = 5, 6, \cdots .\end{aligned}$$

Comparing coefficients of corresponding powers of α leads to a reproduction of the explicit formulas (3.28) for the functions h_n.

4. Some special distributions; continuity of the sample paths.

4.1. *The maximum function.* We now study the distribution of the maximum function for $x(t)$, which is defined as

$$\sigma(\alpha, T) = \lim_{n\to\infty} P\{\max_{0 \le t = i/n \le T} x(t) < \alpha\} \,,$$

where $x(0) = 0$. It is assumed that the limit exists. Clearly, replacing $x(t)$ by $x^+(t) \equiv \max\,[x(t), 0]$ makes no difference, since the process starts at zero. For fixed n, we let

$$\sigma_n(\alpha, T) = P\{\max_{0 \le t = i/n \le T} x^+(t) < \alpha\}$$

and

$$\varphi_n(\lambda, T) = \int_0^\infty e^{-\lambda\alpha} \, d_\alpha \sigma_n(\alpha, T) \,.$$

Following the technique of Baxter and Donsker [1], we now apply an identity of Spitzer [14] to the double Laplace transform of $\sigma_n(\alpha, T)$. Note that Spitzer's identity is applicable here, for it is a purely combinatorial result which applies to any symmetric distribution, positive or not, as long as the total measure is one. If $\{x_k\}$ is a sequence of independent, identically distributed quantities with partial sums $s_k = x_1 + x_2 + \cdots + x_k$, and if

$$\varphi_n(\lambda) = \int_0^\infty e^{-\lambda\alpha} \, dP\{\max_{k \le n} s_k^+ < \alpha\}$$
$$\psi_n(\lambda) = \int_0^\infty e^{-\lambda\alpha} \, dP\{s_n^+ < \alpha\} \,,$$

then Spitzer's identity states that

$$\sum_{n=0}^{\infty} t^n \varphi_n(\lambda) = \exp\left\{\sum_{n=1}^{\infty} \frac{t^n}{n} \psi_n(\lambda)\right\}. \tag{4.1}$$

Multiplying both sides by $\exp\{-\sum_{n=1}^{\infty} t^n/n\} = 1 - t$ we get

$$(1-t)\sum_{n=0}^{\infty} t^n \varphi_n(\lambda) = \exp\left\{-\sum_{n=1}^{\infty} \frac{t^n}{n}(1-\psi_n(\lambda))\right\}; \tag{4.2}$$

i.e.,

$$\begin{aligned}(1-t)\sum_{n=0}^{\infty} t^n E\{\exp\{-\lambda \max_{k\le n} s_k{}^+\}\}\\ = \exp\left\{-\sum_{n=1}^{\infty} \frac{t^n}{n}(1 - E(\exp\{-\lambda s_n{}^+\}))\right\}.\end{aligned} \tag{4.3}$$

Applying this formula with

$$x_k = x(k/n) - x((k-1)/n), \qquad k = 1, 2, \cdots,$$
$$t = \exp(-u/n),$$

and

$$\psi(\lambda, T) = \int_0^\infty e^{-\lambda\alpha}\, d_\alpha P\{x^+(T) < \alpha\},$$

we have, for $u > 0$,

$$\begin{aligned}\lim_{n\to\infty} u \int_0^\infty \int_0^\infty e^{-uT-\lambda\alpha}\, d_\alpha \sigma_n(\alpha, T)\, dT\\ = \lim_{n\to\infty} u \int_0^\infty e^{-uT}\varphi_n(\lambda, T)\, dT\\ = \lim_{n\to\infty} \{1 - e^{u/n}\} \sum_{k=0}^{\infty} \varphi_n(\lambda, k/n) e^{-uk/n}\\ = \exp\left\{\lim_{n\to\infty} \sum_{k=1}^{\infty} \frac{(\psi(\lambda, k/n) - 1)}{k} \exp\{-uk/n\}\right\}\\ = \exp\{\textstyle\int_u^\infty \int_0^\infty e^{-sT}(\psi(\lambda, T) - 1)\, dT\, ds\}\\ = \exp\left\{\frac{1}{2\pi}\int_u^\infty \int_{-\infty}^{\infty} \frac{\lambda}{\xi(\xi - i\lambda)} \frac{(-\xi^4)}{s(s+\xi^4)}\, d\xi\, ds\right\}.\end{aligned} \tag{4.4}$$

(For details of the last equality, see Baxter and Donsker [1], pages 78–79.)

Using the technique of contour integration with contour bounded by the real axis from $-n$ to n and the lower half of the semicircle $|z| = n$, a simple residue calculation yields

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\lambda}{\xi(\xi - i\lambda)} \frac{(-\xi)^4}{s(s+\xi^4)}\, d\xi = \sum_{k=1}^{2} \frac{i\lambda}{z_k(s)[z_k(s) - i\lambda]} \frac{dz_k(s)}{ds},$$

where

$$z_1(s) = \exp\left\{-\frac{\pi i}{4}\right\} s^{\frac{1}{4}} \quad \text{and} \quad z_2(s) = \exp\left\{-\frac{3\pi i}{4}\right\} s^{\frac{1}{4}}.$$

Integration with respect to s yields

$$\frac{1}{2\pi}\int_u^\infty \int_{-\infty}^{\infty} \frac{\lambda}{\xi(\xi - i\lambda)} \frac{(-\xi^4)}{s(s+\xi^4)}\, d\xi\, ds = -\log\left\{\left\{1 - \frac{i\lambda}{z_1(u)}\right\}\left\{1 - \frac{i\lambda}{z_2(u)}\right\}\right\},$$

and hence

$$(4.5)\quad \begin{aligned}
&\int_0^\infty \int_0^\infty \exp\{-uT - \lambda\alpha\}\, d_\alpha \sigma(\alpha, T)\, dT \\
&\quad = \frac{z_1(u) z_2(u)}{u[z_1(u) - i\lambda][z_2(u) - i\lambda]} \\
&\quad = u^{-\frac{1}{2}}(u^{\frac{1}{2}} + 2^{\frac{1}{2}}\lambda u^{\frac{1}{4}} + \lambda^2)^{-1} \\
&\quad = u^{-\frac{1}{2}}\left[\lambda + \left\{\frac{u^{\frac{1}{4}}}{2^{\frac{1}{2}}} + \frac{iu^{\frac{1}{4}}}{2^{\frac{1}{2}}}\right\}\right]^{-1}\left[\lambda + \left\{\frac{u^{\frac{1}{4}}}{2^{\frac{1}{2}}} - \frac{iu^{\frac{1}{4}}}{2^{\frac{1}{2}}}\right\}\right]^{-1}
\end{aligned}$$

Upon inversion of the Laplace transform with respect to α we get

$$(4.6)\quad \int_0^\infty e^{-uT} \frac{\partial}{\partial\alpha} P\{\max_{s \leq T} x(s) < \alpha\}\, dT = 2^{\frac{1}{2}} u^{-\frac{3}{4}} \exp\{-u^{\frac{1}{4}}\alpha/2^{\frac{1}{2}}\} \sin(u^{\frac{1}{4}}\alpha/2^{\frac{1}{2}}),$$

assuming the existence of $\sigma = \lim \sigma_n$ and the interchange of this limit and the Laplace transform.

Since we do not know that $x(t)$ distributed by P is almost surely continuous, we cannot assert that the first passage time of $x(t)$ to any point α will be finite. If we could, however, we would be able to apply the reflection principle of D. André, as is done in the case of the Wiener process, to obtain

$$\begin{aligned}
P\{\max_{0 \leq s \leq t} x(s) < \alpha\} &= 1 - 2P\{x(t) \geqq \alpha\} \\
&= \frac{1}{\pi} \int_0^\alpha \int_{-\infty}^\infty \exp\{ix\xi - \xi^4 t\}\, d\xi\, dx .
\end{aligned}$$

Then the density of the distribution of the maximum would be

$$\frac{\partial}{\partial\alpha}\left\{\frac{1}{\pi}\int_0^\alpha \int_{-\infty}^\infty \exp\{ix\xi - \xi^4 t\}\, d\xi\, dx\right\} = \frac{1}{\pi}\int_{-\infty}^\infty \exp\{i\alpha\xi - \xi^4 t\}\, d\xi ,$$

whose Laplace transform in t is

$$(4.7)\quad \begin{aligned}
&\int_0^\infty e^{-uT}\left\{\frac{1}{\pi}\int_{-\infty}^\infty \exp\{i\alpha\xi - \xi^4 t\}\, d\xi\right\} dT \qquad u > 0 \\
&\quad = \frac{1}{\pi}\int_{-\infty}^\infty \exp\{i\alpha\xi\} \frac{1}{u + \xi^4}\, d\xi \\
&\quad = \frac{\exp\{-\alpha u^{\frac{1}{4}}/2^{\frac{1}{2}}\}}{u^{\frac{3}{4}} 2^{\frac{1}{2}}}\left\{\cos\frac{\alpha u^{\frac{1}{4}}}{2^{\frac{1}{2}}} + \sin\frac{\alpha u^{\frac{1}{4}}}{2^{\frac{1}{2}}}\right\} .
\end{aligned}$$

This result is very close to, though not identical with, the actual transform in t of the density of the maximum function, which was shown in (4.6) to be

$$\frac{2^{\frac{1}{2}} \exp\{-\alpha u^{\frac{1}{4}}/2^{\frac{1}{2}}\}}{u^{\frac{3}{4}}} \sin\frac{u^{\frac{1}{4}}\alpha}{2^{\frac{1}{2}}} .$$

Hence, it seems reasonable to conclude that the sample paths of $x(t)$ are not continuous as Brownian paths are, but are perhaps not terribly discontinuous either. We will return to this matter shortly.

4.2. *Feynmann-Kac formula.* Note that if we let $T_t f = E_x\{f(x(t))\}$, then T_t is a semigroup operator, whose infinitesimal generator is $A = -\partial^4/\partial x^4$. This is the setup required for the next theorem, whose proof will not be repeated here; see, for example, [6] and [8].

THEOREM 4.1 (Feynmann-Kac formula). *Let $V(x)$ be a bounded piecewise continuous function, and let $f(x) \in C^4$. Then the solution to*

$$\frac{\partial u}{\partial t} = -\frac{\partial^4 u}{\partial x^4} - V(x)u \tag{4.8}$$

$$u(0, x) = f(x)$$

is given by

$$u(t, x) = E_x\{\exp\{-\textstyle\int_0^t V(x(\sigma))\, d\sigma\} f(x(t))\}\,, \tag{4.9}$$

where the expectation is computed by replacing $\int_0^t V\, d\sigma$ by a Riemann sum and passing to the limit outside, the existence of the limit being part of the assertion.

We now use the Feynmann-Kac formula to verify equation (4.6) for the Laplace transform of the density of the distribution $\sigma(\alpha, T)$ of the maximum function. Letting

$$\begin{aligned} V(x) &= 1\,, \quad x > \alpha \\ &= 0\,, \quad x \leqq \alpha \end{aligned}$$

we should have

$$\begin{aligned} \sigma(\alpha, T) &= \lim_{n\to\infty} P\{\max_{0\leqq t=i/n\leqq T} x(t) < \alpha\} \\ &= \lim_{\lambda\to\infty} E\{\exp\{-\lambda \textstyle\int_0^T V(x(s))\, ds\}\}\,. \end{aligned}$$

Let

$$\psi_\lambda(s, x) = \int_0^\infty u_\lambda(T, x) e^{-sT}\, dT$$

where

$$u_\lambda(T, x) = E_x\{\exp\{-\lambda \textstyle\int_0^T V(x(s))\, ds\}\}\,.$$

The latter is the solution of $\partial u/\partial t = -\partial^4 u/\partial x^4 - \lambda V(x)u$ with initial function $f(x) = 1$, so $\psi = \psi_\lambda$ satisfies

$$\begin{aligned} &\psi'''' + (\lambda + s)\psi = 1\,, && x > \alpha \\ &\psi'''' + s\psi = 1\,, && x \leqq \alpha \\ &\psi^{(n)} \text{ continuous at } \alpha\,, && n = 0, 1, 2, 3 \\ &\psi \text{ bounded as } x \to \pm\infty\,. \end{aligned} \tag{4.10}$$

Thus, ψ_λ is of the form

$$\begin{aligned} \psi_\lambda(s, x) &= (\lambda + s)^{-1} + a\exp\{(\lambda + s)^{\frac14}\omega_3 x\} \\ &\qquad + b\exp\{(\lambda + s)^{\frac14}\omega_4 x\}\,, \qquad x > \alpha \\ &= s^{-1} + c\exp\{s^{\frac14}\omega_1 x\} + d\exp\{s^{\frac14}\omega_2 x\}\,, \quad x \leqq \alpha\,, \end{aligned} \tag{4.11}$$

where $\omega_1 = e^{\pi i/4}$, $\omega_2 = e^{-\pi i/4}$, $\omega_3 = e^{-3\pi i/4}$, and $\omega_4 = e^{3\pi i/4}$. Solving for a, b, c and d, evaluating $\psi(s, 0)$ for $\alpha > 0$, and letting $\lambda \to \infty$, we get, after some computation,

$$\lim_{\lambda\to\infty} \psi_\lambda(s, 0) = \frac{1}{s}\left[1 - \exp\left\{-\frac{s^{\frac14}\alpha}{2^{\frac12}}\right\}\left\{\sin\frac{s^{\frac14}\alpha}{2^{\frac12}} + \cos\frac{s^{\frac14}\alpha}{2^{\frac12}}\right\}\right],$$

which is the Laplace transform of $\sigma(\alpha, T)$ in T. Differentiating this with respect to α, we get the transform of the density of the maximum distribution:

$$\int_0^\infty e^{-sT} \frac{\partial}{\partial\alpha} P\{\max_{t\leq T} x(t) < \alpha\}\, dT = 2^{\frac{1}{2}} s^{-\frac{3}{4}} \exp\left\{-\frac{s^{\frac{1}{4}}\alpha}{2^{\frac{1}{2}}}\right\} \sin\frac{s^{\frac{1}{4}}\alpha}{2^{\frac{1}{2}}} .$$

This agrees with result (4.6) obtained earlier using Spitzer's identity.

4.3. *The arc-sine law.* The arc-sine law for Brownian motion particles says that the measure of the set of continuous paths starting at the origin and staying on the positive half-line for a proportion of time less than or equal to α is given by

$$(4.12)\qquad P\{\tfrac{1}{2}\int_0^1 [1 + \operatorname{sgn} x(t)]\, dt \leqq \alpha\} = 0 , \qquad \alpha \leqq 0$$
$$= \frac{2}{\pi} \arcsin \alpha^{\frac{1}{2}} , \quad 0 \leqq \alpha \leqq 1 .$$

Spitzer [14] proved the arc-sine law as a consequence of his identity (4.1). We have already pointed out that Spitzer's identity is a purely combinatorial result applying to any symmetric distribution whose total measure is one, so that derivation of the arc-sine law applies here as well.

An alternative method is to apply the Feynmann–Kac formula with $V(x) = \lambda((1 + \operatorname{sgn} x)/2)$ for $\lambda > 0$ and $f(x) = 1$, similar to what was done by Krylov [8]. Then, the solution $u(t, x)$ in (4.9) evaluated at $x = 0$ is

$$E\left\{\exp\left\{-\frac{\lambda}{2} \int_0^t [1 + \operatorname{sgn} x(\sigma)]\, d\sigma\right\}\right\} = \int_0^\infty e^{-\lambda\alpha}\, d_\alpha P\{\tfrac{1}{2}\int_0^t [1 + \operatorname{sgn} x(\sigma)]\, d\sigma \leqq \alpha\} .$$

With this V, the Laplace transform $\psi(s, x) = \int_0^\infty u(t, x)e^{-st}\, dt$ of the solution $u(t, x)$ in (4.9) satisfies the ordinary differential equation (4.10) with $\alpha = 0$, whose solution is of the form (4.11). Solving for the constants c and d we find that

$$\psi(s, 0) = s^{-1}\left\{1 + \frac{\lambda}{\lambda + s}\, \frac{-2i[(\lambda + s)^{\frac{1}{4}} + s^{\frac{1}{4}}]^2(\lambda + s)^{\frac{3}{4}} s^{\frac{1}{4}}}{2i[(\lambda + s)^{\frac{1}{4}} + s^{\frac{1}{4}}]^2[(\lambda + s)^{\frac{1}{2}} + s^{\frac{1}{2}}]s^{\frac{1}{4}}(\lambda + s)^{\frac{1}{4}}}\right\}$$
$$= (s(\lambda + s))^{-\frac{1}{2}} .$$

The desired result (4.12) now follows upon inversion of the Laplace transforms and evaluating at $t = 1$.

4.4. *Distribution of the eigenvalues.* Following the lead of Kac [7], Rosenblatt [13], and Ray [12], who used probabilistic techniques to prove Weyl's classical result on the distribution of the eigenvalues λ_j of the Laplace operator Δ, we would like to prove the generalization of this result for higher-order operators using properties of the related processes. For the heat equation, the argument goes essentially as follows. A Brownian particle starting at time zero at point x in an N-dimensional bounded region Ω will, as $t \to 0$, have had no time to "feel" the boundary by time t. Thus, the probability of starting at $x_0 \in \Omega$ and returning at time t to $x \in \Omega$, is, as $t \to 0$, well approximated by the unrestricted fundamental solution of the heat equation, from which one easily gets the desired result via the Hardy–Littlewood Tauberian theorem. The proof is inherently based upon the almost sure continuity of the Brownian paths.

We now show that, at least heuristically, one can use the same approach to obtain, correctly, the asymptotic behavior of the eigenvalues of the operator $L = -\Delta^2$. From the Hölder continuity condition (2.12), as well as from the transform (4.6) of the distribution of the maximum function, it appears that the absorbing barrier problem in one dimension should be described by the boundary value problem for $u_t = -u_{xxxx}$ with $u = u' = 0$ at the endpoints of the one-dimensional interval. We now generalize to a bounded N-dimensional region with sufficiently regular boundary Γ, and let λ_j and μ_j be the discrete eigenvalues and normalized eigenfunctions corresponding to the eigenvalue problem $Lu + \lambda u = 0$ in Ω with $u = \partial u/\partial n = 0$ on Γ. Then,

$$p_1(t, x, y) = \sum_j \exp\{-\lambda_j t\} u_j(x) u_j(y), \qquad x, y \in \Omega,\ t > 0,$$

is the fundamental solution to

$$\begin{aligned} u_t &= Lu \quad \text{in} \quad \Omega \\ u &= \partial u/\partial n = 0 \quad \text{on} \quad \Gamma \\ u(0, x) &= \delta(x - y). \end{aligned}$$

If we now imitate the Brownian argument and approximate $p_1(t, x, y)$ by $p(t, x, y) = p(t, y - x)$, where $p(t, x)$ is the fundamental solution to the unrestricted problem, we get

$$\begin{aligned} \sum_j \exp\{-\lambda_j t\} u_j^2(x) &\sim [(2\pi)^{-N} \int_{R^N} \exp\{-|\xi|^4 t\}\, d^N\xi] \\ &= \frac{1}{2^{N+1}\pi^{N/2} t^{N/4}} \frac{\Gamma(N/4)}{\Gamma(N/2)}, \qquad t \to 0. \end{aligned}$$

Integrating over Ω yields, as $t \to 0$,

$$\sum_j \exp\{-\lambda_j t\} = \int_0^\infty e^{-\lambda t}\, d\{\sum_{\lambda_j < \lambda} 1\} \sim \frac{|\Omega|}{2^{N+1}\pi^{N/2} t^{N/4}} \frac{\Gamma(N/4)}{\Gamma(N/2)}$$

where $|\Omega|$ denotes the measure of the region Ω, from which, via the Hardy–Littlewood Tauberian theorem, we obtain

$$N(\lambda) = \sum_{\lambda_j < \lambda} 1 \sim \frac{|\Omega|}{2^{N-1}\pi^{N/2} N \Gamma(N/2)} \lambda^{N/4}, \qquad \lambda \to \infty. \tag{4.13}$$

This result can be obtained using the classical methods of operator theory; in particular, for the 1-dimensional case $\Omega = \{x : 0 \leqq x \leqq L\}$ with $u(0) = u(L) = u'(0) = u'(L) = 0$, the eigenvalues λ_j satisfy the equation $1 = \cos \lambda_j^{\frac{1}{4}} L \cdot \cosh \lambda_j^{\frac{1}{4}} L$, and thus $\lambda_j \sim [(2j + 1)\pi/2L]^4$ as λ_j becomes infinite, so

$$\begin{aligned} N(\lambda) = \sum_{\lambda_j < \lambda} 1 &\sim \text{number of integers that are less than or equal to } \frac{L}{\pi} \lambda^{\frac{1}{4}} - \tfrac{1}{2} \\ &\sim \frac{L}{\pi} \lambda^{\frac{1}{4}} \end{aligned}$$

which agrees with (4.13) for $N = 1$.

Earlier, when discussing the distribution of the maximum function, we argued that the sample paths were not a.s. continuous as Brownian paths are. What we might conjecture from the results of this action is that the paths are "locally continuous with a high 'probability'" in the sense that we may invoke the probabilistic principle that an interior particle will not "feel the boundary" as $t \to 0$.

5. Central limit theorem. In this section we prove two versions of a central limit theorem for independent random variables, identically distributed by a signed measure, whose first five moments exist as absolutely convergent integrals and which are normalized to have first three moments equal to zero and fourth moment equal to $(-4!)$.

THEOREM 5.1 (central limit theorem). *Let x_i, $i = 1, 2, \cdots$, be independent random variables, identically distributed by a signed measure μ, such that*

$$\begin{aligned} &\textstyle\int_{-\infty}^{\infty} d\mu = 1\,, \qquad \int_{-\infty}^{\infty} x^j\, d\mu = 0 \quad \text{if } j = 1, 2, 3 \\ &\qquad\qquad\qquad\qquad\qquad\;\; = -4! \quad \text{if } j = 4\,, \\ &\textstyle\int_{-\infty}^{\infty} |x|^5 |d\mu| < \infty\,. \end{aligned} \tag{5.1}$$

Let $s_n = x_1 + x_2 + \cdots x_n$. Then, for any function $\varphi \in C_{\downarrow}^{\infty}$ whose Fourier transform $\hat{\varphi}$ has compact support, we have

$$\lim_{n\to\infty} E\left\{\varphi\left(\frac{s_n}{n^{\frac{1}{4}}}\right)\right\} = \int_{-\infty}^{\infty} \varphi(x) p(1, x)\, dx\,. \tag{5.2}$$

PROOF. Let $x_1 = x$. Then

$$E\{\exp\{i\xi s_n/n^{\frac{1}{4}}\}\} = [E\{\exp\{i\xi x/n^{\frac{1}{4}}\}\}]^n\,.$$

Upon expanding $\exp\{i\xi x/n^{\frac{1}{4}}\}$ about $x = 0$ and taking expectations, we get

$$E[\exp\{i\xi s_n/n^{\frac{1}{4}}\}] = \{E[\exp\{i\xi x/n^{\frac{1}{4}}\}]\}^n = \left[1 - \frac{\xi^4}{n} + O\left\{\frac{|\xi|^5}{n^{\frac{5}{4}}}\right\}\right]^n .$$

For $\varphi \in C_{\downarrow}^{\infty}$ whose Fourier transform $\hat{\varphi}$ has compact support,

$$\begin{aligned} E\left\{\varphi\left(\frac{s_n}{n^{\frac{1}{4}}}\right)\right\} &= \int_{-\infty}^{\infty} \hat{\varphi}(\xi) E\{\exp\{i\xi s_n/n^{\frac{1}{4}}\}\}\, d\xi \\ &= \int_{-\infty}^{\infty} \hat{\varphi}(\xi)\left[1 - \frac{\xi^4}{n} + O\left\{\frac{|\xi|^5}{n^{\frac{5}{4}}}\right\}\right]^n d\xi \\ &\to \int_{-\infty}^{\infty} \hat{\varphi}(\xi) \exp\{-\xi^4\}\, d\xi \\ &= \int_{-\infty}^{\infty} \varphi(x) p(1, x)\, dx\,. \end{aligned} \tag{5.3}$$

□

REMARK 5.1. The eligible functions φ for this central limit theorem are entire functions of finite exponential type, i.e., functions φ such that $|\varphi(x)| \leqq$ constant $\times\, e^{T|x|}$ for some finite T and any complex x.

THEOREM 5.2 (central limit theorem). *Let x_i $(i = 1, 2, \cdots)$, s_n, and μ be as in Theorem 5.1, and let the signed measure μ have a density $f(x)$ with Fourier transform*

$\hat{f}$ such that $|\hat{f}| \leqq (2\pi)^{-1}$. Then, for any $\varphi \in C_{\downarrow}^{\infty}$, we have

$$\lim_{n\to\infty} E\left\{\varphi\left(\frac{s_n}{n^{\frac{1}{4}}}\right)\right\} = \int_{-\infty}^{\infty} \varphi(x)p(1, x)\,dx\,. \tag{5.2}$$

PROOF. Since $|\hat{f}| \leqq (2\pi)^{-1}$,

$$\begin{aligned}|E\{\exp\{i\xi x/n^{\frac{1}{4}}\}\}| &= |\textstyle\int_{-\infty}^{\infty} \exp\{i\xi x/n^{\frac{1}{4}}\}f(x)\,dx| \\ &= 2\pi|\hat{f}(\xi n^{-\frac{1}{4}})| \leqq 1\,,\end{aligned}$$

so

$$\begin{aligned}|E\{\exp\{i\xi s_n/n^{\frac{1}{4}}\}\}| &= |E\{\exp\{i\xi x/n^{\frac{1}{4}}\}\}|^n \\ &= |1 - \xi^4/n + O(|\xi|^5/n^{\frac{5}{4}})|^n \leqq 1\end{aligned}$$

for all ξ, thus assuring the convergence

$$\int_{-\infty}^{\infty} \hat{\varphi}(\xi)E\{\exp\{i\xi s_n/n^{\frac{1}{4}}\}\}\,d\xi \to \int_{-\infty}^{\infty} \hat{\varphi}(\xi)\exp\{-\xi^4\}\,d\xi\,. \qquad \square$$

REMARK 5.2. There are many such functions; for example,

$$f(x) = (k/\pi)^{\frac{1}{2}}[(-12k^2 + \tfrac{15}{8}) + (48k^3 - 5k/2)x^2 + (-16k^4 + k^2/2)x^4]\exp\{-kx^2\}$$

satisfies conditions (5.1) where k is any positive constant, and its Fourier transform

$$\hat{f}(\xi) = (2\pi)^{-1}[1 + \xi^2/4k + (\tfrac{1}{32}k^2 - 1)\xi^4]\exp\{-\xi^2/4k\}$$

has modulus at most $(2\pi)^{-1}$ as long as k lies between zero and $\frac{1}{4}\cdot 2^{\frac{1}{2}}$.

6. The general even-order equation. We now present the analogues of formulas contained in Sections 2 through 5, generalized to the process related to the partial differential equation

$$\frac{\partial u}{\partial t} = (-1)^{n+1}\frac{\partial^{2n}u}{\partial x^{2n}}\,, \qquad -\infty < x < \infty,\ 0 \leqq t < \infty\,. \tag{2.1$'$}$$

Their demonstrations are, of course, similar to those already presented. The number to the left of each formula is the same (modulo a "prime" symbol) as the number of the corresponding formula found in the previous sections.

$$p(t, x) = [\exp\{-\xi^{2n}t\}]^{\wedge} = (2\pi)^{-1}\int_{-\infty}^{\infty} e^{ix\xi}\exp\{-\xi^{2n}t\}\,d\xi\,. \tag{2.3$'$}$$

$$p(t, x) = p(1, t^{-1/2n}x)t^{-1/2n}\,. \tag{2.4$'$}$$

$$\begin{aligned}p(1, x) = kx^{(1-n)/(2n-1)}&\exp\{-ax^{2n/(2n-1)}\}\cos(bx^{2n/(2n-1)}) \\ &+ \text{lower order terms.}\end{aligned} \tag{2.5$'$}$$

$$\lambda_k = c(n)k^{1-1/2n} + O\{k^{-1/2n}\} \tag{2.6$'$}$$

where $c(n)$ is a constant depending on n.

$$\int x^{2n}p(t, x)\,dx = (-1)^n\frac{d^{2n}}{d\xi^{2n}}\exp\{-\xi^{2n}t\}\bigg|_{\xi=0} = (-1)^{n+1}(2n)!\,t\,. \tag{2.7$'$}$$

$$\int x^j p(t, x)\,dx = 0\,, \qquad j = 1, 2, \cdots, 2n - 1\,. \tag{2.8$'$}$$

In Proposition 2.2, (i), (ii), and (iii) remain unchanged, while (iv) and (v) become

(iv)′ $$t^{1/n}x(1/t)\,,$$

(v)′ $$cx(t/c^{2n})\,.$$

(3.7)′ $$\int_0^1 \varphi(x(t))(dx)^{2n}(t) = (-1)^{n+1}(2n)!\int_0^1 \varphi(x(t))\,dt\,.$$

(3.8)′ $$(dx)^{2n}(t) = (-1)^{n+1}(2n)!\,dt\,.$$

(3.11)′ $$f(x(b)) - f(x(a)) = \sum_{k=1}^{2n-1}\frac{1}{k!}\int_a^b f^{(k)}(x(t))(dx)^k(t) + (-1)^{n+1}\int_a^b f^{(2n)}(x(t))\,dt\,.$$

(3.13)′ $$df = \sum_{k=1}^{2n-1}\frac{1}{k!}f^{(k)}(dx)^k + (-1)^{n+1}f^{(2n)}\,dt\,.$$

(3.14)′ $$E\left\{\sum_{k=1}^m\left[x\left(\frac{k}{m}\right) - x\left(\frac{k-1}{m}\right)\right]^j\right\} \begin{array}{ll}\to 0\,, & j\neq 2n\\ \to(-1)^{n+1}(2n)!\,, & j = 2n\end{array} \quad \text{as } m\to\infty\,.$$

(3.29)′ $$y(t) = \exp\left\{\sum_{k=1}^{2n}(-1)^{k+1}\frac{\alpha^k}{k}\int_0^t (dx)^k(s)\right\} = \exp\{\xi(t)\} = \sum\frac{[\xi(t)]^n}{n!}$$

satisfies $dy = \alpha y\,dx$, $y(0) = 1$.

(4.5)′ $$\int_0^\infty\int_0^\infty e^{-uT}e^{-\lambda\alpha}\,d_\alpha\sigma(\alpha, T)\,dT = \frac{1}{u}\prod_{k=1}^n\frac{z_k(u)}{[z_k(u) - i\lambda]} = \frac{i^n}{u^{\frac{1}{2}}}\prod_{k=1}^n\frac{\exp\{-(2k-1)\pi i/2n\}}{[\lambda + iz_k(u)]}$$

where $z_k(s) = [\exp\{-(2k-1)\pi i\}s]^{1/2n}$, $k = 1, 2, \cdots, n$.

(4.6)′ $$\int_0^\infty e^{-uT}\frac{\partial}{\partial\alpha}P\{\max_{s\leqq T}x(s)\leqq\alpha\}\,dT = \frac{i}{u^{\frac{1}{2}}}\prod_{k=1}^n\exp\{-(2k-1)\pi i/2n\}\left\{\sum_{j=1}^n\frac{\exp\{-z_j(u)\alpha\}}{\prod_{k=1;k\neq j}^n[z_k(u) - z_j(u)]}\right\}.$$

The conditions (5.1) for the central limit theorem become

(5.1)′ $$\begin{array}{lll}\int d\mu = 1\,, & \int x^j\,d\mu = 0\,, & j = 1, 2, \cdots, 2n-1\,,\\ & \qquad\;\; = (-1)^{n+1}(2n)!\,, & j = 2n\,,\\ \int |x|^{2n+1}|d\mu| < \infty\,, & & \end{array}$$

and the conclusion is that

(5.2)′ $$\lim_{k\to\infty}E\left\{\varphi\left(\frac{S_k}{k^{1/2n}}\right)\right\} = \int_{-\infty}^\infty\varphi(x)p(1, x)\,dx\,.$$

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