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ON HITTING PROBABILITIES FOR AN ANNIHILATING PARTICLE MODEL[1]

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Erdös and Ney introduced a discrete time annihilating particle model on the integer lattice and conjectured that, starting from an initial state of a particle at each lattice site except the origin, the probability a particle ever hits the origin is 1. This paper proves this conjecture for the continuous time version of their model.

1. Introduction. In [2] Erdös and Ney introduced the following annihilating particle model on the integer lattice, Z. Start with a particle at each lattice site of Z except 0. Let $0 < p < 1$. At each unit of time a particle at x jumps to $x - 1$ with probability p and jumps to $x + 1$ with probability $1 - p$. If two particles collide or cross each other's path then both particles are annihilated. A one-sided variant of this model is at each unit of time let a particle at x jump to $x - 1$ with probability p and remain in place with probability $1 - p$. In this case if a particle attempts to land on an occupied site then both particles are annihilated. For both models Erdös and Ney [2] conjectured but could not prove that

$$P[\text{the origin is ever hit}] = 1 .$$

In this paper we prove Erdös and Ney's conjecture for the continuous time versions of these models. By the continuous time version we mean that each partile waits an exponentially distributed random time (mean 1) before jumping, where the random times are independent for each particle and for each jump. The method of proof is to identify the continuous time annihilating particle model as a transformation of Holley and Liggett's [4] voter model and then use Harris's [3] results on representing certain additive processes as random graphs.

Adelman [1] has written a paper in which he argues that the Erdös–Ney conjecture (discrete time) is true whenever $p = q = \frac{1}{2}$. Adelman bases his proof on symmetry relationships and hence our proof is quite different and also extends to the case where $p \neq q$.

2. Identifying the annihilating particle model with the voter model. The voter model that we consider is a strong Markov process η_t on $\{0, 1\}^Z$ (product topology) whose infinitesimal generator, when restricted to cylinder functions, is given by

$$\begin{aligned}(2.1)\qquad \Omega f(\eta) = \lambda \textstyle\sum_{x \in Z} &\{\eta(x)(1 - \eta(x + 1)) + (1 - \eta(x))\eta(x + 1)\}[f({}_x\eta) - f(\eta)] \\ &+ (1 - \lambda) \textstyle\sum_{x \in Z} \{\eta(x)(1 - \eta(x - 1)) + (1 - \eta(x))\eta(x - 1)\} \\ &\times [f({}_x\eta) - f(\eta)]\end{aligned}$$

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where $0 \leqq \lambda \leqq 1$, ${}_x\eta(u) = \eta(u)$ if $u \neq x$ and ${}_x\eta(x) = 1 - \eta(x)$. Then η_t can be described as a configuration of states 0 and 1 on Z with site x being in state $\eta_t(x)$ at time t. Each site remains in its present state for an exponential random time (mean 1), then with probability λ it aligns itself with its right neighbor ($01 \to 11$, $10 \to 00$) and with probability $1 - \lambda$ it aligns itself with its left neighbor ($10 \to 11$, $01 \to 00$). It is possible that there is no resulting change if a site and its neighbor are already in the same state.

Think of η_t as describing a random string of 0's and 1's on the integer lattice. At each time t place a particle at site x if $\eta_t(x) \neq \eta_t(x-1)$. Let $\xi_t = \{x \in Z \mid \eta_t(x-1) \neq \eta_t(x)\}$. Then ξ_t describes a configuration of particles on Z and we show below that ξ_t is the annihilating particle model with initial configuration $\xi_0 = \{x \mid \eta_0(x-1) \neq \eta_0(x)\}$. If $0 < \lambda < 1$ this is the two-sided version and if $\lambda = 1$ or $\lambda = 0$ this is the one-sided version (cf. Erdös and Ney's random intervals [2]).

Using the convention that $\xi(x) = 1$ if there is a partcle at site x and $\xi(x) = 0$ if not, the annihilating particle model is a strong Markov process ξ_t on $\{0, 1\}^Z$ (product topology) whose infinitesimal generator, restricted to cylinder functions, is given by

$$\mathscr{A}g(\xi) = \lambda \textstyle\sum_{x \in Z} [f({}_{x+1,x}\xi) - f(\xi)] + (1 - \lambda) \sum_{x \in Z} [f({}_{x,x+1}\xi) - f(\xi)]$$

where $0 \leqq \lambda \leqq 1$ and

$$\begin{aligned} {}_{ab}\xi(u) &= \xi(u) && \text{if} \quad u \neq a, \quad u \neq b \\ &= 0 && \text{if} \quad u = a \\ &= \xi(b)[1 - \xi(a)] && \text{if} \quad u = b . \end{aligned}$$

Hence ${}_{ab}\xi$ describes the change in ξ after a particle jumps (and is possibly annihilated) from a to b. Let $\theta\colon \{0, 1\}^Z \to \{0, 1\}^Z$ be given by $\theta(\eta)(x) = 1$ if and only if $\eta(x-1) \neq \eta(x)$. Then the claimed correspondence between ξ_t (annihilating particle model) and η_t (voter model) is $\xi_t = \theta(\eta_t)$. To prove this we note that

$$\{\eta(a)(1 - \eta(b)) + (1 - \eta(a))\eta(b)\}[g(\theta(\eta))] = g({}_{b,a}\xi) - g(\xi)$$

whenever $a = x$ and $b = x + 1$ or $a = x$ and $b = x - 1$. Therefore $\Omega g(\theta(\eta)) = \mathscr{A}g(\xi)$ for all cylinder functions g and hence $\{\theta(\eta_t)\}_{t \geqq 0}$ and $\{\xi_t\}_{t \geqq 0}$ have the same finite distributions.

Next suppose that η_0 is that element of $\{0, 1\}^Z$ satisfying $\eta_0(x) = 1$ if $x > 0$ and x is an odd integer, $\eta_0(x) = 1$ if $x < 0$ and x is an even integer, and $\eta_0(x) = 0$ elsewhere. Then $\xi_0 = \{x \in Z \mid x \neq 0\}$. Hence Erdös and Ney's conjecture in continuous time becomes

$$P_{\eta_0}[\eta_t(0) \neq \eta_t(-1) \text{ for some } t] = 1 . \tag{2.2}$$

Since $\{\eta_t(0) = 1 \text{ for some } t\} \subset \{\eta_t(0) \neq \eta_t(-1) \text{ for some } t\}$ P_{η_0}-a.s. when η_0 is as above, the proof of (2.2) will be a consequence of the proof that

$$P_{\eta_0}[\eta_t(0) = 1 \text{ for some } t] = 1 .$$

3. Constructing η_t as a graphical process. In this section we follow Harris [3] and construct η_t (the voter model) using a random graph on

$$S = \{(x, t) \mid x \in Z, t \in [0, \infty)\} .$$

Let $N_x(t)$, $x \in Z$ be independent Poisson processes with parameter 1. Fix a realization of these Poisson processes. Let $\tau_x(k)$ be the kth jump time of the $N_x(t)$ process. At each $(x, t) \in S$ such that $\tau_x(k) = t$ for some $k \geqq 1$ place a directed line segment from (x, t) to $(x - 1, t)$ with probability λ (i.e., an arrow with head at $(x - 1, t)$ and tail at (x, t)) and with probability $1 - \lambda$, place a directed line segment from (x, t) to $(x + 1, t)$. An *active path* (*up*) from (x, s) to (y, t) is a sequence of alternately vertical and horizontal line segments from (x, s) to (x, t_1) to (y_1, t_1) to (y_1, t_2) to (y_2, t_2) to $\cdots$ to (y, t_N) to (y, t) satisfying

(i) $s < t_1 < t_2 < \cdots < t_N \leqq t$.
(ii) There is no arrowhead in the interior of any vertical line segment.
(iii) Each horizontal line segment coincides in extent and direction with one of the directed arrows on the random graph S.

An active path (up) follows the graph upwards along vertical lines and horizontally along the direction of the arrows (tail to head). Using the random graph define the process η_t on $\{0, 1\}^Z$ as follows: $\eta_t(y) = 1$ if and only if there exists an active path (up) from $(x, 0)$ to (y, t) from some $x \in Z$ such that $\eta(x) = 1$. From [3] (Section 4.c and Section 9) we know that η_t is what Harris calls an additive process and that η_t has an infinitesimal generator given by (2.1). That the generator is (2.1) can be verified directly.

In order to continue we define an *active path* (*down*) from (y, t) to (x, s) as a sequence of alternating vertical and horizontal line segments from (y, t) to (y_1, s_1) to (x_1, s_1) to (x_2, s_2) to $\cdots$ to (x, s_N) to (x, s) satisfying

(i) $t > s_1 > s_2 > \cdots > s_N \geqq s$.
(ii) There is no arrowhead in the interior of any vertical segment.
(iii) Each horizontal line segment coincides in extent and in *opposite* direction with one of the directed arrows on the random graph S.

An active path (down) follows the graph downwards along vertical lines and horizontally along the arrows from head to tail.

For any fixed realization of the random graph S there is an active path (down) from (y, t) to (x, s) if and only if there is an active path up from (x, s) to (y, t). For the process η_t started at η we have $\eta_t(y) = 1$ if and only if there exists an active path (down) from (y, t) to $(x, 0)$ for some x such that $\eta(x) = 1$. We point out that it is possible that there is no active path (up) from $(x, 0)$ to (y, t) for any $y \in Z$ but, in contrast, for each $y \in Z$, $t > 0$ and $s < t$ there is a (unique) $x \in Z$ such that there is an active path (down) from (y, t) to (x, s). In particular, there is always an active path (down) from (y, t) to $(x, 0)$ for some $x \in Z$.

4. The dual process. Fix a realization of the random graph S. For each t

let $(Y_s^t)_{0 \leqq s \leqq t}$ be a process on Z defined as follows.

$$Y_0^t = 0 \qquad Y_s^t = x$$

if there exists an active path down from $(0, t)$ to $(x, t - s)$. Then $(Y_s^t)_{0 \leqq s \leqq t}$ is a stopped continuous time birth and death process on Z with $p(x, x - 1) = 1 - \lambda$ and $p(x, x + 1) = \lambda$. From our comments at the end of Section 3 we see that

$$P_\eta[\eta_t(0) = 1 \text{ for some } t \geqq 0] = \text{Prob}[Y_t^t \in \{u \mid \eta(u) = 1\} \text{ for some } t \geqq 0] .$$

In Section 5 we will prove that these probabilities are 1 and hence that the Erdös and Ney conjecture is true. First we need the following lemmas.

(4.1) LEMMA. *Let $C = \{x \in Z \mid x > 0$ and x is odd$\} \cup \{x \in Z \mid x < 0$ and x is even$\}$. Then $\lim_{t\to\infty} \text{Prob}[Y_t^t \in C] = \frac{1}{2}$.*

PROOF. Let X_t be a continuous time jump process on Z starting at 0 with $p(x, x + 1) = \lambda$ and $p(x, x - 1) = 1 - \lambda$. Then, for each t, X_t and Y_t^t have the same distribution. If $\lambda > \frac{1}{2}$ then $\lim_{t\to\infty} \text{Prob}[X_t \in C] = \lim_{t\to\infty} \text{Prob}[X_t$ has made an odd number of jumps$] = \frac{1}{2}$. Similarly if $\lambda < \frac{1}{2}$. If $\lambda = \frac{1}{2}$ then $\lim_{t\to\infty} \text{Prob}[X_t \in C \mid X_0 = 0] = \lim_{t\to\infty} \text{Prob}[X_t \in C \mid X_0 = -1]$. By symmetry $\text{Prob}[X_t \in C \mid X_0 = -1] = \text{Prob}[X_t \in C^c \mid X_0 = 0]$ and hence the desired result follows.

(4.2) LEMMA. *Let $\varepsilon > 0$ and $t > 0$. Then there exists a $T > 0$ such that for any $r_1 < r_2 < \cdots < r_n \leqq t$ and $A_i \subset Z$, $A \subset Z$*

$$\begin{aligned}|\text{Prob}[Y_{r_i}^{r_i} \in A_i,\ i = 1, \cdots, n,\ Y_{t+T}^{t+T} \in A] \\ - \text{Prob}[Y_{r_i}^{r_i} \in A_i,\ i = 1, \cdots, n]\,\text{Prob}[Y_{t+T}^{t+T} \in A]| < \varepsilon .\end{aligned}$$

PROOF. Fix a realization of the random graph. Let

$$W_0^t = 0 , \qquad U_0^t = 0 ,$$

$W_s^t = x$ if there exists an *active path down* from $(0, t)$ to $(x, t - s)$ using only the directed arrows which go from left to right (i.e., tail at (y, r) and head at $(y + 1, r)$),

and

$U_s^t = x$ if there exists an *active path down* using only the directed arrows which go from right to left (i.e., tail at $(y + 1, r)$ and head at (y, r)).

Then $(W_s^t)_{0 \leqq s \leqq t}$ is a jump process on Z which jumps only one integer step to the left at a time with rate $1 - \lambda$. Similarly $(U_s^t)_{0 \leqq s \leqq t}$ jumps to the right one step at a time with rate λ. Hence for a fixed t and ε there exists an M such that

$$\text{Prob}[|U_s^t| < M \text{ and } |W_s^t| < M \text{ for all } 0 \leqq s \leqq t] > 1 - \varepsilon .$$

By construction $W_s^t \leqq Y_s^t \leqq U_s^t$ for all $0 \leqq s \leqq t$. Therefore with probability at least $1 - \varepsilon$, Y_t^t is defined in the terms of the subset of the random graph given by

$$D = \{(x, s) \mid |x| \leqq M, 0 \leqq s \leqq t\} .$$

From the construction of Y_s^r we see that if $r \leqq t$ then Y_r^r is also defined in terms of D. We now select a $T > 0$ so that with probability at least $1 - \varepsilon$ Y_{t+T}^{t+T} is defined in terms of the subset $S \backslash D$. This can be done by choosing T satisfying

$$\text{Prob}\,[|X_s| \geqq M + 1 \quad \text{for all} \quad T \leqq s \leqq T + t] > 1 - \varepsilon$$

where X_s is a continuous time jump process on Z starting at 0 with jump rates $p(x, x + 1) = \lambda$ and $p(x, x - 1) = 1 - \lambda$. Since $(Y_s^{t+T})_{T \leqq s \leqq T+t}$ and $(X_s)_{T \leqq s \leqq T+t}$ have the same distribution and since $\{|Y_t^{t+T}| \geqq M + 1$ for all $T \leqq s \leqq T + t\}$ says that the active path (down) which determines Y_s^{t+T} does not intersect D, the lemma follows.

(4.3) LEMMA. *Let* $N > 0$, *and* $\varepsilon > 0$. *Let* $A \subset Z$. *Then there exists* $t_1 < t_2 < \cdots < t_N$ *such that*

$$|\text{Prob}\,[Y_{t_i}^{t_i} \in A \ \textit{for each} \ i = 1, \cdots, N] - \textstyle\prod_{i=1}^N \text{Prob}\,[Y_{t_i}^{t_i} \in A]| < \varepsilon .$$

PROOF. Use Lemma (4.2) and an induction argument.

5. The main result.

(5.1) THEOREM. *Let* $C = \{x \in Z \mid x > 0$ *and* x *is odd*$\} \cup \{x \in Z \mid x < 0$ *and* x *is even*$\}$. *Then* $P_\eta[\eta_t(0) = 1$ *for some* $t \geqq 0] = 1$ *whenever* $\eta(x) = 1$ *for* $x \in C$.

PROOF. From Section 4 we need only prove that $\text{Prob}\,[Y_t^t \in C$ for some $t \geqq 0] = 1$. From Lemma (4.1) $\text{Prob}\,[Y_r^r \notin C] < \frac{2}{3}$ for sufficiently large r. From Lemma (4.3) given $\varepsilon > 0$ and N there exist $r < t_1 < t_2 < \cdots < t_N$ such that $|\text{Prob}\,[Y_{t_i}^{t_i} \notin C$ for each $i] - \prod_{i=1}^N \text{Prob}\,[Y_{t_i}^{t_i} \notin C]| < \varepsilon$ and hence that

$$\begin{aligned}\text{Prob}\,[Y_t^t \in C \ \text{ for some } \ t \geqq 0] &\geqq 1 - \text{Prob}\,[Y_{t_i}^{t_i} \notin C, i = 1, \cdots, N] \\ &\geqq 1 - (\tfrac{2}{3})^N - \varepsilon .\end{aligned}$$

Since N and ε are arbitrary this gives the desired result.

REMARK. Theroem (5.1) will be true for any set C satisfying

$$\liminf_{t \to \infty} \text{Prob}\,[Y_t^t \in c] < 1 .$$

Hence the Erdös and Ney conjecture can be proved for other initial configurations.

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