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ADDITIVE SET-VALUED MARKOV PROCESSES AND GRAPHICAL METHODS[1]

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Let Z be a countable set, Ξ the set of subsets of Z. A Ξ-valued Markov process $\{\xi_t\}$ with transition function $P(t, \xi, \Gamma)$ is called additive if there exists a family $\{\xi_t^A, t \geqq 0, A \in \Xi\}$ such that for each A, $\{\xi_t^A\}$ is Markov with transition function P and $\xi_0^A = A$, and such that $\xi_t^{A \cup B} = \xi_t^A \cup \xi_t^B$, $A, B \in \Xi$, $t \geqq 0$. Additive processes include symmetric simple exclusion, voter models and all contact processes having associates. The structure of such processes is studied, their construction from sets of independent Poisson flows, and their representations by random graphs. Applications for the case $Z = Z_d$, the d-dimensional integers, include individual ergodic theorems for certain cases as well as lower bounds for growth rates, and some results about different kinds of criticality when $d = 1$.

1. Introduction. Let Z be a countable set of elements $x, y, z, \cdots$ and Ξ the set of subsets of Z with elements $\xi, \eta, \zeta, \cdots$ and sometimes $A, B, \cdots$. Let $|\xi|$ denote the cardinality of ξ. Numerous studies have dealt with Ξ-valued Markov processes $\{\xi_t\}$; see [20] for many references to recent literature.

The present paper studies a class of Ξ-valued processes that will be called *additive* (so will their transition functions, etc.). If $\{\xi_t\}$ is a Ξ-valued process with transition function $P(t, \xi, \Gamma)$, additivity means that there exists a family $\{\xi_t^A, t \geqq 0, A \in \Xi\}$ of Markov random functions, each with the transition function P, all defined on the same probability space, such that $\xi_0^A = A$ and

$$\xi_t^{A \cup B} = \xi_t^A \cup \xi_t^B, \qquad t \geqq 0, \quad A, B \in \Xi. \tag{1.1}$$

(Note that A and B need not be disjoint and independence plays no role in the definition.) We will always take $\xi_t^\varnothing = \varnothing$.

The additive processes include all contact processes that have associates ([13]), symmetric simple exclusion (see, e.g., [23]), voter models ([17], as well as the "biased" case of [22]), and others. Thus many genuine interactions are included, in spite of the simple law of composition. There are close relations to percolation processes and they will be exploited.

If $|Z| < \infty$ and at least in many cases where $|Z|$ is countably infinite, additive processes have useful representations as functions of countable products of Poisson flows. They also have graphical representations which, in the case where Z is the set of 1-dimensional integers, are helpful in studying the process. In fact it was the examination of graphical representations for certain special processes in discrete time (see Toom [25], Vasil'ev [26]) that suggested the property (1.1)

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to the author. Clifford and Sudbury used other graphical representations in [2] for certain processes, called "swapping" and "invasion," which are cases of simple exclusion and the voter model respectively. Gray and Griffeath [6] have used a related construction for proximity processes. In Section 9 we shall give a graphical representation that applies to all additive processes.

We shall show that for certain additive processes (in particular contact processes with appropriate parameter values) we have

$$P_\xi \left\{ \inf_{t>0} \frac{|\xi_t|}{t} > 0 \,\middle|\, \xi_t \text{ never } \varnothing \right\} = 1$$

if $\xi \neq \varnothing$. As far as the author knows, lower bounds for growth rates have not previously been obtained except when ξ_t is always increasing; see, Richardson [21] for results in this case. Using the above lower bound, we obtain, for certain additive processes, an individual ergodic result of the form[2]

$$P_\xi \left\{ \frac{1}{T} \int_0^T f(\xi_t)\, dt \to \int_\Xi f(\eta)\nu(d\eta) \right\} = 1$$

for certain explicitly described ξ. In the last section we investigate the question, raised in Section 10 of [13], of the relations among different notions of "extinction" or of "critical value" for self-associate contact processes.

Every additive process has an associate (see, [13] and Section 2 below). The association relation has a particularly simple graphical representation for additive processes (Section 9). This is also true for the graphical representations used in [2] and [26]. All the processes studied in [13] having associates turn out to be additive, but we shall give an example of a process with an associate that is not additive. However, every binary process with an associate is additive (Section 6).

An alternative approach to additive processes was sketched in Harris [14]; this is related to Theorem 6.1. Related material, including the graphical analysis of Section 9, was sketched in Harris [12].[3]

REMARK. Since this paper was submitted Holley and Liggett [18] have found a very good upper bound for the critical parameter of the simplest contact process. By combining their result (not derived by percolation or graphical methods) with the methods of Sections 11 and 12, applied to the graphical structure of Section 9 rather than to discrete-time processes, it appears that Theorem 13.5 can be improved, "sufficiently large" being replaced by a good bound. Lemma 13.1 would then be unnecessary if we are interested only in continuous time. This may be included in some current work of J. Jameson.

NOTATION AND TERMINOLOGY. Z_d is the set of d-dimensional integers. In this

[2] I am indebted to Claude Kipnis for proposing the question of individual ergodicity in connection with simple exclusion, where the problem still seems to be unsolved.

[3] In [1] Bertein and Galves give other results about additive processes.

case $|x - y|$ is the Euclidean distance from x to y. O is the origin of coordinates in Z_d and Ξ_d denotes Ξ if $Z = Z_d$. $\xi(A) = |\xi \cap A|$ if $A \subset Z$; $\xi(x)$ means $\xi(\{x\})$ and is a *coordinate* of ξ, and we may regard ξ as an element of $\{0, 1\}^Z$. Then Ξ, with the topology of simple convergence, is compact and metrizable. $\mathscr{B}(\Xi)$ and $\mathscr{B}_d$ are the Borel sets in Ξ and R_d, C the continuous real functions on Ξ, and C_0 the cylinder functions in C. If $A \subset Z$, C_A is the set of continuous functions depending on coordinates $\xi(x)$, $x \in A$. The semigroup on C of a Ξ-valued process is denoted by T_t, the generator by $\mathscr{A}$. P_ξ is the measure for a process with initial value ξ and $\mathscr{E}_\xi$ is the corresponding expectation. If $|Z| < \infty$ we may regard $\mathscr{A}$ as a matrix; then $\mathscr{A}(\xi, \eta)$ is the intensity $\xi \to \eta$ if $\xi \neq \eta$. We sometimes write $\xi \cup x$ rather than $\xi \cup \{x\}$, etc. δ_ξ is the unit mass in Ξ concentrated at ξ. If ν is a Borel measure on $\mathscr{B}(\Xi)$, $\nu(f)$ means $\int f\, d\nu$ if the integral exists.

The special notation $\xi \# \eta$ means $\xi \cap \eta \neq \emptyset$. We define $\theta_\xi(\eta) = 1$ if $\xi \# \eta$, 0 if $\xi \cap \eta = \emptyset$.

2. Structure of additive processes ($|Z| < \infty$).

NOTE. In Sections 2—7, $|Z| < \infty$. In this section definitions are with respect to a fixed Z and Ξ.

(2.1) DEFINITIONS. A mapping $W: \Xi \to \Xi$, written on the right for later convenience, is called *additive* if $\phi W = \emptyset$ and $(\xi \cup \eta)W = \xi W \cup \eta W$, whether or not ξ and η are disjoint. The identity is W_I. The set of all additive transformations is $\mathscr{W}$.

(2.2) DEFINITION. If $W \in \mathscr{W}$, define $W^* \in \mathscr{W}$ by $\xi W^* = \{x: xW \# \xi\}$. We say that W^* is *associate* to W. Then $\xi W \# \eta$ iff $\xi \# \eta W^*$.

Taking $Z = \{x_1, \cdots, x_n\}$ and representing ξ as a row vector $(\xi(x_1), \cdots, \xi(x_n))$, where $\xi(x_i) = 1$ if $x_i \in \xi$ and 0 if not, we identify W with a matrix, also called W, whose ith row has 1 in the jth column if $x_j \in x_i W$ and 0 if not. Then ξW has the usual matrix meaning except that integers > 1 are replaced by 1, and the same applies to matrix products; thus $(\xi W_1)W_2 = \xi(W_1 W_2)$. The associate matrix W^* is just the transpose of the matrix W, and we see that $(W^*)^* = W$.

We have already defined additive processes in Section 1.

(2.3) THEOREM. *Let $\{\xi_t\}$ be additive, with generator $\mathscr{A}$. Then there are additive transformations $W_1, W_2, \cdots, W_m$ and constants $\rho_1, \rho_2, \cdots, \rho_m > 0$ such that*

$$\mathscr{A}f(\xi) = \sum_{i=1}^{m} \rho_i \left(f(\xi W_i) - f(\xi)\right) . \tag{2.4}$$

Conversely a process with the generator (2.4) *is additive.*

We can always assume W_I does not appear in (2.4). Of course ξW_i may equal ξ for certain ξ and i.

Referring to (2.4), we may say that *W_i is applied with intensity ρ_i.*

PROOF. Suppose $\{\xi_t\}$ is additive. Let the processes $\xi_t{}^A$, $A \in \Xi$ be defined on some probability space with measure Q. Then

$$P(t, A, \{\eta\}) = \textstyle\sum'_{(A)} Q\{\xi_t^x = \eta_x, x \in Z\} \tag{2.5}$$

where the sum $\sum'_{(A)}$ has a term for each set $\{\eta_x, x \in Z\}$ such that $\bigcup_{x \in A} \eta_x = \eta$. A given set $\{\eta_x, x \in Z\}$ corresponds in a 1 : 1 fashion to some $W \in \mathscr{W}$ such that $xW = \eta_x$, $x \in Z$. Putting $Q\{\xi_t^x = \eta_x, x \in Z\} = Q_t(W)$, (2.5) becomes

$$P(t, A, \{\eta\}) = \textstyle\sum_{W \in \mathscr{W}} Q_t(W)\delta_{AW}(\{\eta\}) \, . \tag{2.6}$$

For each $W' \neq W_I$ there is an $A' \in \Xi$ such that $A'W' \neq A'$. Put $A = A'$, $\eta = A'W'$ in (2.6), divide by t, and let $t \downarrow 0$. Since the left side approaches the generator element $\mathscr{A}(A', A'W')$, $Q_t(W')/t$ is bounded. Hence we can find $t_n \downarrow 0$ such that for each $W \neq W_I$ $\lim Q_{t_n}(W)/t_n = \rho(W)$ exists. It follows that $\mathscr{A}f(\xi) = \sum_{W \in \mathscr{W}} \rho(W)(f(\xi W) - f(\xi))$, and (2.4) is proved.

Now suppose $\{\xi_t\}$ is a Ξ-valued process whose generator $\mathscr{A}$ satisfies (2.4). Take m independent Poisson processes $0 < \tau_{i1} < \tau_{i2} < \cdots$, $i = 1, \cdots, m$ with rates $\rho_1, \cdots, \rho_m$. Let $0 < \tau_1 < \tau_2 < \cdots$ be a single ordering of all the τ_{in}, where we can assume ties do not occur. Put $\tau_0 = 0$. Construct ξ_t^A as follows. Let $\xi_t^A = A$ for $0 \leqq t < \tau_1$. Assuming ξ_t^A has been determined for $0 \leqq t < \tau_n$, $n \geqq 1$ and is constant on $[\tau_{j-1}, \tau_j)$, $1 \leqq j \leqq n$, put $\xi_t^A = \xi_{\tau_{n-1}} W(\tau_n)$ for $\tau_n \leqq t < \tau_{n+1}$, where $W(\tau_n) = W_i$ if τ_n is one of $\tau_{i1}, \tau_{i2}, \cdots$. Then $\xi_t^A = AW(\tau_1)\, W(\tau_2) \cdots$, the product being continued over all $\tau_n \leqq t$. Since the product is in $\mathscr{W}$, the additivity of ξ_t^A is shown. From the properties of Poisson processes we see that the generator is given by (2.4). Note that we can write

$$\xi_t^A = AW(t) \tag{2.7}$$

where $W(t)$ is a random element of $\mathscr{W}$ with the distribution $Q_t(W) = \text{Prob}\,\{W(t) = W\}$. □

WARNING. The generator of an additive process is not determined by the values of $\mathscr{A}(x, \eta)$, $x \in Z$, $\eta \in \Xi$.

From [13] we recall that the Ξ-valued process $\{\xi_t^*\}$ with semigroup T_t^* and generator $\mathscr{A}^*$ is *associate to* $\{\xi_t\}$ if $T_t\theta_\xi(\eta) = T_t^*\theta_\eta(\xi)$, $\xi, \eta \in \Xi$, or equivalently if $\mathscr{A}\theta_\xi(\eta) = \mathscr{A}^*\theta_\eta(\xi)$. (If $|Z| = \infty$ we require $T_t\theta_\xi(\eta) = T_t^*\theta_\eta(\xi)$ if ξ or η is finite.) We call $\{\xi_t\}$ *self-associate* if $T_t^* = T_t$. We shall use the *—notation only for association. For related notions of duality see [7], [19] and [22].

(2.8) THEOREM. *If* $\{\xi_t\}$ *is additive with generator* (2.4), *then* $\{\xi_t^*\}$ *exists and has the generator* $\mathscr{A}^*$ *given by the same expression but with* W_i^* *instead of* W_i.

PROOF. It is easily verified that $\mathscr{A}^*\theta_\xi(\eta) = \mathscr{A}\theta_\eta(\xi)$, which proves the result. □

(2.9) REMARK. From (2.4) we see that the class of additive generators is a convex cone.

3. Recognition of additive processes. To determine whether a given generator is additive we must see whether there are numbers $\rho(W) \geqq 0$, $W \neq W_I$, such that $\mathscr{A}(\xi, \eta) = \sum_{W : \xi W = \eta} \rho(W)$, $\xi \neq \eta$. This is a big computational problem unless Z is small but in special cases, particularly if $\mathscr{A} = \sum \mathscr{A}_x$ where each $\mathscr{A}_x$ "lives" on a few points of Z, we can sometimes recognize additivity.

4. Examples. The following examples for finite Z can be used as building blocks when $Z = Z_d$; see Section 8.

(a) *Symmetric simple exclusion.* For each $i = 1, 2, \cdots, n$ let x_i, y_i be points of Z, $x_i \neq y_i$. Define W_i by $x_i W_i = y_i$, $y_i W_i = x_i$, $zW_i = z$ otherwise. Let $\{\xi_t\}$ have W_i applied with intensity $\rho_i > 0$. Then $\{\xi_t\}$ is a simple exclusion process with intensity ρ_i for a particle at x_i to jump to y_i if y_i is unoccupied and the same intensity for a jump from y_i to x_i. Since $W_i = W_i^*$, this process is self-associate.

(b) *Contact processes.* Let $Z = O \cup N$, where O is a distinguished point, $O \notin N$, and $1 \leqq |N| = n < \infty$. Let $\lambda(\xi)$ depend only on $\xi(x)$, $x \in N$, with $\lambda(\phi) = 0$, $\lambda(\xi) \geqq 0$. Let $\mu \geqq 0$ be constant and consider $\{\xi_t\}$ with the generator

$$\mathscr{A}f(\xi) = \mu(f(\xi \backslash O)) - f(\xi)) + \lambda(\xi)(f(\xi \cup O) - f(\xi)) .$$

$\{\xi_t\}$ was called a "contact process" in [13], extending the definition given in [11]. In the present paper we shall assume, unless the contrary is stated, that λ depends only on $|\xi \cap N|$, say $\lambda(\xi) = \lambda_k$ if $|\xi \cap N| = k$.

It was shown in [13], Section 6, that $\{\xi_t\}$ has an associate iff

$$\sum_{r=0}^{k} (-1)^{1+r}\binom{k}{r}\lambda_{n-k+r} \geqq 0 , \qquad 1 \leqq k \leqq n . \tag{4.1}$$

If $n = 2$, this means $\lambda_1 \leqq \lambda_2 \leqq 2\lambda_1$. Let q_k denote the left side of (4.1).

(4.2) THEOREM. *If a contact process has an associate, both are additive.*

PROOF. For each $\eta \subset N$ with $|\eta| = k$, $k = 1, 2, \cdots, n$, let W_η be the additive transformation defined by $xW_\eta = x \cup O$ if $x \in \eta$, $xW_\eta = x$ if $x \notin \eta$. Define W_O by $xW_O = \varnothing$ if $x = O$, $xW_O = x$ otherwise. Some routine calculations show that $\{\xi_t\}$ is the additive process having W_O applied with intensity μ and W_η with intensity q_k if $\eta \subset N$, $|\eta| = k$, $1 \leqq k \leqq n$. In fact $\{\xi_t\}$ is a "proximity process" as defined in [17]. Since $(W_O)^* = W_O$ and $O(W_\eta)^* = O \cup \eta$, $\{\xi_t^*\}$ is a "branching process with interference" as defined in [17]; this was shown from a different point of view in [13]. □

(c) *The voter model.* Let $Z = \{x_1, x_2\}$ and consider

$$W_1 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} , \qquad W_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} , \qquad W_3 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} , \qquad W_4 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} .$$

Let $\{\xi_t\}$ have W_1 and W_2 applied with rate α, W_3 and W_4 with rate β. Representing ξ as $(\xi(x_1), \xi(x_2))$, we see that $(1, 0)$ and $(0, 1)$ both have intensity α for a transition to $(0, 0)$ and $\alpha + \beta$ for a transition to $(1, 1)$, while $(1, 1)$ and $(0, 0)$ are absorbing. If $\beta = 0$ we have a component of the voter model of [17]; if $\beta > 0$ we have a biased voter model ([22]).

5. Extremal additive generators. Continuing to assume $|Z| < \infty$, let $\mathscr{G}_a$ be the class of generators of additive processes. We include the 0-generator in $\mathscr{G}_a$, which is thus a closed convex cone. Each $\mathscr{A} \in \mathscr{G}_a$ is a positive linear combination of generators $\mathscr{A}_W$, where

$$\mathscr{A}_W f(\xi) = f(\xi W) - f(\xi), \qquad W \in \mathscr{W}. \tag{5.1}$$

If $W = W_I$, $\mathscr{A}_W$ is 0. Note $(\mathscr{A}_W)^* = \mathscr{A}_{W^*}$.

$\mathscr{A} \in \mathscr{G}_a$ is called *extremal* if $\mathscr{A} = \mathscr{A}' + \mathscr{A}''$, where $\mathscr{A}', \mathscr{A}'' \in \mathscr{G}_a$, implies $\mathscr{A}'$ and $\mathscr{A}''$ are proportional to $\mathscr{A}$. The determination of the extremal $\mathscr{A}$'s is connected with a certain partial ordering of $\mathscr{W}$.

(5.2) DEFINITIONS. If $W_1, W_2 \in \mathscr{W}$, $W_1 < W_2$ means that for each ξ either $\xi W_1 = \xi W_2$ or $\xi W_1 = \xi$. We say W_2 *dominates* W_1 (or W_1 is subordinate to W_2). The domination is *strong* if also $W_1^* < W_2^*$; in this case we write $W_1 \ll W_2$. The domination is *proper* if W_1 is neither W_I nor W_2. Both $<$ and $\ll$ are partial orderings.

For $|Z| = 2$ it can be verified that $\binom{0\ 1}{0\ 1} < \binom{0\ 1}{0\ 0}$, but $\binom{0\ 1}{0\ 1}^* = \binom{0\ 0}{1\ 1} \not< \binom{0\ 0}{1\ 0} = \binom{01}{00}^*$, so the definition of $\ll$ is not superfluous. Also $\binom{1\ 1}{1\ 1}$ strongly dominates $\binom{1\ 0}{1\ 1}$ and $\binom{1\ 1}{0\ 1}$, and these are the only cases of proper strong domination when $|Z| = 2$.

(5.3) THEOREM. *A generator $\mathscr{A} \in \mathscr{G}_a$ is extremal* iff $\mathscr{A} = c\mathscr{A}_W$, *where W has no proper strong subordinate.*

The proof is omitted.

6. Additivity and association. A Ξ-valued process is called *deterministic* if the embedded jump process is deterministic. Such a process is described by an intensity function $q(\xi)$ and a transformation $V: \Xi \to \Xi$ (in general not additive) such that a jump from ξ is to ξV. Our convention is that $q(\xi) = 0$ iff $\xi V = \xi$.

(6.1) THEOREM. ($|Z| < \infty$). *Let $\mathscr{G}$ be the convex hull of the set of generators $\mathscr{A}$ such that $\mathscr{A}$ has a Markov associate $\mathscr{A}^*$ and both $\mathscr{A}$ and $\mathscr{A}^*$ are deterministic. Then $\mathscr{G} = \mathscr{G}_a$.*

This result will not be used and the proof is omitted. A generator of the sort mentioned in the theorem may have an intensity $q(\xi)$ taking 2 or more strictly positive values.

Here is an example of a process with $Z = \{x, y\}$ which is not additive but has an associate. Let the transitions $x \to y$, $y \to x$, $Z \to x$ have intensity 1, with no other transitions. This process is deterministic. The associate process, which is not deterministic, has intensity 1 for $x \to y$, $y \to \emptyset$, and $y \to Z$, and has no other transitions. It can be verified that these processes are not additive.

It can be verified that if $|Z| = 2$, every self-associate process is additive. This property carries over to self-associate generators $\mathscr{A} = \sum \mathscr{A}_x$ in Z_d (see, Section 8) if each generator $\mathscr{A}_x$ lives on 2 points.

7. Correlation inequalities. ($|Z| < \infty$). Let Ξ be partially ordered by inclusion. Let C_i be the set of increasing $f: \Xi \to R_1$. A process is called *monotone* if $T_t C_i \subset C_i$. It is said to have *positive correlations* (PC) if $\mathscr{E}_\xi f(\xi_t) g(\xi_t) \geqq \mathscr{E}_\xi f(\xi_t) \mathscr{E}_\xi g(\xi_t)$, $f, g \in C_i$, $\xi \in \Xi$. It was shown in [15] that a monotone process in any finite partially ordered state space has PC iff each possible jump $\xi \to \eta$ is between comparable states; i.e., $\xi < \eta$ or $\xi > \eta$. Since from (2.7) every additive process is monotone, it follows that *an additive process has* PC iff *for each* W_i *in* (2.4) *we have*

$$(7.1) \qquad \forall \xi, \quad \xi W_i \subset \xi \quad \text{or} \quad \xi W_i \supset \xi .$$

Later we shall need a correlation inequality for a process $\eta_t = \{\eta_{1t}, \cdots, \eta_{kt}\}$, where $\eta_{it} = A_i W(t)$, $W(t)$ is as in (2.7), and $A_1, \cdots, A_k \in \Xi$. The monotone process η_t has the state space Ξ^k, with the product partial ordering, and initial state $(A_1, \cdots, A_k)$.

(7.2) LEMMA. *Suppose* $k \geqq 2$. *Then* $\{\eta_t\}$ *has* PC iff *each* W_i *in* (2.4) *satisfies*

$$(7.3) \qquad \text{Either} \quad \xi W_i \subset \xi \quad \forall \xi \quad \text{or} \quad \xi W_i \supset \xi \quad \forall \xi .$$

Furthermore $\{\xi_t\}$ *and* $\{\xi_t^*\}$ *simultaneously have* PC iff (7.3) *holds for each* W_i *in* (2.4).

REMARKS. In the same process we may have $\xi W_i \subset \xi$ and $\xi W_j \supset \xi$. If $|Z| = 2$, $\binom{0\ 0}{1\ 1}$ satisfies (7.1) but not (7.3).

PROOF. If (7.3) holds, each jump of $\{\eta_t\}$ is between comparable states, so $\{\eta_t\}$ has PC. Conversely suppose $\{\eta_t\}$ has PC; then so does $\{\xi_t\}$ and hence (7.1) holds. Suppose some $W_i = W$ in (2.4) does not satisfy (7.3). Then Z is a disjoint union $Z_1 \cup Z_2 \cup Z_3$ where $Z_1 \neq \emptyset$, $Z_2 \neq \emptyset$, $xW = \emptyset$ if $x \in Z_1$, $xW \supsetneqq x$ if $x \in Z_2$, $xW = x$ if $x \in Z_3$. Moreover $xW \supset Z_1$ if $x \in Z_2$. Take $\eta_0 = (Z_1, Z_2, A_3, \cdots, A_k)$ where $A_3, \cdots, A_k$ are arbitrary. There is a positive probability that the first jump is to $(Z_1 W, Z_2 W, \cdots)$ which is not comparable with η_0, but this is impossible if $\{\eta_t\}$ has PC.

We omit the proof of the second assertion which will not be used in what follows. □

8. Additive processes in Z_d. Going to countably infinite Z, we restrict ourselves to Z_d, although some of the results have obvious analogues for more general cases.

The generator $\mathscr{A}$ is said to "live" on the finite subset V of $Z = Z_d$ iff $f \in C_0 \cap C_{Z \setminus V}$ or $f \in C_V$ implies $\mathscr{A} f = 0$ or $\mathscr{A} f \in C_V$ respectively. If $\mathscr{A}_0$ is a generator living on the finite subset V of Z_d, let $\mathscr{A}_x$, the translate of $\mathscr{A}_0$ to x, be defined by

$$(8.1) \qquad \mathscr{A}_x f(\xi) = \mathscr{A}_0 f_x(\xi - x), \qquad f_x(\xi) = f(\xi + x).$$

From [16] or other known results it follows that $\mathscr{A} = \sum_{x \in Z_d} \mathscr{A}_x$ is the generator of a well-defined process. See Sections 5 and 7 of [13] for further discussion.

The transformation $W \colon \Xi_d \to \Xi_d$ is said to "live" on the finite subset V of Z_d if $\xi \subset V$ or $\xi \subset Z_d \setminus V$ implies $\xi W \subset V$ or $\xi W = \xi$ respectively. Additivity is defined as in the finite case, and $\mathscr{W}$ is the set of additive W such that W lives on some finite set. The generator $\mathscr{A} f(\xi) = f(\xi W) - f(\xi)$ then lives on the same set as W.

If $W_1, \cdots, W_m \in \mathscr{W}$, put

$$\mathscr{A}_O f(\xi) = \sum_{i=1}^m \rho_i(f(\xi W_i) - f(\xi)) , \tag{8.2}$$

$$\mathscr{A} = \sum \mathscr{A}_x , \tag{8.3}$$

where $\mathscr{A}_x$ is defined in (8.1). Note that $\mathscr{A}_x$ is defined by (8.2) with W_i replaced by W_{ix} where

$$\xi W_{ix} = (\xi - x)W_i + x . \tag{8.4}$$

We shall say that a process with the generator $\mathscr{A}$ is additive. In Section 9 we shall see that such processes satisfy the definition of additivity of Section 1.

Each of the examples in Section 4 produces, by means of (8.3), a process in Z_d which we call by the same name. $\mathscr{A}_O$ for a *contact process* in Z_d is defined with a finite nonempty set N, $O \notin N$, called the *basic neighborhood.* Then $\mathscr{A}_O$ acts on $O \cup N$ as in 4(b). From Theorem 4.2 and Section 7b of [13], *a contact process in Z_d is additive* iff *it has an associate.*

9. Basic flow; graphical representation. We shall define additive processes in Z_d on a probability space appropriate for showing (1.1). We still use Poisson flows but now in a different manner in order to provide the graphical construction that will be important to us. In order to establish precise relations for later use it seems necessary to give some lengthy definitions. The partly verbal description below (9.3) may help visualize the construction.

Start with given $W_1, \cdots, W_m \in \mathscr{W}$, $\rho_1, \cdots, \rho_m > 0$, no $W_i = W_I$. In what follows the indices i, j, k, n, N are integers. Note also that we hold fixed the integer $m \geqq 1$.

For $x \in Z_d$, $1 \leqq i \leqq m$, let Ω_{ix} be the set of atomic Borel measures ω_{ix} in $(R_1, \mathscr{B}_1)$ with atoms of weight 1, such that $\omega_{ix}(0, \infty) = \omega_{ix}(-\infty, 0) = \infty$, $\omega_{ix}(-n, n) < \infty$ for $n \geqq 1$. Let $\mathscr{F}_{ix}$ be the σ-field in Ω_{ix} generated by the sets $\{\omega_{ix}(B) \leqq u\}$, $B \in \mathscr{B}_1$, $u \in R_1$. Let $\mathscr{F}_{ix}^{st}$ be the σ-field if B is restricted to the interval $(s, t]$, $s < t$, writing $\mathscr{F}_{ix}^{t}$ if $s = -\infty$; let $\mathscr{F}_{ix}^{tt} = \{\phi, \Omega_{ix}\}$. Let P_{ix} be the probability measure on $\mathscr{F}_{ix}$ corresponding to a Poisson process with intensity ρ_i. Letting $(\Omega', \mathscr{F}', P')$ be the product $(\prod \Omega_{ix}, \prod \mathscr{F}_{ix}, \prod P_{ix})$, we next obtain Ω by removing from Ω' each ω' having two or more different ω_{ix}'s with an atom at the same point. We then *complete* $\Omega \cap \mathscr{F}'$ with respect to the trace of P' on it, obtaining our basic probability space $(\Omega, \mathscr{F}, P)$. The σ-fields $\mathscr{F}^{st}$ and $\mathscr{F}^{t}$ are obtained by corresponding completion operations.

Let $0 < \tau_{ix1} < \tau_{ix2} < \cdots$ be the atoms of ω_{ix} in $(0, \infty)$. For $T \in R_1$ let $T > \tau_{ix1}^T > \tau_{ix2}^T > \cdots$ be the atoms in $(-\infty, T)$. We may write $\tau_{ixn} = \tau(ixn)$.

If $t \in R_1$ the transformation U_t on Ω which subtracts t from the coordinate of each atom of each ω_{ix} is measure-preserving; it is also mixing because the individual Poisson flows are mixing ([3], page 367). Let $U_t f(\omega) = f(U_t \omega)$.

(9.1) DEFINITIONS. For $N \geqq 1$ and $x, y \in Z_d$ let $S_N(x, y)$ be the set of ordered triples (X, Y, ι) where

$$X = (x_0, x_1, \cdots, x_N), \quad x_0 = x, \quad x_N = y,$$
$$Y = (y_1, y_2, \cdots, y_N),$$
$$\iota = (i_1, i_2, \cdots, i_N), \quad 1 \leqq i_1, \cdots, i_N \leqq m$$

and where we also require $x_k \neq x_{k-1}$ and $x_k \in x_{k-1} W_{i_k y_k}$, $1 \leqq k \leqq N$. If $n_1, \cdots, n_N \geqq 1$, let I_k, $0 \leqq k \leqq N$ be random intervals defined as follows:

$$I_0 = (0, \tau(i_1 y_1 n_1)),$$
$$I_k = (\tau(i_k y_k n_k), \tau(i_{k+1} y_{k+1} n_{k+1})) \qquad 1 \leqq k \leqq N-1,$$
$$I_N = (\tau(i_N, y_N, n_N), t],$$

or ϕ if the indicated interval is not proper. Let

$$\Gamma(z) = \{(i, z'): 1 \leqq i \leqq m, z' \in Z_d, z \notin zW_{iz'}\}, \qquad z \in Z_d.$$

If $x \neq y$, the event $\{\xi_t^x(y) = 1\}$ is defined as

$$\text{(9.2)} \qquad \bigcup_{N \geqq 1} \bigcup_{(X,Y,\iota) \in S_N(x,y)} \bigcup_{n_1,\cdots,n_N \geqq 1} \{[0 < \tau(i_1 y_1 n_1) < \cdots < \tau(i_N y_N n_N) \leqq t] \cap \bigcap_{0 \leqq k \leqq N} \bigcap_{(i,z) \in \Gamma(x_k)} \bigcap_{n \geqq 1} (\tau(izn) \notin I_k)\}.$$

If $x = y$, $\{\xi_t^x(y) = 1\}$ denotes the union of (9.2) with

$$\bigcap_{(i,z) \in \Gamma(x)} \{\tau(iz1) > t\}.$$

Put $\xi_0^x(y) = 1$ iff $y = x$. Now define

$$\text{(9.3)} \qquad \xi_t^x = \xi_t^x(\omega) = \{y : \xi_t^x(y) = 1\},$$
$$\xi_t^A = \bigcup_{x \in A} \xi_t^x, \quad A \in \Xi_d.$$

To visualize this, let $\mathbb{Z} = Z_d \times R_1 = \{(x, t): x \in Z_d, t \in R_1\}$. "Up" in $\mathbb{Z}$ means in the direction of increasing t. Put a symbol D (for "death") at each point (x, t) such that some ω_{iz} has an atom at t and $x \notin xW_{iz}$. (If $t > 0$, then $t =$ some τ_{izn}.) We may imagine that D causes the death of a particle if one is present at (x, t). For each $x \neq y$, $t \in R_1$ such that some ω_{iz} has an atom at t and $y \in xW_{iz}$, put an arrow with tail at (x, t) and head at (y, t). We may imagine that if a particle is present at (x, t), it generates another one at (y, t). Note that several arrows may overlap. Let $\mathscr{X}$ be $\mathbb{Z}$ together with the above D's and arrows.

An *active path* (up) in $\mathscr{X}$ from $(x, 0)$ to (y, t), where $t > 0$, is a sequence of alternately vertical and horizontal directed segments $(x_0, 0)$ to (x_0, t_1) to (x_1, t_1) to $(x_1, t_2) \cdots$ to (x_{N-1}, t_N) to (x_N, t_N), and finally to (x_N, t) if $t_N < t$, where $N \geqq 1$, satisfying the following requirements:

(a) $0 < t_1 < \cdots < t_N \leqq t$; $x_0 = x$, $x_N = y$, $x_k \neq x_{k-1}$ for $1 \leqq k \leqq N$.

(b) There is no D in the *interior* of any vertical segment nor at the point (y, t).

(c) Each horizontal segment (x_{k-1}, t_k) to (x_k, t_k) coincides with an arrow whose tail is at (x_{k-1}, t_k) and whose head is at (x_k, t_k), $k = 1, 2, \cdots, N$.

If $x = y$ an active path may also be the single vertical segment $(x, 0)$ to (x, t) if no D lies in the interval $(0, t]$. We agree there is always an active path from $(x, 0)$ to $(x, 0)$. If $s < t$, an active path from (s, x) to (t, y) is constructed similarly. We then have

(9.4) $$\xi_t^A = \{y: \exists \text{ an active path from } A \times \{0\} \text{ to } (y, t)\}, \qquad t \geqq 0.$$

Assuming for the moment that we have verified the appropriate properties of ξ_t^A, let us note that we can construct associate processes on the same $\mathscr{X}$. Fix some $T > 0$ and let the Poisson sequences $T > \tau_{ix1}^T > \tau_{ix2}^T > \cdots$ be defined as earlier in this section. If $t \geqq 0$, an active path *down* from (x, T) to $(y, T - t)$ is constructed like a path up except that we go down and use the transformation W_{ix}^* instead of W_{ix} at the time τ_{ixn}^T. We ignore any $\tau_{ijn} = T$ but include $T - t$. Then for fixed T and t an active path down from (x, T) to $(y, T - t)$ exists a.s. iff an active path up exists from $(y, T - t)$ to (x, T), the two events being possibly different if some $\tau_{ijn} = T$ or $T - t$. Hence if we put

(9.5) $${}_T\xi_t^{*A} = \{y: \exists \text{ an active path down from } A \times \{T\} \text{ to } (y, T - t)\}, \qquad t \geqq 0$$

we obtain a process with the same properties as $\{\xi_t^A\}$ but using W_{ix}^* instead of W_{ix}. It is clear that

$$\{{}_T\xi_T^{*A} \# B\} = \{\xi_T^B \# A\} \quad \text{a.s.},$$

giving a graphical exhibition of association.

Vasil'ev [26] and Clifford and Sudbury [2] have noted this kind of relationship for certain graphical representations.

10. Properties of ξ_t^A. The following properties can be verified by routine arguments depending on the construction.

(10.1) $\xi_t^A(\omega)$ is right continuous in t and for each $s \geqq 0$ is $\mathscr{B}(\Xi) \times (\mathscr{B}_1 \cap [0, s]) \times \mathscr{F}^{0s} \to \mathscr{B}(\Xi)$ measurable.

(10.2) Putting $F(A, t, \omega) = \xi_t^A(\omega)$, we have

$$F(A, s + t, \omega) = F(F(A, s, \omega), t, U_s\omega), \qquad s, t \geqq 0,$$

where U_s was defined in Section 9. The Markov property of ξ_t^A follows from this because $F(A, s, \omega) \in \mathscr{F}^{0s}$ and $F(\xi, t, U_s\omega) \in \mathscr{F}^{s,s+t}$. We let $P(t, \xi, \Gamma)$ be the transition function.

(10.3) If $A \in \Xi_d$, $0 < |A| < \infty$, let $H = \{y: xW_{iy} \neq x$ for some i and some $x \in A\}$. Let $\tau = \tau_A = \inf\{\tau_{iy1}: 1 \leqq i \leqq m, y \in H\}$. Then $\{\tau \leqq t\} \in \mathscr{F}^t$, $t \geqq 0$. Let $W_\tau = W_{iy}$ if $\tau = \tau_{iy1}$. If F is as in (10.2), then

$$\begin{aligned} F(A, t, \omega) &= A, & \tau > t \\ &= F(AW_\tau, t - \tau, U_\tau\omega), & 0 \leqq \tau \leqq t. \end{aligned}$$

(10.4) There are constants $c_1, c_2 > 0$ (depending on the process) such that

$$P\{\xi_t^x(y) = 1\} \leqq \sum_{N \geqq c_2|x-y|} \frac{(c_1 t)^N}{N!}, \qquad t \geqq 0.$$

In fact if $x \neq y$, $\{\xi_t^x(y) = 1\}$ is contained in the event in (9.2) without the conditions $\tau(izn) \notin I_k$, and the required estimate follows from properties of Poisson processes.

(10.5) From (10.4) it follows that $P(t, \xi, \Gamma)$ has the Feller property. Another consequence of (10.4) is that if $|A| < \infty$,

$$P\{\bigcup_{0 \leq s \leq t} \xi_s^A \text{ is a finite set}\} = 1 .$$

(10.6) From (10.3) we find that if g is a cylinder function and $A \in \Xi_d$, $|A| < \infty$, then

$$\lim_{t \downarrow 0} \mathscr{E} t^{-1}(g(\xi_t^A) - g(A)) = \sum_{ix} \rho_i(g(AW_{ix}) - g(A)) .$$

It follows from (10.4) that this equation holds for cylinder functions even if $|A| = \infty$, and hence $\mathscr{A}$ is given by (8.3) for cylinder functions. According to a result of [16], this uniquely determines $\mathscr{A}$.

(10.7) REMARK. We shall use P_ξ to denote probabilities for any right-continuous process with initial value ξ having the transition function $P(t, \eta, \Gamma)$. Any such process is strong Markov with respect to $\{\mathscr{F}^{0t}, t \geqq 0\}$ and $\{\mathscr{F}^{0,t+0}, t \geqq 0\}$ (See, [4], Chapter 3). We can always assume right-continuity for all our processes.

(10.8) DEFINITION. If $x \in Z_d$ and $\omega \in \Omega$ let $V_x\omega$ be the element of ω having each ω_{iy} replaced by $\omega_{i,y+x}$. Let $V_x f(\omega) = f(V_x\omega)$. Then V_x is measure-preserving.
It can be verified that if $A, B \in \mathscr{F}$ and $\varepsilon > 0$ there exists $R > 0$ such that $|x| \geqq R$ implies $|P(A \cap V_x B) - P(A)P(B)| \leqq \varepsilon$. From this follows

(10.9) LEMMA. *Let $f(\omega)$ have the values 0 or 1 only and suppose $P(f = 1) > 0$. If $x_1, \cdots, x_n$ are distinct points of Z_d then*

$$P\{\textstyle\sum_{i=1}^n V_{x_i} f \geqq 1\} \geqq 1 - \varepsilon_n ,$$

where ε_n depends only on n and f, and $\varepsilon_n \to 0$.

This is because, given $\varepsilon > 0$, we can pick m of the x_i, say $y_1, \cdots, y_m$, where $m > c_n$ and c depends only on the dimension, so that the variance of $\sum V_{y_i} f/m$ is $< \varepsilon$.
From (10.9) we get the following corollary.

(10.10) LEMMA. *Let $\{\xi_t\}$ be an additive process such that*

$$P_O\left\{\inf_{t>0} \frac{|\xi_t|}{t} > 0\right\} > 0 .$$

Then

$$P_A\left\{\inf_{t>0} \frac{|\xi_t|}{t} > 0 \,\middle|\, |\xi_t| \to \infty\right\} = 1 , \qquad 0 < |A| < \infty .$$

PROOF. Let f be the indicator of $\{\inf |\xi_t^O|/t > 0\}$. Since $\xi_t^A = \bigcup_{x \in A} \xi_t^x$, it follows from (10.9) that $P_A\{\inf |\xi_t|/t > 0\}$ is arbitrarily close to 1 if $|A|$ is sufficiently large. The lemma follows from the strong Markov property and the condition $|\xi_t| \to \infty$. □

11. Results in discrete time. In Sections 11 and 12 we establish some estimates for discrete-time processes in Ξ_1 that will give lower bounds for the growth rates of certain continuous-time additive processes in Ξ_d. Some of the methods are familiar from the theory of percolation processes. In Sections 11 and 12 t is an integer $\geqq 0$.

Let $Z = \{(x, y): x, y = 0, 1, 2, \cdots\}$. From each (x, y) go directed bonds to $(x, y+1)$ and to $(x+1, y+1)$. The bonds are independently "active" or "passive" with probabilities p or $1-p$. A *path* is a finite or infinite sequence of directed bonds, each beginning at the end of the proceding one, with no self-intersections. It is active if all its bonds are active. A single point (x, y) is considered an active path.

Let A_i be the event that there is an infinite active path from $(i, 0)$, $i = 0, 1, 2, \cdots$.

(11.1) LEMMA. *If* $p \geqq p_0 > \frac{8}{9}$, *then*

$$P\{A_0 \cup A_1 \cup \cdots \cup A_{k-1}\} \geqq 1 - c_1[9(1-p)]^{k+1}, \qquad k = 1, 2, \cdots, \tag{11.2}$$

where $c_1 > 0$ *depends only on* p_0.

PROOF. This extends the result of Hammersley [8] that if $p > p_c$, where it is known that $.6 < p_c < .85$, then $P(A_0) > 0$. The method here is essentially the same but the proof is included because [8] is not widely available and the result is essential to what follows. Embed Z in R_2. Construct vertical lines with abscissae $x/2$, $x = -1, 1, 3, 5, \cdots$, and lines with slope 1 through $(0, y/2)$, $y = \pm 1, \pm 3, \pm 5, \cdots$. Each $(x, y) \in Z$ is now the center of one of the parallelogram areas π (interior and boundary) cut out by the lines. Let ω be a realization of the assignments "active" and "passive." Fix $k \geqq 1$ and let

$$T_0 = \{(i, 0): i = 0, 1, \cdots, k-1\},$$

and

$$T_{n+1}(\omega) = \{(x, y) \in Z: (x, y) \notin \textstyle\bigcup_{j=0}^{n} T_j, \exists \text{ an active bond to } (x, y) \text{ from some } (x', y') \in T_n\}, \qquad n = 0, 1, \cdots.$$

If $T_n = \emptyset$ we take $T_{n+1} = \emptyset$. Let $T = \bigcup T_n$. We now consider an ω such that $|T(\omega)| < \infty$.

Let S_j be the union of the π's centered at the points of T_j, except that S_0 is first extended downward slightly at each point $(j/2, 0)$, $j = 1, 3, \cdots, 2k-3$, so that the boundary of S_0 becomes a simple closed curve. If $S_j \neq \emptyset$, each π in S_j has a side in common with some π in S_{j-1}, $j \geqq 1$; hence the outer boundary of $\bigcup S_j$ is a simple closed curve J. A clockwise circuit of J beginning at $(-\frac{1}{2}, -1)$ goes straight up on U segments (i.e., sides of a π), straight down on U, to the right slanting up on R segments and to the left slanting down on R, where $R \geqq k$ and $U \geqq 1$. Also each segment traversed to the right or straight down is intersected by a passive bond. Such a circuit will be called "passive."

Let n_{RU} be the number of circuits with R segments to the right and U up. Then $n_{RU} \leqq 3^{2R+2U}$, since there are $\leqq 3$ choices for continuing each step of the circuit (one could do better; see, [8]). Hence the expected number of passive circuits is

$$\sum_{R \geqq k, U \geqq 1} n_{RU}(1-p)^{R+U} \leqq \frac{[9(1-p)]^{k+1}}{(1-9(1-p))^2} .$$

Then

$$\begin{aligned} P(A_0 \cup \cdots \cup A_{k-1}) &= 1 - P(|T(\omega)| < \infty) \\ &= 1 - P\{\exists \text{ a passive circuit}\} \geqq 1 - \frac{[9(1-p)]^{k+1}}{(1-9(1-p))^2} , \end{aligned}$$

and the lemma is proved. □

(11.3) LEMMA. *Let ν_k, $k = 1, 2, \cdots$ be the number of integers $j = 1, 2, \cdots, k$ such that A_j occurs. If $p \geqq p_0 > \frac{8}{9}$, there are numbers $c_2, c_3 > 0$, $0 < c_4 < 1$, depending on p_0, such that*

(11.4) $$P(\nu_k < c_3 k) \leqq c_2(c_4)^k , \quad k \geqq 1 .$$

PROOF. Let L_n be the section $\{(x, n): x = 0, 1, \cdots\}$ in Z. Fix $N \geqq 1$ and let B_j be the event that there is an active path from $(j, 0)$ to L_N. Let ν_k^N be defined like ν_k but with B_j replacing A_j. We shall show that $P(\nu_k^N < c_3 k) \leqq$ right side of (11.4); the desired result follows because $\nu_k^N \downarrow \nu_k$.

Let Λ_i be the set of paths λ from $(i, 0)$ to L_N, $i = 0, 1, \cdots$; $\Lambda = \bigcup \Lambda_i$. For $\lambda \in \Lambda$ let $\mathscr{F}_\lambda$ be the σ-field for the set of bonds between L_0 and L_N, on or to the left of λ. If $n \geqq 1$, $k \geqq 1$, $\lambda \in \Lambda_k$, and $t_1, \cdots, t_n$ are integers $\geqq 1$ whose sum is k, let $D_{k\lambda n}(t_1, \cdots, t_n)$ be the event that λ is active, no other path from $(k, 0)$ to L_N consisting of bonds on or to the left of λ is active, the events B_i, $i = t_1, t_1 + t_2, \cdots, t_1 + t_2 + \cdots + t_{n-1}$ occur, and no other B_i occurs for $1 \leqq i < k$. Then $D_{k\lambda n}(t_1, \cdots, t_n) \in \mathscr{F}_\lambda$. Let $\tau_1 = \inf\{i: i \geqq 1, B_i \text{ occurs}\}$; then $\tau_1 < \infty$ a.s. from Lemma 11.1. Assuming $\tau_1 + \cdots + \tau_n < \infty$ for some ω, let τ_{n+1} be the smallest $i \geqq 1$ such that $B_{\tau_1 + \cdots + \tau_n + i}$ occurs. Assuming we have shown $\tau_1, \cdots, \tau_n < \infty$ a.s., we have, putting $t_1 + \cdots + t_n = k$,

$$P\{\tau_1 = t_1, \cdots, \tau_n = t_n, \tau_{n+1} \leqq t\} = \sum_{\lambda \in \Lambda_k} P(D_{k\lambda n}(t_1, \cdots, t_n), \tau_{n+1} \leqq t) .$$

Let $E_{k\lambda t}$ be the event that there is an active path from some $(k + i, 0)$ to either λ or the part of L_N to the right of λ, $1 \leqq i \leqq t$. Then $E_{k\lambda t}$ is independent of $D_{k\lambda n}$ and the above expression is, from (11.2)

$$\begin{aligned} \sum P(D_{k\lambda n}(t_1, \cdots, t_n) \cap E_{k\lambda t}) &= \sum P(D_{k\lambda n}(t_1, \cdots, t_n)) \cdot P(E_{k\lambda t}) \\ &\geqq \sum P(D_{k\lambda n}(t_1, \cdots, t_n)) \cdot P(A_1 \cup \cdots \cup A_t) \\ &\geqq P(\tau_1 = t_1, \cdots, \tau_n = \tau_n)(1 - c_1[9(1-p_0)]^{t+1}) . \end{aligned}$$

Hence

(11.5) $$P(\tau_{n+1} > t \mid \tau_1 = t_1, \cdots, \tau_n = t_n) \leqq c_1[9(1-p_0)]^{t+1} \qquad t = 1, 2, \cdots ,$$

and $P(\tau_1 > t)$ is also $\leqq$ the right side of (11.5). Hence taking $c > 0$ so that

$9(1-p_0)e^c < 1$, we have $\mathscr{E}e^{c(\tau_1+\cdots+\tau_n)} \leqq (c_2)^n$, $n \geqq 1$, where $c_2 > 1$. Take $c_3 > 0$ such that $(c_2)^{c_3}e^{-c} < 1$. Then if $c_3 j$ is an integer,

$$P(\nu_j^N < c_3 j) = P(\tau_1 + \cdots + \tau_{c_3 j} > j) \leqq [(c_2)^{c_3}e^{-c}]^j = (c_4)^j, \qquad j = 1, 2, \cdots .$$

If $c_3 j$ is not an integer we require the extra factor c_2 in (11.4). □

(11.6) DEFINITION. Let $[u]$ denote the integer part of u. If $\beta > 0$ let $A_{t,\beta}$, $t = 1, 2, \cdots$, be the event that there is an active path from some $(i, 0)$ to some (j, t) with $0 \leqq i \leqq j \leqq [\beta t]$.

(11.7) LEMMA. *If* $\frac{8}{9} < p_0 \leqq p \leqq p_1 < 1$, *we can find* $\beta > 0$ *and* $0 < \alpha < 1$, *depending only on* p_0 *and* p_1, *such that*

$$P(A_{t,\beta}) \leqq (\alpha)^t, \quad t \geqq 1 .$$

PROOF. Let $X_0, X_1, \cdots$ be a random walk with $X_0 = 0$; given $X_t = k$ $(k = 0, 1, \cdots)$, let $X_{t+1} = k$ or $k+1$ according as the vertical bond out of (k, t) is active or passive. We shall show that if $X_T > k$, there is no active path from any $(i, 0)$ to any (j, T) with $0 \leqq i, j \leqq k$. Suppose R were such a path. Connect the points (X_t, t), $t = 0, 1, \cdots, T$, by straight line segments to make a path S from $(0, 0)$ to (k', T), $k' > k$. Going down on R from (j, T) let (m, s) be the first point on S. Then (m, s) is a lattice point and necessarily $(m+1, s) \in R$, since (m, s) must be at the bottom of a vertical bond of R. Since every bond of R is active, this implies $(m+1, s) \in S$, a contradiction. It follows that

$$P(A_{t,\beta}) \leqq P(X_t \leqq [\beta t]) \leqq P(e^{-X_t} \geqq e^{-\beta t}) \leqq [e^{\beta}((1-p_1)e^{-1} + p_1)]^t .$$

Pick β so that the quantity in the bracket (which we call α) is < 1, and the proof is finished. □

(11.8) DEFINITION. Taking β as in Lemma 11.7, for $t \geqq 1$ let μ_t be the number of integers i, $1 \leqq i \leqq [\beta t]$, such that there is an active path from $(i, 0)$ to some (j, t), $j > [\beta t]$.

(11.9) LEMMA. *If* $\frac{8}{9} < p_0 \leqq p \leqq p_1 < 1$, *then*

$$P(\mu_t \geqq c_3\beta t - c_3) \geqq 1 - \alpha^t - c_2(c_4)^{\beta t - 1},$$

where α, β, c_2, c_3, c_4 *are as in Lemmas* 11.3 *and* 11.7.

PROOF. Using Lemmas 11.3 and 11.7, if βt is an integer then

$$P(\mu_t \geqq c_3\beta t) \geqq P\{(\mu_t \geqq c_3\beta t) \cap (A_{t,\beta})^c\} \geqq P\{(\nu_t \geqq c_2\beta t) \cap (A_{t,\beta})^c\}$$
$$\geqq 1 - P(\nu_t < c_3\beta t) - P(A_{t,\beta}) \geqq 1 - c_2(c_4)^{\beta t} - \alpha^t .$$

If βt is not an integer, a slight adjustment gives the desired result. □

12. Growth rate of a discrete time process. We shall get a lower bound on the growth rate of a "two-sided" process in discrete time. Let Z_2 be the set of integer pairs (x, t); consider Z_2 as embedded in R_2. From each $(x, t) \in Z_2$, directed bonds go to $(x-1, t+1)$, $(x, t+1)$, $(x+1, t+1)$. They are called respectively *left*, *vertical*, and *right* bonds. A *path* is as in Section 11 with three kinds of bonds instead of two. A path with only vertical and left (right) bonds is a *left* (*right*) *path*. The bonds are independently active with probability p. Let A_i^r be the event that there is an infinite active *right* path from $(i, 0)$, $i = 0, \pm 1, \cdots$. Then A_i^r has the same probability as A_i of Section 11, and similarly for several events $A_i^r, A_j^r, \cdots$ jointly.

Let Ξ_1 be the set of subsets of Z_1. For each $\eta \in \Xi_1$, define a Ξ_1-valued Markov random function ξ_t^η, $t = 0, 1, \cdots$, with $\xi_0^\eta = \eta$, as follows. Given $\xi_t^\eta = \xi$, then $x \in \xi_{t+1}^\eta$ iff there is an active bond from some (y, t), $y \in \xi$ to $(x, t+1)$. From Section 11 we see that if $p > \frac{8}{9}$ then $P(\xi_t^0 \text{ never } \emptyset) \geqq P(A_0^r) > 0$.

(12.1) LEMMA. *Suppose $p \geqq p_0 > \frac{8}{9}$. Given $\varepsilon > 0$, there exists N depending only on p_0 and ε such that if $i_1, i_2, \cdots, i_N$ are distinct integers, then*

$$P(A_{i_1}^r \cup A_{i_2}^r \cup \cdots \cup A_{i_N}^r) > 1 - \varepsilon .$$

PROOF. Suppose $p = p_0$, using a domination argument for the case $p > p_0$. Let Y_i be the indicator of A_i^r. The stationary process $\cdots Y_{-1}, Y_0, Y_1, \cdots$ is mixing. It follows that the variance of $(Y_{i_1} + \cdots + Y_{i_n})/n$ is $\leqq \delta_n$ where $\delta_n \to 0$ and δ_n depends only on n. The lemma follows from this. $\square$

(12.2) THEOREM. *If $p > \frac{8}{9}$, then*

$$P\left\{\inf_{t \geqq 1} \frac{|\xi_t^\eta|}{t} > 0 \,\middle|\, \xi_t^\eta \text{ never } \emptyset\right\} = 1 , \qquad 0 < |\eta| < \infty . \tag{12.3}$$

PROOF. The result is obvious if $p = 1$. We now assume $\frac{8}{9} < p_0 \leqq p \leqq p_1 < 1$ as in Lemma 11.7. Let μ_t' be the number of integers i, $-[\beta t] \leqq i \leqq -1$, such that there is an active left path from some $(j, 0)$ with $j \geqq 0$ to (i, t), $t = 1, 2, \cdots$, where β is as in Lemmas 11.7 and 11.9.

For simplicity we prove the theorem with $\eta = O$, and we write ξ_t for ξ_t^O. From the construction we have $\{|\xi_t| \geqq k\} \cap A_0^r \supset (\mu_t' \geqq k) \cap A_0^r$. From Lemma 11.9, since μ_t' and μ_t have the same distribution, putting $k = c_3\beta t - c_3$

$$\begin{aligned} P\{|\xi_t| \geqq c_3\beta t - c_3, A_0^r\} \geqq P\{\mu_t' \geqq k, A_0^r\} &\geqq P(\mu_t' \geqq c_3\beta t - c_3) \cdot P(A_0^r) \\ &\geqq (1 - \alpha^t - c_2(c_4)^{\beta t - 1}) \cdot P(A_0^r) , \end{aligned}$$

where the correlation inequality can be seen directly or deduced from Lemma 4.1 of [9]. From the Borel–Cantelli lemma,

$$P(|\xi_t| < c_3\beta t - c_3 \text{ i.o.} \mid A_0^r) = 0 ,$$

and from this

$$P\left(\inf_{t \geqq 1} \frac{|\xi_t|}{t} > 0 \,\middle|\, A_0^r\right) = 1 . \tag{12.4}$$

Given $\varepsilon > 0$, let N be determined as in Lemma 12.1, $\tau = \inf\{t\colon |\xi_t| \geqq N\}$ or ∞ if not defined as an integer, $B = \{\xi_t \text{ never } \emptyset\}$. Then (taking only finite η)

$$P\left\{\inf \frac{|\xi_t|}{t} = 0, B\right\} \leqq \sum_{0 \leqq s < \infty} \sum_{\eta : |\eta| \geqq N} P(\tau = s, \xi_\tau = \eta) \cdot P\left\{\inf \frac{|\xi_t^\eta|}{t} = 0\right\} .$$

From (12.4) and Lemma 12.1 we have, for $|\eta| \geqq N$,

$$P\left\{\inf \frac{|\xi_t^\eta|}{t} = 0\right\} \leqq P\{(\bigcup_{i\in\eta} A_i^r)^c\} \leqq \varepsilon ,$$

showing that $P\{\inf |\xi_t|/t = 0, B\} = 0$. □

13. Growth in continuous time. In this section a discrete time process $\{\xi_n\}$ with the transition law of one of the random functions $\{\xi_n^\eta\}$ of Section 12 will be called a "basic process with parameter p."

A Ξ_1-valued process $\eta_0, \eta_1, \cdots$ is said to *dominate* a Ξ_1-valued process $\zeta_0, \zeta_1, \cdots$ if for arbitrary initial η_0 and ζ_0 with $\eta_0 \supset \zeta_0$ there is a joint realization with $\eta_n \supset \zeta_n$, $n = 0, 1, \cdots$. A similar definition holds for continuous time. Actually domination refers to transition laws rather than processes. A process does not necessarily dominate itself, but an additive process does.

In this section $\{\eta_t, t \geqq 0\}$ will be a contact process in Z_1 with $\mu = 1$, basic neighborhood[4] $N = \{-1, 1\}$, and $\lambda_k = k\lambda$, $k = 0, 1, 2$. This process will be used to get lower bounds for others. It is additive and self-associate.

(13.1) LEMMA. *For each $0 < p < 1$, there exist $\Delta > 0$ and $\lambda > 0$ such that the discrete skeleton $\{\eta_{n\Delta}, n = 0, 1, \cdots\}$ dominates the basic process $\{\xi_n\}$ with parameter p.*

PROOF. We first construct an intermediate process ζ_n dominated by $\eta_{n\Delta}$. Let p be given; Δ and λ will be determined later. We use a construction for $\{\eta_t\}$ like that in Section 9 of [11], but it seems advisable to give details here because of a substantial difference due to the different nature of the dominated process and because the lemma is essential to what follows.

For $x \in Z_1$ and $i = 1, 2, \cdots$ let $\alpha_i'(x)$ be independent exponential random variables with mean 1. For $x \in Z_1$, $i = 1, 2, \cdots$ and $j = 1, 2$, or 3 let $\alpha_{ij}(x)$ be independent of each other and the α's, $\text{Prob}\{\alpha_{ij}(x) \leqq u\} = (1 - e^{-u})^{\frac{1}{3}}$, $u \geqq 0$; then $a_i(x) = \max_j \alpha_{ij}(x)$ is exponential with mean 1. Let the integer $N \geqq 1$ be fixed until further notice. We will define $\eta_t^{(N)} = \eta_t$ so that $\eta_t(x) \equiv \eta_0(x)$ for $|x| > N$. If $\eta_0(x) = 0$, let $\sigma_1(x) = \alpha_1'(x)$, $\sigma_2(x) = \alpha_1'(x) + \alpha_2(x)$, $\sigma_3(x) = \alpha_1'(x) + \alpha_2(x) + \alpha_3'(x)$, etc., alternating α and α'. If $\eta_0(x) = 1$, let $\sigma_1(x) = \alpha_1(x)$, $\sigma_2(x) = \alpha_1(x) + \alpha_2'(x)$, $\cdots$, again alternating. Associate with each x, $|x| \leqq N$, a clock function $S_x(t)$ with $S_x(0) = 0$ and

$$(13.2)\qquad \frac{dS_x}{dt} = \eta_t(x) + (1 - \eta_t(x))\lambda(\eta_t(x - 1) + \eta_t(x + 1)) , \quad |x| \leqq N .$$

The kth jump of the coordinate $\eta_t(x)$ occurs at the smallest t such that $S_x(t) = \sigma_k(x)$. The details for an essentially identical situation are given in Section 4 of [10].

Let $V_1(x) = \min(\alpha_{11}(x), \Delta)$, $V_3(x) = \min(\alpha_{13}(x), \Delta)$. Assuming $\zeta_0 = \eta_0$ and letting 1(A) be the indicator of A, define, for $x \in Z_1$,

$$(13.3)\qquad \zeta_1(x) = \eta_0(x)1(\alpha_{12}(x) > \Delta) + 1(\alpha_2(x) > \Delta)\cdot(1 - \eta_0(x))\cdot(I + J - IJ) ,$$

[4] See the end of Section 8.

where

$$I = \eta_0(x-1)1(\lambda V_3(x-1) > \alpha_1'(x)) ,$$
$$J = \eta_0(x+1)1(\lambda V_1(x+1) > \alpha_1'(x)) .$$

Checking the variables involved in the definition, we find that the random variables $\zeta_1(x)$, $|x| \leqq N$, are independent. If $|x| \leqq N$, then $\eta_\Delta(x) \geqq \zeta_1(x)$. For example suppose $\eta_0(x) = 0$, $\eta_0(x-1) = 0$, $\eta_0(x+1) = 1$ for some x, $|x| \leqq N$. If $\zeta_1(x) = 1$, then $\lambda\alpha_1(x+1) \geqq \lambda\alpha_{11}(x+1) > \alpha_1'(x)$, $\lambda\Delta > \alpha_1'(x)$, and $\alpha_2(x) > \Delta$. It follows that $\eta_t(x)$ becomes 1 before Δ and remains 1 until after Δ. Similar arguments hold for other cases. Moreover, letting N go to ∞, using the fact that $\{\eta_t^{(N)}\}$ converges weakly in law to $\{\eta_t\}$, we find that $\eta_\Delta \supset \zeta_1$. From the nature of the ζ-process the same is true whenever $\eta_0 \supset \zeta_0$.

If $\eta_0(x) = 1$ then $\text{Prob}\{\zeta_1(x) = 1\} = 1 - (1 - e^{-\Delta})^{\frac{1}{3}}$; if $\eta_0(x) = 0$ and $\eta_0(x-1) + \eta_0(x+1) = k$ then $\text{Prob}\{\zeta_1(x) = 1\} = r_k$, $k = 0, 1, 2$, where

$$\begin{aligned} &r_0 = 0 \\ (13.4)\qquad &r_1 = e^{-\Delta}\,\text{Prob}\{\lambda V_3(x-1) > \alpha_1'(x)\} \\ &r_2 = e^{-\Delta}\,\text{Prob}\{\max(\lambda V_3(x-1), \lambda V_1(x+1)) > \alpha_1'(x)\} \geqq r_1 . \end{aligned}$$

Now pick Δ and λ so that $\text{Prob}\{\zeta_1(x) = 1 \mid \zeta_0(x) = 1\} = 1 - (1 - e^{-\Delta})^{\frac{1}{3}} \geqq 1 - (1-p)^3$ and $r_1 \geqq 1 - (1-p)^3$.

Since the $\zeta_1(x)$ are independent for different x's, and so are the $\xi_1(x)$, this choice of Δ and λ ensures that the pair ζ_0, ζ_1 dominates ξ_0, ξ_1. The lemma is a consequence of this. □

(13.5) THEOREM. *Let $\{\zeta_t\}$ be a contact process in Z_d with basic neighborhood N and parameters $\mu = 1$, $0 = \lambda_0, \lambda_1, \lambda_2, \cdots, \lambda_{|N|}$. Suppose N contains a unit vector (say, $(1, 0, \cdots, 0)$) and its negative. Then for sufficiently large $\lambda_1, \cdots, \lambda_{|N|}$ we have, for $\zeta \neq \emptyset$,*

$$(13.6)\qquad P_\zeta\{\zeta_t \text{ never } \emptyset\} > 0 ,$$

$$(13.7)\qquad P_\zeta\left\{\inf_{t>0}\frac{|\zeta_t|}{t} > 0 \,\middle|\, \zeta_t \text{ never } \emptyset\right\} = 1 .$$

PROOF. We first prove the result for the process $\{\eta_t\}$ in Z, defined above Lemma 13.1. Choose $p = p_0$, $\frac{8}{9} < p_0 < 1$ and then Δ and λ as in Lemma 13.1. From Lemma 13.1, the statement just above Lemma 12.1, and Theorem 12.2 we see that

$$(13.8)\qquad P_\eta\left\{\inf_{n\geqq 1}\frac{\eta_{n\Delta}}{n} > 0\right\} > 0 , \quad \eta \neq \emptyset .$$

From the independence of the death processes at different x's, there are constants $0 < c_1, c_2 < 1$ such that

$$(13.9)\qquad P_\eta\{\inf_{0\leqq t\leqq\Delta}|\eta_t| \leqq c_1|\eta|\} \leqq (c_2)^{|\eta|} , \qquad 0 < |\eta| < \infty .$$

This means $|\eta_t|$ will a.s. not shrink by a factor $\leqq c_1$ on infinitely many intervals $[n\Delta, (n+1)\Delta]$, and hence $P_\eta\{\inf_{t>0}|\eta_t|/t > 0\} > 0$. Since $\{\eta_t \text{ never } \emptyset\} \subset \{|\eta_t| \to \infty\}$ a.s. (see, Lemma 9.3 of [13]), (13.7) holds for $\{\eta_t\}$ by Lemma 10.10.

Now consider $\{\eta_t\}$ in Z_1 embedded in Z_d on the set of integer multiples of $(1, 0, \cdots, 0)$. Arguing almost as Lemma 5.8 of [11], we see that $\{\eta_t\}$ and $\{\zeta_t\}$ can be defined on the same probability space so that $\eta_t \subset \zeta_t$ provided $\lambda_1 \geqq \lambda$ and $\lambda_k \geqq 2\lambda$ for $k \geqq 2$, if $\eta_0 \subset \zeta_0$, and (13.6) follows. If $|\zeta| \geqq N^2$ then either a line parallel to the unit vector $(1, 0, \cdots)$ has N points of ζ or N such lines have at least one point of ζ. Hence if $|\zeta_0| \geqq N^2$, ζ_t contains either a process η_t with $|\eta_0| \geqq N$ or N independent processes η_t on different lines, and (13.7) follows. □

REMARK. In d dimensions we could probably replace t by t^d in the denominator in (13.4).

Letting $p_t(y) = P(\xi_t^O(y) = 1)$, we find from (10.4) that for a suitable constant $c > 0$ we have

$$\sum_{n>0;(n \text{ integer})} \sum_{y:|y|\geqq cn} p_n(y) < \infty$$

showing that $\sup_{n\geqq 1} |\xi_n^O|/n^d < \infty$ a.s. The result can be extended to continuous time and finite initial sets so that

$$P_\xi \left\{\sup_{t\geqq 1} \frac{|\xi_t|}{t^d} < \infty\right\} = 1\,, \qquad |\xi| < \infty\,. \tag{13.10}$$

This is to be expected from the results of Richardson [21].

14. Individual ergodic theorem. If x_t is a Markov process in a state space E with an invariant probability measure μ, and if $f \in L_1(\mu)$, then for a.e. $(\mu)x$, $P_x\{\lim (1/T) \int_0^T f(x_t)\, dt \text{ exists}\} = 1$. Further information is needed before it can be asserted that the limit is constant, and without some absolute continuity properties that do not usually hold for Ξ-valued processes if Z is infinite, we cannot assume that for some specified x the limit exists. In the case of the processes considered in Section 8 of [11], where there is a unique invariant probability measure, the limit can be shown to exist and be a.s. constant for all initial ξ, if f is continuous, by means of the special coupling used there.[5]

For processes with two or more invariant measures the situation is more complicated. Theorem 14.3 below applies to additive processes having an invariant measure besides δ_ϕ. The result is in terms of the set D^* of Definition 14.1, which relates to the associate process. Later in this section some elements of D^* will be identified for certain processes. The condition (c) of Theorem 14.5 indicates why the growth rate established in Section 13 is important.

(14.1) DEFINITION. If $\{\xi_t\}$ is any Ξ-valued process with an associate $\{\xi_t^*\}$, taken to be right continuous, let D^* be the set of $\eta \in \Xi$ with the following property: for each ξ, $0 < |\xi| < \infty$, we have

$$P_\xi^*\{\xi_t^* \# \eta \;\; \forall \text{ sufficiently large } \; t \,|\, \xi_t^* \;\text{ never }\; \varnothing\} = 1\,. \tag{14.2}$$

[5] Two copies of the process with different initial values are defined on the same probability space with a contact process, in such a way that when a coordinate of the latter is 0, the corresponding coordinates of the other processes are equal.

(As the referee has noted, (14.2) implies that $P_\eta\{\xi_t \# \xi\} - P_Z\{\xi_t \# \xi\} \to 0$ if $\eta \in D^*$ and $|\xi| < \infty$.)

(14.3) THEOREM. *Let $\{\xi_t\}$ be a right continuous additive process in Z_d. Then weak $\lim_{t\to\infty} P(t, Z_d, \cdot) = \nu$ exists, and for each $\eta \in D^*$ and each $f \in C$ we have*

$$P_\eta\left\{\frac{1}{T}\int_0^T f(\xi_t)\,dt \to \int_\Xi f(\xi)\,d\nu\right\} = 1\,. \tag{14.4}$$

PROOF. The existence of ν is known for processes with associates, since if $|\xi| < \infty$, $P_Z\{\xi_t \# \xi\} = P_\xi^*\{\xi_t^* \# Z\}$ is a decreasing function of t.

We shall use the representation of Section 9 for $\{\xi_t\}$ and its associate.

Fix $\eta \in D^*$. For the proof we first take $f(\xi) = \xi(O)$. We construct the random function $\{\xi_t^\eta, t \geqq 0\}$ as in Section 9, and also for each $t \geqq 0$ we construct $\{{}_t\xi_s^{*O}, s \geqq 0\}$. Put $\xi_t = \xi_t^\eta$, ${}_t\xi_s^* = {}_t\xi_s^{*O}$. Let $\Omega_t = \{\omega\colon$ no ω_{ix} has an atom at $t\}$. Then

$$\{\xi_t \# O\} = \{{}_t\xi_t^* \# \eta\}\,, \quad \omega \in \Omega_t \cap \Omega_0\,, \qquad t \geqq 0\,.$$

Let X_{tb} be the indicator of the event $\{{}_t\xi_s^* \# \eta\ \forall s \geqq b\}$, $t \geqq 0$, $b \geqq 0$. Then $X_{tb} = U_t X_{0b}$ where U is the shift defined in Section 9, and $\{{}_t\xi_t^* \# \eta\} \supset \{X_{tb} = 1\}$, $0 \leqq b < t < \infty$. Letting S_ω be the set of all atoms of every ω_{ix} of ω, we have for $T \geqq b$, $\omega \in \Omega_0$,

$$\frac{1}{T}\int_b^T \xi_t(O)\,dt = \frac{1}{T}\int_{(b,T)\cap(S_\omega)^c} \xi_t(O)\,dt \geqq \frac{1}{T}\int_{(b,T)\cap(S_\omega)^c} X_{tb}\,dt = \frac{1}{T}\int_b^T U_t X_{0b}\,dt\,.$$

From the Birkhoff theorem and the mixing property of the flow we have

$$\liminf_{T\to\infty} \frac{1}{T}\int_0^T \xi_t(O)\,dt \geqq P_O^*\{\xi_s^* \# \eta\ \ \forall s \geqq b\}\,, \qquad b > 0\,.$$

Since $\eta \in D^*$, we can take b sufficiently large so that, for a given $\varepsilon > 0$, we have

$$\liminf_{T\to\infty} \frac{1}{T}\int_0^T \xi_t(O)\,dt \geqq p_\infty^*(O) - \varepsilon$$

where $p_\infty^*(\xi) = P_\xi^*\{\xi_t^*$ never $\varnothing\}$. Since ε is arbitrary and since $p_\infty^*(O) = \lim_{t\to\infty} P_O^*(\xi_t^* \# Z_d) = \lim_{t\to\infty} P_{Z_d}(\xi_t \# O) = \int \xi(O)\nu(d\xi)$, we see that

$$P_\eta\left\{\liminf \frac{1}{T}\int_0^T \xi_t(O)\,dt \geqq \int \xi(O)\,d\nu\right\} = 1\,, \quad \eta \in D^*\,.$$

Let $Y_b(\omega)$ be the indicator of $\{{}_0\xi_s^* \neq \varnothing, 0 \leqq s \leqq b\}$. Then

$$\begin{aligned}\limsup_{T\to\infty} \frac{1}{T}\int_0^T \xi_t(O)\,dt &\leqq \limsup \frac{1}{T}\int_{(b,T)\cap(S_\omega)^c} U_t Y_b\,dt \\ &= P_O^*({}_0\xi_s^* \neq \varnothing, 0 \leqq s \leqq b)\,.\end{aligned}$$

Since the last quantity $\to p_\infty^*(O)$ as $b \to \infty$, the theorem is proved for $f(\xi) = \xi(O)$. A similar argument applies if $f = \theta_\gamma$, γ finite, and the proof is completed by taking limits. □

Call η *R-dense* if each ball in Z_d of radius R contains a point of η. Call η *dense* if it is R-dense for some $R > 0$.

(14.5) THEOREM. *Let $\{\xi_t\}$ be an additive process such that*

(a) $\{\xi_t \text{ never } \emptyset\} \subset \{|\xi_t| \to \infty\}$ a.s.;
(b) *for each x, y, and $t > 0$, $P_x\{\xi_t(y) = 1\} > 0$;*
(c) $P_\xi\{\inf_{t>0} |\xi_t|/t > 0\} > 0$ *whenever $\xi \neq \emptyset$.*

Then for each dense η and each $\xi \neq \emptyset$

$$P_\xi\{\xi_t \# \eta \quad \forall \text{ sufficiently large } \; t \,|\, \xi_t \; \text{ never } \; \emptyset\} = 1 \, .$$

REMARK. Hence $\eta \in D^*$ if (a), (b), and (c) are satisfied by $\{\xi_t{}^*\}$.

PROOF. We omit the details. Since $P_\xi\{\inf |\xi_t|/t > 0 \,|\, \xi_t \text{ never } \emptyset\} = 1$, given $\varepsilon > 0$ and $\xi \neq \emptyset$ we can find $\alpha > 0$ such that (n is an integer) $P_\xi\{|\xi_n| \geqq \alpha n$ $\forall\, n \,|\, \xi_t \text{ never } \emptyset\} > 1 - \varepsilon$. We can also find $\beta > 0$ such that

$$P_\xi\{|\xi_n| \geqq \alpha n,\, |\xi_{n+1} \cap \eta| \leqq \beta n \;\text{ i.o.}\} = 0 \, ,$$

implying $P_\xi\{|\xi_n \cap \eta| \leqq \beta n \text{ i.o.} \,|\, \xi_t \text{ never } \emptyset\} = 0$, and it is easy to extend the result to continuous t. □

EXAMPLE. Let $\{\xi_t\}$ be an additive contact process with basic neighborhood N and parameters μ and $0 = \lambda_0, \lambda_1, \cdots, \lambda_n$, $n = |N|$. Suppose $\lambda_1 > 0$, and N contains each of the $2d$ unit vectors. Using Theorem 13.5, we see that $\{\xi_t\}$ and $\{\xi_t{}^*\}$ satisfy (a), (b) and (c) of Theorem 14.5 provided the ratios λ_k/μ, $k = 1, \cdots, n$, are sufficiently large, or $\mu = 0$. It follows from Theorems 14.3 and 14.5 that (14.4) is true for $\{\xi_t\}$ provided η is dense.

15. Survival and extinction. Following Section 9 of [11] we call a Ξ-valued process *permanent* if

(15.1) $$\liminf_{t\to\infty} P_\xi\{\xi_t(x) = 1\} > 0 \, , \qquad \xi \neq \emptyset \, , \quad x \in Z \, .$$

This obviously implies

(15.2) $$P_\xi\{\xi_t \;\text{ never }\; \emptyset\} > 0 \, , \quad \xi \neq \emptyset \, ,$$

but there are processes for which (15.2) holds but not (15.1); see, [13], Section 10. We shall see that for certain self-associate contact processes in Z_1, the parameter regions where (15.1) or (15.2) hold have the same interior (in one simple case the regions are identical). For Z_1 this goes beyond Therem 10.1 of Harris [13], because $|\xi| = \infty$ is allowed in (10.1 c) of [13], as we see by examining the statements negating (15.1) and (15.2).

For simplicity we state the result for the self-associate contact process $\{\xi_t\}$ in Z_1 with $\mu = 1$, and basic neighborhood $N = \{-K, -K+1, \cdots, -1, 1, 2, \cdots, K\}$, where $K \geqq 1$, and with $\lambda_j = j\lambda$, $0 \leqq j \leqq 2K$. We suppose K fixed and consider the effect of varying λ. We use the notation $\xi_t{}'$ if $\lambda = \lambda'$, etc. For $\{\xi_t\}$ the basic W's of (8.2) are W_i, $-K \leqq i \leqq K$, $i \neq 0$ (each with intensity λ) where $iW_i = O \cup i$

and $xW_i = x$ otherwise; and W_0 (intensity 1), where $OW_0 = \varnothing$ and $xW_0 = x$ otherwise.

(15.3) THEOREM. *If* (15.2) *holds for* $\{\xi_t'\}$ *with parameter* λ', *then* (15.1) *holds for* $\{\xi_t''\}$ *with parameter* λ'' *provided* $\lambda'' > \lambda'$. *If* $K = 1$, (15.2) *implies* (15.1) *for the same* λ.

PROOF. In [26] it was shown that for an analogous discrete-time process, (15.1) and (15.2) hold in a certain parameter region, although the relation between (15.1) and (15.2) was not studied. To do this we use some ideas of [26] as well as correlation inequalities from Section 7. In case $K \geqq 2$ we must handle the additional problem that when two active paths cross, it is not always possible to form a single active path using part of each.

In the language of [5], Section 4, we are showing that weak and strong large-range connectivity are almost equivalent.

We shall carry out the proof for $K = 2$, which shows the basic ideas. There is also no harm in assuming $\xi_0 = O$.

We need some definitions and a lemma.

(15.4) DEFINITIONS. Let $\delta = \inf_{t \geqq 0} P_O\{\xi_t \neq \varnothing\}$. Let L_t, L_t^+, L_t^- be the set of all points (x, t), $x \in Z_1$, or those with $x \geqq 0$ or $x \leqq 0$ respectively. Let Z^+ (Z^-) be the nonnegative (nonpositive) points of Z_1.

(15.5) DEFINITIONS. We use the probability space of Section 9. Fix $T > 0$. Let D be the event that there exists an A-path (i.e., active path; see, Section 9) from L_0 to $(0, T)$ containing an arrow $(x + 2, s)$ to (x, s) for some $x \in Z_1$, $0 < s < T$ and there exists an A-path from $(0, 0)$ to L_T containing a vertical interval $(x + 1, s')$ to $(x + 1, s'')$, $s' < s < s''$ (this interval need not be maximal). Let D_1 be the same as D except that the phrases "L_0 to $(0, T)$" and "$(0, 0)$ to L_T" are interchanged. D^- is the same as D except that $x + 2$ and $x + 1$ are replaced by $x - 2$ and $x - 1$ respectively. D_1^- is related to D^- as D_1 is to D. Let E_{00} be the event that there is an A-path from $(0, 0)$ to $(0, T)$. Then $E_{00} = \{\xi_T{}^O(O) = 1\} = \{{}_T\xi_T{}^{*O}(O) = 1\}$ (see, (9.4) and (9.5)). *We will henceforth put* $\xi_t{}^O = \xi_t$ *and* ${}_T\xi_t{}^{*O} = \xi_t{}^*$.

REMARK. From symmetry, $P(D) = P(D_1) = P(D^-) = P(D_1^-)$. (Note that D becomes D_1 if we reverse all arrows and turn the graph upside down.)

(15.6) LEMMA. $P\{\xi_T(O) = 1\} + 4P(D) \geqq \delta^4/16$.

PROOF. Assume $\delta > 0$. From symmetry

$$P\{A\text{-path from } (0, 0) \text{ to } L_T{}^+\} \geqq \delta/2\,. \tag{15.7}$$

From the construction of $\{\xi_t\}$

$$\begin{aligned} &P\{A\text{-path from } (0,0) \text{ to } L_T{}^+,\ A\text{-path from } (0,0) \text{ to } L_T{}^-,\ A\text{-path} \\ &\quad \text{from } L_0{}^+ \text{ to } (0,T),\ A\text{-path from } L_0{}^- \text{ to } (0,T)\} = P\{E\} \\ &\quad = P\{\xi_T \# Z^+, \xi_T \# Z^-, (\textstyle\bigcup_{x\geqq 0} \xi_T{}^x) \# O, (\bigcup_{x \leqq 0} \xi_T{}^x) \# O\}\,, \end{aligned} \tag{15.8}$$

where E is defined by the context in (15.8). From symmetry and Lemma 7.2 (transferred by limits to the case of countable Z) we have (note that $\bigcup_{x\geq 0} \xi_T^x = \xi_T^{Z+}$, etc.)

$$P(E) \geqq (\delta/2)^4 . \tag{15.9}$$

Suppose $\omega \in E$. Then some A-path, say π, from (0, 0) to L_T intersects some A-path, say π^*, from L_0 to $(0, T)$. Note that this does not imply the existence of an A-path from (0, 0) to $(0, T)$, although it would in the case $K = 1$. Let (x, s) be the last point of π^*, going up from L_0, which is on π. Suppose now $\omega \in E \cap (E_{00})^c$. Then necessarily $0 < s < T$.

CASE I. (x, s) is interior to a vertical segment of π. Then (x, s) lies on an arrow σ^* of π^* from, say, (x', s) to (x'', s) where $|x' - x''| = 1$ or 2. We cannot have x' or $x'' = x$ since $x' = x$ contradicts $\omega \in (E_{00})^c$ and $x'' = x$ contradicts the definition of (x, s). Hence we must have $|x' - x''| = 2$ with x strictly between x' and x''. Hence $\omega \in D \cup D^-$.

CASE II. (x, s) lies on an arrow σ of π from, say, (y', s) to (y'', s). As in case I, x must be strictly between y' and y''. Hence $\omega \in D_1 \cup D_1^-$. Since I and II exhaust all possibilities we have

$$E \subset E_{00} \cup D \cup D^- \cup D_1 \cup D_1^- , \tag{15.10}$$

whence, using (15.9),

$$\begin{aligned} \frac{\delta^4}{16} &\leqq P(E_{00}) + 4P(D) \\ &= P\{\xi_T(O) = 1\} + 4P(D) , \end{aligned} \tag{15.11}$$

proving Lemma 15.6.

In what follows we use the familiar fact that if $\{u_i\}$ is a Poisson point process in R_1 with intensity ρ and if each point independently of the others is allowed to remain with probability p or is "deleted" with probability $1 - p$, the remaining points $\{u_i'\}$ and the deleted points $\{u_i''\}$ are independent Poisson point processes with intensities $p\rho$ and $(1 - p)\rho$. It is hoped that readers familiar with Poisson processes will accept this and similar statements, which can easily be rigorized by an enlargement of our probability space.

Let W^+ be the additive transformation defined by $yW^+ = y \cup O$ if $y = 1$ or 2; $yW^+ = y$ otherwise. Let W_x^+ be W^+ shifted to x as in (8.4). We defined W above Theorem (15.3).

CONCLUSION OF PROOF OF THEOREM 15.3. Let $\{\xi_t'\}$, $\{\xi_t''\}$, and $\{\bar{\xi}_t\}$ be processes like $\{\xi_t\}$ with parameters $\lambda' < \lambda''$ and $\bar{\lambda} = \frac{1}{2}(\lambda' + \lambda'')$, constructed as in Section 9 with initial state O. Define p by $(1 - p)\bar{\lambda} = \lambda'$. Construct a process $\{\eta_t\}$, with $\eta_0 = O$, by modifying $\{\bar{\xi}_t\}$ as follows: whenever W_{2x} is applied in the construction of $\{\bar{\xi}_t\}$, replace it by W_x^+ with probability p, and leave it unchanged with probability $1 - p$. This means that in the graph construction of $\{\bar{\xi}_t\}$ a single arrow from $(x + 2, s)$ to (x, s) has probability p of being replaced by a double arrow from both $x = 1$ and $x = 2$ to x. We show

$$\text{Prob}\,\{\eta_T(O) = 1\} \geqq p\bar{\delta}^4/80\,, \tag{15.12}$$

where $\bar{\delta}$ is the left side of (15.2) if $\lambda = \bar{\lambda}$ and $\xi = O$. In fact (15.6) implies $\max\,\{\text{Prob}\,(\bar{\xi}_T(O) = 1), \text{Prob}\,(\bar{D})\} \geqq \bar{\delta}^4/80$, where $\bar{D}$ refers to the construction of $\{\bar{\xi}_t\}$ (see (15.5)). Since $\{\bar{\xi}_T(O) = 1\} \subset \{\eta_T(O) = 1\}$, (15.12) is true if $\text{Prob}\,\{\bar{\xi}_T(O) = 1\} \geqq \bar{\delta}^4/80$. If it is $< \bar{\delta}^4/80$, then $P(\bar{D}) \geqq \bar{\delta}^4/80$. We observe that we may construct $\{\eta_t\}$ by first constructing $\{\xi_t\}$, later making certain arrows double. If $\bar{D}$ occurs, every arrow which applies in the definition of $\bar{D}$ has a probability p of becoming double; with a little care in the formulation one can say that the transformations to double arrows are independent events. After any of these arrows becomes double, an active path exists from (0, 0) to (0, T). This proves (15.12).

We complete the proof by showing that if $f(\xi)$ is a continuous increasing function of ξ, then

$$\mathscr{E}f(\xi_t'') \geqq \mathscr{E}f(\eta_t)\,, \tag{15.13}$$

implying $\text{Prob}\,\{\xi_T''(O) = 1\} \geqq p\bar{\delta}^4/80$ and proving the theorem. To show (15.13), note that the generator of $\{\eta_t\}$ is $\sum \mathscr{A}_x$ (see, (8.1)), where (N was defined previously in this section)

$$\mathscr{A}_O f(\xi) = \Delta^- f(\xi) + \lambda' \Delta f(\xi) \cdot \xi(N) + (\bar{\lambda} - \lambda')\Delta f(\xi) \cdot \xi(N') + (\bar{\lambda} - \lambda')\Delta f(\xi) \cdot \theta_{\{1,2\}}(\xi)\,,$$

$\Delta^- f(\xi) = f(\xi \backslash O) - f(\xi)$, $\Delta f(\xi) = f(\xi \cup O) - f(\xi)$, and $N' = \{-2, -1, 1\}$. The generator of $\{\xi_t''\}$ is $\sum \mathscr{A}_x''$, where $\mathscr{A}_O'' f(\xi) = \Delta^- f(\xi) + \lambda'' \xi(N) \Delta f(\xi)$. Then $\mathscr{A}_O'' f(\xi) = \Delta^- f(\xi) + \Lambda''(\xi) \cdot \Delta f(\xi)$, $\mathscr{A}_O f(\xi) = \Delta^- f(\xi) + \Lambda(\xi) \cdot \Delta f(\xi)$, where Λ'' and Λ are positive increasing functions of ξ and $\Lambda''(\xi) \geqq \Lambda(\xi)$. Using the method of proof of Lemma 5.8 of [11] or using Theorem 9.9 of Sullivan [24], we see that (15.13) holds, proving the theorem except for the special result when $K = 1$. This follows readily if we note that if $K = 1$, the event E of (15.1) implies the existence of an active path from (0, 0) to (0, T). □

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